University of Victoria
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Course Notes
Math 422 Combinatorial Mathematics

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Chapter 1

Introduction

1.1 Factorials and Binomial Coefficients

One way of defining *counting numbers*, meaning expressions that represent the number of some sort of configuration, is to first give a formula and then later show what sort of configurations it counts. For example, the symbol $n!$ can be defined as:

$$0! = 1, \quad \text{and } n! = n(n-1)(n-2)\cdots 1 \text{ when } n \geq 1,$$

and then later it can be shown that this equals (for example) the number of permutations of n distinguishable objects. A challenge in this approach is justifying why $0! = 1$. Two possible justifications are (i) it is the definition! and (ii) $0!$, being the product of all integers k with $1 \leq k \leq 0$, is an empty product and an empty product is defined to equal 1 (why?). Similarly, if the symbol $\binom{n}{k}$ is defined for $0 \leq k \leq n$ by the formula

$$\binom{n}{k} = \frac{n!}{k!(n-k)!},$$

then there is a struggle to explain what the symbol might mean if k is outside of the allowed range. Technically, the symbol is not defined at all in these situations, even though (as will be seen below) there is no reason for that.

Another approach is to define counting numbers *combinatorially*, that is, in terms of something they count, and then applying arguments involving counting principles to obtain formulas later. This is the approach we take.

For $n \geq 0$, define n *factorial*, written $n!$, to be the number of arrangements of n distinguishable objects in a line (that is, *permutations*, of n distinguishable objects). Thus $0!$ is the number of linear arrangements of zero distinguishable objects, which is one (the empty arrangement).

For $n \geq 0$, define $\binom{n}{k}$, read "n choose k", to be the number of ways to select a subset of k distinguishable objects from a set of $n \geq 0$ distinguishable objects (without regard for order).

The following are immediate consequences of the definition:

1. $\binom{n}{0} = 1 = \binom{n}{n}$;
2. $\binom{n}{1} = n = \binom{n}{n-1}$;
3. $\binom{n}{k} = 0$, if $k < 0$ or $k > n$.

1.2 Counting Principles

In this section we recall the main "principles" that arise in counting arguments, and give pointers as to how they might be formulated as theorems.

The first principle to be discussed arises from the theorem that if there is a 1-1 correspondence (*i.e.* a 1-1 and onto function) between two sets, then they have the same number of elements. For that reason, this how we define infinite sets having the same cardinality.

Principle of 1–1 Corespondence. If the collection of outcomes of two processes can be put into 1-1 correspondence, then the two processes have the same number of outcomes.

Recall that a 1–1 correspondence between two sets A and B is a 1–1, onto function $f : A \to B$. To show that such a correspondence exists, three things are required:

1. Explain how each element of A gives rise to a unique element of B, that is, describe the function;
2. Explain why different elements of A give rise to different elements of B, that is, argue that the function is 1–1; and
3. Explain why each element of B arises from some element of A, that is, argue that the function is onto.

These three steps can be accomplished by arguing that the reverse of the procedure for obtaining an element of B from an element of A can be applied to an arbitrary element of B. If the reverse procedure is a function, then the given function must have been 1–1 and onto. A different way to proceed is to remember that if A and B are finite sets, then the existence of 1–1 functions $f : A \to B$ and $g : B \to A$ implies $|A| = |B|$, so it suffices to carry out the first two points for both f and g.

One application of the principle is the argument that the number of non-negative integer solutions to $x_1 + x_2 + \cdots + x_n = k$, equals the number of sequences of k ones and $n-1$ plus signs ("+"). These two collections can be put into 1-1 correspondence: the number of ones before the first plus sign equals x_1, the number between the first and second plus signs equals x_2, and so on until, finally, the number of ones after the $(n-1)$-st plus sign equals x_n. For example, the non-negative integer solution $x_1 = 2, x_2 = 0, x_3 = 3$ to $x_1 + x_2 + x_3 = 5$ corresponds to the sequence $11 + +111$. The number in question is $\binom{k+(n-1)}{k}$.

Addition Principle. If the outcomes of a process can be separated into a collection of disjoint cases, then the number of outcomes of the process equals the sum of the number of outcomes of the cases.

Suppose the outcomes of the process can be separated into n disjoint cases (saying that the cases are *disjoint* means that no two of them can arise simultaneously). For $i = 1, 2, \ldots, n$, let A_i be the set of outcomes that can occur in case i. Since the cases are disjoint, $A_i \cap A_j = \emptyset$ if $i \neq j$. The Addition Principle is the statement that, in this situation,

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n|.$$

The Principle of Inclusion and Exclusion, which we will study later, can be though of as generalizing the Addition Principle to the case where the sets need not have pairwise empty intersection.

Multiplication Principle. If a process is carried out as a sequence of steps *in which the number of outcomes of each step does not depend on the outcome of previous steps*, then the number of outcomes of the process equals the product of the number of outcomes of the steps.

Notice the careful wording. It is the *number* of outcomes of subsequent steps that must not change depending on the outcome of previous steps. For example, if we consider forming a line of the three people A, B and C as a process involving the three steps: (i) choose the first person in line; (ii) choose the second person in line; and (iii) choose the third person in line, then there are 3 possible outcomes for the first step, and no matter which of these occurs, there are 2 possible outcomes of the second step, and no matter what outcomes have occurred in steps one and two, there is 1 possible outcome for the third step.

The outcome of a process which is a sequence of n steps can be recorded by successively writing down the outcome of each step and, in so doing, forming a 1-1 correspondence between the collection of outcomes and a particular collection of n-tuples. For $i = 1, 2, \ldots, n$, let A_i be the set of outcomes of step i. Viewed in this way, the Multiplication Principle is the statement that

$$|A_1 \times A_2 \times \cdots \times A_n| = |A_1| \times |A_2| \times \cdots \times |A_n|.$$

In the example above, the collection of triples is

$$\{(x_1, x_2, x_3) : x_i \in \{A, B, C\}, 1 \leq i \leq 3, \text{ and } x_i \neq x_j \text{ if } i \neq j\}.$$

What are the sets A_1, A_2 and A_3? (Is $A_2 = \{x_2 \in \{A, B, C\} : x_2 \neq x_1\}$?)

Since instances when addition and multiplication arise in counting arguments were mentioned above, for completeness this section concludes with a brief discussion of when subtraction and division arise.

Subtraction normally occurs in counting arguments when *counting the complement*. This technique is used when the outcomes you want to count are part of a larger collection that includes some outcomes in which you are not interested. Sometimes it is easier to count the number of outcomes in the larger collection and the number of outcomes in which you are not interested, and then obtain the number in which you are interested by subtraction. For example, the number of arrangements of the letters XXXXYYY with at least one pair of adjacent Y's equals the total number of arranagements of these seven letters minus the number where no two Y's are adjacent, that is, $\binom{7}{4} - \binom{5}{3}$.

Division arises when each desired outcome has been counted the same number of times. If n counts each outcome t times, then the number of outcomes equals n/t. This situation often comes up when counting the number of equivalence classes of objects under some equivalence relation. That is, counting the number of groups of objects where all objects in the same group are deemed to be "the same" in some way. An example can be extracted from the proof of Proposition 1.3.2. The number of ways to select k out of n people equals the number of equivalence classes of lineups of k of the n people, where we deem two lineups to be "the same" (equivalent) if they contain the same collection of people. This equivalence relation partitions collection of $\frac{n!}{(n-k)!}$ lineups into equivalence classes that each contain $k!$ lineups, so that the number of equivalence classes is $\frac{n!}{k!(n-k)!}$.

1.3 Introduction to Combinatorial Arguments

We will try to adhere to the view that equalities involving counting numbers are best proved using *combinatorial arguments* (counting arguments), and consequently will try to offer a combinatorial proof in addition to any proof that uses other methods. When using a combinatorial argument to prove an equality, the main idea is to use the technique of *counting something in two ways*, that is, to show that the LHS and RHS of the expression count the same thing, possibly in different ways.

Proposition 1.3.1 $\binom{n}{k} = \binom{n}{n-k}$.

Proof. The number of outcomes of deciding which k out of n elements get selected is the same for as deciding which $n-k$ elements are not selected. ■

In the above proposition, we are assuming the symbol $\binom{n}{k}$ is defined, that is $n \geq 0$. Note that if $k < 0$ or $k > n$, the equality in the proposition is $0 = 0$.

Proposition 1.3.2 *For* $0 \leq k \leq n$, $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

Proof. Suppose we have a group of n people and must arrange k of them in a line, $0 \leq k \leq n$. We count the number of outcomes of doing this in two different ways.

On the one hand, we can form the line directly. There are n choices for the first person in line, $n-1$ choices for the second person in line, $n-2$ choices for the third person in line, and so on until, finally, there are $n-(k-1)$ choices for the last person in line. The number of lines is therefore $n(n-1)(n-2)\cdots(n-(k-1)) = \frac{n!}{(n-k)!}$. (This equality uses the hypothesis $0 \leq k \leq n$.)

On the other hand, we can first choose the k people who will form the line in $\binom{n}{k}$ ways, and then arrange them in a line in $k!$ ways. Thus, the number of lines is $\binom{n}{k}k!$.

Since the LHS and RHS count the same thing, they are equal. That is, $\frac{n!}{(n-k)!} = \binom{n}{k}k!$. Hence $\binom{n}{k} = \frac{n!}{(n-k)!k!}$. ■

Proposition 1.3.3 *[Pascal's Identity: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.*

Proof. The LHS counts the number of outcomes of selecting k out of n children to go on a trip. We argue that the RHS counts the same thing.

Let Georgie be one of the children. A selection of k children either includes Georgie or it doesn't. If Georgie is included there is 1 way to include him and then $\binom{n-1}{k-1}$ ways to select $k-1$ of the remaining $n-1$ children to go on the trip. If Georgie is not included, all k children going on the trip must be selected from the remaining $n-1$ children. There are $\binom{n-1}{k}$ outcomes of doing this. Thus, the number of selections euqls $\binom{n-1}{k-1} + \binom{n-1}{k}$.

Since the LHS and RHS count the same thing, they are equal. ■

The RHS of Pascal's identity is a sum, which suggests that the Addition Principle would be used in the counting argument that gives it. A reasonable rule of thumb is that a summation will arise from counting something as a collection of disjoint cases, and the number of cases equals the number of summands.

Another reasonable rule of thumb is that an expression which is a product will arise from counting the number of outcomes of a process which is a sequence of steps. This idea is illustrated in the proof of the next proposition.

Proposition 1.3.4 $\binom{n}{k+1} = \frac{n-k}{k+1}\binom{n}{k}$.

Proof. We need to argue that $(k+1)\binom{n}{k+1} = (n-k)\binom{n}{k}$. Suppose we have n people and must select a team of k players and a team manager, who does not play.

On the one hand, there are $\binom{n}{k+1}$ outcomes of choosing the $k+1$ people involved, and then $(k+1)$ outcomes of selecting which of those $(k+1)$ people will be the team manager (the remaining k being the team). Thus, the number of selections equals $(k+1)\binom{n}{k+1}$.

On the other hand, there are $\binom{n}{k}$ outcomes of selecting the k players for the team, and then $(n-k)$ outcomes of selecting one of the remaining $(n-k)$ people to be the team manager. Thus, the number of selections equals $(n-k)\binom{n}{k}$.

Therefore, $(k+1)\binom{n}{k+1} = (n-k)\binom{n}{k}$, or $\binom{n}{k+1} = \frac{n-k}{k+1}\binom{n}{k}$. ■

It is interesting to note that Proposition 1.3.4 can be used to prove that the sequence of binomial coefficients $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{k}$ is *unimodal*, meaning that it first increases and then decreases. If $n = 2t$ is even then the middle term $\binom{2t}{t}$ is the largest, and if $n = 2t+1$ is odd then the middle two terms, $\binom{2t+1}{t}$ and $\binom{2t+1}{t+1}$, are the largest.

1.4 Exercises

1. (a) Formulate the Addition Principle as a statement about the cardinality of the union of n disjoint sets.

   (b) Formulate the Multiplication Principle as a statement about the cardinality of the Cartesian product of n sets.

   (c) Formulate and prove a Subtraction Principle as a statement about the difference of finite sets.

   (d) Formulate and prove a Division Principle as a statement about counting equivalence classes. (You can assume that the equivalence classes partition the set on which the equivalence relation is defined.)

2. (a) Explain in detail why the number of ways to select k objects from a collection of n distinct objects with repetition allowed is $\binom{n+k-1}{k}$.

   (b) A collection of c distinguishable cars leave downtown Victoria destined for the ferry from Swarts Bay to Tsawwassen. How many ways can these cars go through the t booths at which the ferry fare can be paid if both the booth that each car goes through, and the order that cars go through each booth, matters?

3. (a) Give a combinatorial definition of the quantity $P(n,k) = \frac{n!}{(n-k)!}, n \geq 0$. Use this to provide, with justification, values for $P(n,k)$ in the cases (i) $k = 0$, (ii) $k < 0$, and (iii) $k > n$.

   (b) Justify the division in the formula given for $P(n,k)$ by counting the number of equivalence classes in the equivalence relation $\approx$ on the set of all permutations of $\{1, 2, \ldots, n\}$ defined by $p_1p_2 \ldots p_n \approx q_1q_2 \ldots q_n$ if and only if $p_i = q_i$ for $i = 1, 2, \ldots, k$.

4. Let $R(n,k) = \binom{n+k-1}{k}$. Give a combinatorial proof of each of the following:

   (a) $R(n,k) = R(n-1,k) + R(n,k-1)$.

   (b) $R(n,k) = R(1,k-1) + R(2,k-1) + \cdots + R(n,k-1)$.

   (c) $R(n,k) = R(n-1,0) + R(n-1,1) + \cdots + R(n-1,k)$.

5. (a) The number of arrangements of four A's, two B's and three C's is known to equal $\frac{9!}{4!2!3!}$. Justify this by considering arrangements of the nine distinguishable objects $A_1, A_2, A_3, A_4, B_1, B_2, C_1, C_2, C_3$ and defining two arrangements to be equivalent just if erasing the subscript gives the same arrangement of four A's, two B's and three C's. (Also, verify that this is an equivalence relation).

   (b) Generalize part (a) to the case where $n_1 + n_2 + \cdots + n_k = n$ and there are n_i objects of type i, $1 \leq i \leq k$.

6. Prove that the sequence of binomial coefficients $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{k}$ is unimodal. Show also that if $n = 2t$ is even then the middle term $\binom{2t}{t}$ is the largest, and if $n = 2t+1$ is odd then the middle two terms, $\binom{2t+1}{t}$ and $\binom{2t+1}{t+1}$, are the largest.

7. Give a combinatorial proof of Proposition 1.3.4 that involves partitioning a collection of $(n-k)\binom{n}{k}$ outcomes into equivalence classes of size $k+1$.

8. Give a combinatorial proof that
$$\binom{n}{r}\binom{n-r}{k} = \binom{n}{k}\binom{n-k}{r} = \binom{n}{r+k}\binom{r+k}{r}.$$

9. Prove that $\binom{2n}{n}$ is always even, and that $2^n < \binom{2n}{n} < 4^n$ for all $n \geq 2$.

10. Every arrangement of 6 zeros and 3 ones encodes an increasing sequence of 6 terms of the digits $1, 2, 3, 4$. For example
$$001000110 \to 1, 1, 2, 2, 2, 4.$$
Prove that the number of increasing n-letter words formed from a set of m letters is
$$\binom{m+n-1}{n}$$
where a word is called increasing if its letters, except for repetitions, appear in increasing (or, alphabetical) order. For example *aabcccdf* is increasing, whereas *cccddbf* is not.

11. (a) Give a combinatorial proof that $\binom{n-1}{m-1} = \binom{n}{n-m} - \binom{n-1}{n-m-1}$.
    (b) Prove that $\binom{n-1}{m-1} = \binom{n}{m} - \binom{n}{m+1} + \cdots \pm \binom{n}{n}$.

12. Show that the following process describes a 1–1 correspondence between unordered selections of r elements from $\{1, 2, \ldots, n\}$ in which repetition is allowed and r-subsets of $\{1, 2, \ldots, n+r-1\}$. *Suppose the r elements of $\{1, 2, \ldots, n\}$ that have been selected are $a_1, a_2, \ldots, a_r$, where $a_1 \leq a_2 \leq \cdots \leq a_r$. For $i = 1, 2, \ldots, r$, add $i-1$ to a_i.*

13. Prove that the number of ways that m distinct numbers from the set $\{1, 2, \ldots, n\}$ can be arranged in a circle is
$$\frac{n!}{m\,(n-m)!}$$
where arrangements which differ only by rotation are considered the same.

    Let R be a set with $|R| = n$. Determine the value(s) of $d, 1 \leq d \leq n$, for which the probability that a randomly chosen function $f : D \to R$, where $d = |D|$, is one-to-one is maximum; minimum.

Chapter 2

The Binomial Theorem and Friends

2.1 The Binomial Theorem

The numbers $\binom{n}{k}$ are called *binomial coefficients* because of the their appearance in the Binomial Theorem, which gives an expansion for an integer power of the binomial $(x+y)$.

Theorem 2.1.1 [The Binomial Theorem] *Let $n \geq 0$ be an integer. Then*

$$\begin{aligned}(x+y)^n &= \binom{n}{n}x^n + \binom{n}{n-1}x^{n-1}y + \binom{n}{n-2}x^{n-2}y^2 + \cdots + \binom{n}{0}y^n \\ &= \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}.\end{aligned}$$

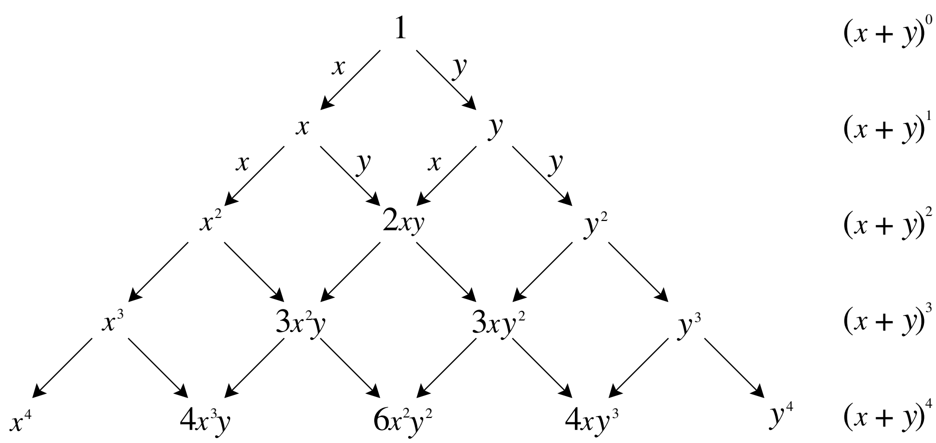
Proof. Consider $(x+y)^n = (x+y)(x+y)\cdots(x+y)$ multiplied out and not simplified. Each monomial term is a product of n factors each of which is an x or a y. A term involving k factors x and $n-k$ factors y arises once for each outcome in which n binomials $(x+y)$ are chosen to contribute an x (the remaining $n-k$ binomials contribute a y). Therefore the coefficient of $x^k y^{n-k}$ is $\binom{n}{k}$. ■

Example 2.1.2

$$\begin{aligned}(x+y)^7 &= \binom{7}{7}x^7 + \binom{7}{6}x^6y + \binom{7}{5}x^5y^2 + \binom{7}{4}x^4y^3 + \binom{7}{3}x^3y^4 \\ &\quad + \binom{7}{2}x^2y^5 + \binom{7}{1}xy^6 + \binom{7}{0}y \\ &= x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7\end{aligned}$$

Now, $(x+y)^8 = (x+y)(x+y)^7$. In this expression a term of the form x^5y^3 can arise in two ways: multiply $35x^4y^3$ by x, or multiply $21x^5y^2$ by y. Therefore the coefficient of x^5y^3 in $(x+y)^8$ is $35+21 = \binom{7}{5} + \binom{7}{4} = \binom{8}{5}$ by Pascal's Identity. This example suggests how mathematical induction can be used to prove the Binomial Theorem.

The example leads to the following triangle:



which is more familiar as Pascal's Triangle:

$$\begin{array}{ccccccccc} &&&&1&&&& \\ &&&1&&1&&& \\ &&1&&2&&1&& \\ &1&&3&&3&&1& \\ 1&&4&&6&&4&&1 \\ \iddots&&\iddots&&\vdots&&\ddots&&\ddots \end{array} \quad \begin{array}{c} \leftrightarrow \\ \leftrightarrow \\ \leftrightarrow \\ \leftrightarrow \\ \leftrightarrow \\ \leftrightarrow \end{array} \quad \begin{array}{ccccccccc} &&&&\binom{0}{0}&&&& \\ &&&\binom{1}{0}&&\binom{1}{1}&&& \\ &&\binom{2}{0}&&\binom{2}{1}&&\binom{2}{2}&& \\ &\binom{3}{0}&&\binom{3}{1}&&\binom{3}{2}&&\binom{3}{3}& \\ \binom{4}{0}&&\binom{4}{1}&&\binom{4}{2}&&\binom{4}{3}&&\binom{4}{4} \\ \iddots&&\iddots&&\vdots&&\ddots&&\ddots \end{array}$$

Proposition 2.1.3 $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$.

Proof 1. $(1+1)^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}$. ■

The convenience of appealing to the Binomial Theorem aside, the proposition gives an identity involving counting numbers. Thus, we offer a combinatorial proof.

Proof 2. The RHS equals the number of length n sequences of zeros and ones. The LHS counts the same thing by cases organized according to the number of ones in the sequence. The number of such sequences with k ones is $\binom{n}{k}$ (choose which k positions hold a one, and then there is only one way to fill in the remaining positions with zeros). Therefore by the Addition Principle, the number of sequences is the LHS. ■

Proposition 2.1.4 $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n\binom{n}{n} = 0$.

Proof 1. $(-1+1)^n = \binom{n}{0} - \binom{n}{1} + \cdots + (-1)^n\binom{n}{n}$. ■

Proof 2. We must show that

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots$$

The LHS counts the number of subsets of $\{1, 2, \ldots, n\}$ with an even number of elements, and the RHS counts the number of such subsets with an odd number of elements. We put these two collections into 1–1 correspondence, which implies that they have the same size.

Let E be the collection of subsets with an even number of elements and O be the collection of subsets with an odd number of elements.

Define $f : E \to O$ by

$$f(X) = \begin{cases} X - \{n\} & \text{if } n \in X \\ X \cup \{n\} & \text{if } n \notin X. \end{cases}$$

Then $f : E \to O$. Since f is invertible (f^{-1} has the same description), it is 1–1 and onto. Therefore, $|E| = |O|$. ■

Proposition 2.1.5 $1\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} = n \cdot 2^{n-1}$.

Proof 1. $(x+1)^n = \binom{n}{n}x^n + \binom{n}{n-1}x^{n-1} + \cdots + \binom{n}{1}x + \binom{n}{0}$

$$\begin{aligned} \therefore \frac{d}{dx}(x+1)^n &= n(x+1)^{n-1} \\ &= n\binom{n}{n}x^{n-1} + (n-1)\binom{n}{n-1}x^{n-2} + \cdots + 1\binom{n}{1}. \end{aligned}$$

The result follows on letting $x = 1$. ■

Proof 2. The RHS counts the number of pairs (x, Y) where $x \in \{1, 2, \ldots, n\}$ and Y is a subset of $\{1, 2, \ldots, n\}$ with $x \in Y$. The LHS counts the same thing by cases organized according to $|Y|, 1 \le |Y| \le n$. If $|Y| = k$ there are $\binom{n}{k}$ outcomes of selecting the k elements of Y and then k outcomes of selecting an element $x \in Y$. Therefore the number of pairs (x, Y) with $|Y| = k$ is $k\binom{n}{k}$. The result now follows by the Addition Principle. ■

Proposition 2.1.6 $1^2\binom{n}{1} + 2^2\binom{n}{2} + 3^2\binom{n}{3} + \cdots + n^2\binom{n}{n} = n2^{n-1} + n(n-1)2^{n-2}$.

Proof. As in Proof 1 of Proposition 2.1.5:

$$\begin{aligned}
n(x+1)^{n-1} &= n\binom{n}{n}x^{n-1} + (n-1)\binom{n}{n-1}x^{n-2} + \cdots + \binom{n}{1} \\
\therefore nx(x+1)^{n-1} &= n\binom{n}{n}x^{n} + (n-1)\binom{n}{n-1}x^{n-1} + \cdots + \binom{n}{1}x \\
\therefore \frac{d}{dx}nx(x+1)^{n-1} &= n(x+1)^{n-1} + n(n-1)x(x+1)^{n-2} \\
&= n^2\binom{n}{n}x^{n-1} + (n-1)^2\binom{n}{n-1}x^{n-2} + \cdots + \binom{n}{1}.
\end{aligned}$$

The result follows on letting $x = 1$. ■

It is left as an exercise to give a combinatorial proof of the above proposition.

Having dealt with identities obtained by differentiating the expansion given by the Binomial Theorem, we now look for identities that could be obtained by integrating it.

Suppose we want to find a closed form for the sum:

$$\frac{1}{n+1}\binom{n}{n} + \frac{1}{n}\binom{n}{n-1} + \cdots + \frac{1}{1}\binom{n}{0}$$

and as our first attempt (which will fail), we try the following:

$$\begin{aligned}
(1+x)^n &= \binom{n}{n}x^n + \binom{n}{n-1}x^{n-1} + \cdots + \binom{n}{0} \\
\therefore \int (1+x)^n dx &= \int \binom{n}{n}x^n + \binom{n}{n-1}x^{n-1} + \cdots + \binom{n}{0} dx \\
\therefore \frac{1}{n+1}(1+x)^{n+1} &= \frac{1}{n+1}\binom{n}{n}x^{n+1} + \frac{1}{n}\binom{n}{n-1}x^{n} + \cdots + \frac{1}{1}\binom{n}{0}x.
\end{aligned}$$

Letting $x = 1$, we get $\frac{1}{n+1}2^{n+1} = \sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k}$.

To test this formula, try $n = 2$: LHS $= \frac{1}{3}2^3 = \frac{8}{3}$. RHS $= \frac{1}{1}\binom{2}{0} + \frac{1}{2}\binom{2}{1} + \frac{1}{3}\binom{2}{2} = 1 + 1 + \frac{1}{3} = \frac{7}{3} \neq$ LHS. What's wrong? These integrals are determined up to a constant. It isn't rasonable to expect that taking the constant to be zero on both sides will give an equality.

To get a method that works, appeal to the Fundamental Theorem of Calculus and use definite integrals instead:

$$\begin{aligned}
\frac{1}{n+1}(1+x)^{n+1}\Big|_a^b &= \frac{1}{n+1}\binom{n}{n}x^{n+1} + \frac{1}{n}\binom{n}{n-1}x^n + \cdots + \frac{1}{1}\binom{n}{0}x\Big|_a^b \\
\frac{1}{n+1}(1+b)^{n+1} - \frac{1}{n+1}(1+a)^{n+1} &= \frac{1}{n+1}\binom{n}{n}b^{n+1} + \frac{1}{n}\binom{n}{n-1}b^n + \cdots + \frac{1}{1}\binom{n}{0}b \\
&\quad - \left[\frac{1}{n+1}\binom{n}{n}a^{n+1} + \frac{1}{n}\binom{n}{n-1}a^n + \cdots + \frac{1}{1}\binom{n}{0}a\right]
\end{aligned}$$

Hopefully it is possible to choose the limits of integration so that the correct sum is obtained on the RHS. In this case, letting $b = 1$ and $a = 0$ (as often works) yields

$$\frac{1}{n+1}2^{n+1} - \frac{1}{n+1} = \frac{1}{n+1}\binom{n}{n} + \frac{1}{n}\binom{n}{n-1} + \cdots + \frac{1}{1}\binom{n}{0}.$$

2.2 Some Standard Combinatorial Arguments

The proofs of the next few propositions illustrate techniques commonly used in counting arguments. The first of these is a standard method of organizing counting by cases: do so by the first (or smallest) object not selected (or sometimes by the first object selected). Notice that if a certain object is the smallest one *not* selected, then all smaller objects must have been selected (and there is only outcome where that happens).

Proposition 2.2.1 $\binom{n}{0} + \binom{n+1}{1} + \cdots + \binom{n+r}{r} = \binom{n+r+1}{r}$.

Proof. The RHS counts the number of ways to select r of the $n+r+1$ distinct objects $x_1, x_2, \ldots, x_{n+r+1}$. We claim that the LHS counts the same thing organized by cases according to the first object not selected. The number of selections that do not contain x_1 is $\binom{n+r}{r}$; the number of selections that contain x_1 but not x_2 is $1 \cdot \binom{n+r-1}{r-1}$; the number of selections that contain x_1 and x_2 but not x_3 is $1 \cdot 1 \cdot \binom{n+r-2}{r-2}$; and so on until, finally, the number of selections that contain $x_1, x_2, \ldots, x_r$ but not x_{r+1} is $1 \cdot 1 \cdots 1\binom{n}{0}$. There are no further cases because only r objects are being selected. Therefore, the number of selections is $\binom{n}{0} + \binom{n+1}{1} + \cdots + \binom{n+r}{r}$. Since the LHS and RHS count the same things, they are equal. ■

The next technique used in proving identities a *committee selection argument*. The main idea is that a committee is to be formed consisting of some number of people, who possibly are from a variety of different groups, and who possibly will take differing roles on the committee. One example of such an argument is the proof of Proposition 1.3.4. Another example is given below.

Proposition 2.2.2 [Vandermonde's Identity]

$$\binom{n+m}{r} = \binom{n}{0}\binom{m}{r} + \binom{n}{1}\binom{m}{r-1} + \cdots + \binom{n}{r}\binom{m}{0}.$$

Proof.

Suppose we must choose a committee of r persons from n men and m women. The number of outcomes of doing this is the LHS $\binom{n+m}{r}$.

The RHS counts the same thing by cases organized according to the number of men chosen. For $0 \le k \le r$, if k men are selected (and therefore $r-k$ women are selected), the number of such committees is $\binom{n}{k}\binom{m}{r-k}$. The result follows from the Addition Principle. ■

The following corollary follows by symmetry of the binomial coefficients, that is, since $\binom{n}{k} = \binom{n}{n-k}$. It is left as an exercise to give a combinatorial proof.

Corollary 2.2.3

$$\binom{2n}{n} = \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2.$$

Another technique for proving identities is *block walking arguments*. There are two different models that arise.

In the first model person walks from $(0,0)$ (say) to (a,b) in the plane by moving one unit Right or one unit Up at each step (necessarily $a \ge 0$ and $b \ge 0$). Any such path corresponds to a sequence of a R's and b U's. There are $\binom{a+b}{a}$ of these.

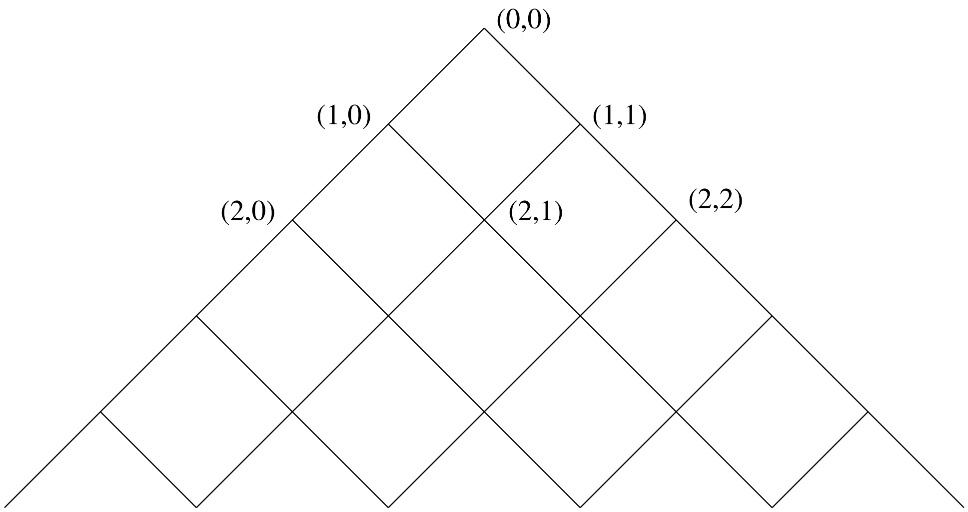
As a example, we give a block-walking proof that

$$\binom{n+m}{n} = \binom{n}{0}\binom{m}{n} + \binom{n}{1}\binom{m}{n-1} + \cdots + \binom{n}{n}\binom{m}{0}.$$

The LHS counts the number of walks in the plane from $(0,0)$ to (m,n). We argue that the RHS counts the same thing by cases organized according to the location after n steps have been taken. If k moves Up have occurred within the first n steps, then $0 \le k \le n$ the remaining $n-k$ moves have been Right, and the location is $(k, n-k)$. The number of paths from $(0,0)$ to $(k, n-k)$ is $\binom{n}{k}$, and the number of paths from $(k, n-k)$ to (m,n) is $\binom{m}{n-k}$. The result follows from the Addition Principle.

We now describe the second block walking model. Consider the diagram shown below. A person starts at $(0,0)$ and moves to (n,k) by choosing either the Left path or the Right path at each intersection. Any such path must involve k R's and $n-k$ L's; consequently the number of such paths from $(0,0)$ to (n,k) equals the number of sequences of k R's and $n-k$ L's, which equals $\binom{n}{k}$.

The main difference between these two models is the way the blocks are co-ordinatized. In the first model the co-ordinates are the usual ones for lattice points (points with integer co-ordinates) in the plane. The number of paths from $(0,0)$ to (a,b) is $\binom{a+b}{a} = \binom{a+b}{b}$. This will be



referred to as the *Plane Model*. In the second model the co-ordinates are as in Pascal's triangle. The number of paths from $(0,0)$ to (a,b) is $\binom{a}{b}$. This will be referred to as the *Pascal's Triangle Model*.

Proposition 2.2.4 $\binom{r}{r} + \binom{r+1}{r} + \cdots + \binom{n}{r} = \binom{n+1}{r+1}$.

Proof.

The RHS counts the number of lattice paths from $(0,0)$ to $(n+1, r+1)$ in the Pascal's Triangle Model. We claim that the LHS counts the same thing according to cases organized by the first time R is chosen.

If R is chosen on step 1, then r of the remaining n moves must be R, and there are $\binom{n}{r}$ outcomes of doing this. If R is not chosen on step 1, but is chosen on step 2, then r of the remaining $n-1$ moves must be R, and there are $\binom{n-1}{r}$ outcomes of doing this. In general, if R is chosen for the first time on move k, with $1 \leq k \leq n+1-r$, then r of the remaining $(n+1)-k$ moves must be R and there are $\binom{n+1-k}{r}$ outcomes of doing this. Therefore, by the Addition Principle, the number of paths equals $\binom{n}{r} + \binom{n-1}{r} + \binom{n-2}{r} + \cdots + \binom{n+1-(n+1-r)}{r}$ $\binom{r}{r}$. ■

2.3 Bertrand's Ballot Problem

The following problem was studied by Bertrand (1822-1900): *An election is held with two candidates, Alice and Bob. Alice wins. How many ways are there to count the ballots so that Alice is always ahead in the counting?* The solution we present gives a clever application of a 1-1 correspondence. It involves paths in the plane, but in neither of the models just described. Translating the solution into the Pascal's Triangle model is an exercise.

Let a be the number of votes for Alice and b be the number of votes for Bob, where $a > b$. The total number of votes cast is $a+b$. Graph the ballot counting as follows: for $i = 1, 2, \ldots, a+b$, plot the point (i, j), where j is Alice's lead after i votes have been counted, that is, the number of votes so far for Alice minus the number of votes so far for Bob.

Each way of counting the ballots describes a path from $(0,0)$ to $(a+b, a-b)$. The paths that represent situations where Alice is always ahead in the counting never touch the x-axis after $(0,0)$. These are what we want to count. If Alice is always ahead in the counting, the first vote counted must be for her. Thus, the paths we want to count correspond to the paths from $(1,1)$ to $(a+b, a-b)$ that never touch the x-axis.

Any path is completely determined by where the votes for Alice are counted. Hence there are $\binom{a+b-1}{a-1}$ paths from $(1,1)$ to $(a+b, a-b)$, including theose that touch the x-axis. We need to determine the number of the latter and subtract.

We claim that there is a 1–1 correspondence between the set of paths that touch the x-axis and the set of paths from $(1,-1)$ to $(a+b, a-b)$. Note that all paths of the latter type necessarily touch the x-axis. Any path that starts at $(1,1)$, goes down to touch the x-axis, and then eventually goes up to $(a+b, a-b)$ can be reflected into a path from $(1,-1)$ to $(a+b, a-b)$: *Reflect the part of the path between* $(0,0)$ *and the first crossing on the x-axis about the x-axis, and leave all other points unchanged.* Reflecting does not change the number of votes for Alice and Bob (only where they are counted).

The reflection sends $(1,1)$ to $(1,-1)$, and any path from $(1,1)$ that eventually meets the x-axis can be reflected into a path that starts at $(1,-1)$. This process is reversible, hence describes a 1–1 correspondence between paths from $(1,1)$ that touch the x-axis and paths from $(1,-1)$ to $(a+b, a-b)$. There are $\binom{a+b-1}{a}$ of the latter.

Therefore the number of ballot arrangements we want is:

$$\begin{aligned}
\binom{a+b-1}{a-1} - \binom{a+b-1}{a} &= \frac{(a+b-1)!}{(a-1)!b!} - \frac{(a+b-1)!}{a!(b-1)!} \\
&= \frac{(a+b-1)!}{a!b!}(a-b) \\
&= \frac{(a+b)!}{a!b!}\frac{(a-b)}{(a+b)} \\
&= \frac{a-b}{a+b}\binom{a+b}{b} = \frac{a-b}{a+b}\binom{a+b}{a}
\end{aligned}$$

Since this is $\frac{\text{A's lead}}{\#\text{ votes}} \times \#$ paths, it follows that probability of Alice always being ahead in the counting is $\frac{a-b}{a+b}$.

2.4 The Multinomial Theorem

Suppose we have four distinguishable boxes, B_1, B_2, B_3, and B_4 and 15 labelled (hence distinguishable) balls. We want to distribute the balls into the boxes so that

B_1 gets 3 balls
B_2 gets 2 balls
B_3 gets 4 balls
B_4 gets 6 balls

The number of distributions is

$$\begin{aligned}\binom{15}{3}\binom{12}{2}\binom{10}{4}\binom{6}{6} &= \frac{15!}{3!\cancel{12!}}\cdot\frac{\cancel{12!}}{2!\cancel{10!}}\cdot\frac{\cancel{10!}}{4!\cancel{6!}}\frac{\cancel{6!}}{6!0!} \\ &= \frac{15!}{3!2!4!6!}\end{aligned}$$

In the solution to the counting problem, we have partitioned the collection of 15 distinguishable balls into four subsets B_1, B_2, B_3, B_4 so that $|B_1| = 3,\ |B_2| = 2,\ |B_3| = 4,$ and $|B_4| = 6$.

The symbol $\binom{n}{r_1 r_2 \cdots r_k}$ is defined to be the number of outcomes of distributing the elements of a set of size $n \geq 0$ among k subsets $R_1, R_2, \ldots, R_k$ so that, for $i = 1, 2, \ldots, k,\ |R_i| = r_i$, and $R_i \cap R_j = \emptyset$ of $i \neq j$.

It follows from the definition that $\binom{n}{r_1 r_2 \cdots r_k} = 0$ if some $r_i < 0$, or if $r_1 + r_2 + \cdots + r_k \neq n$. Hence $\binom{n}{r_1 r_2 \cdots r_k}$ is non-zero if and only if $r_i \geq 0,\ 1 \leq i \leq k$ and $r_1 + r_2 + \cdots r_k = n$.

To distribute the elements as required, first select a subset of r_1 elements, and then a subset of r_2 elements from the remaining $n - r_1$ elements, and so on until finally we select r_k elements from the remaining $n - r_1 - r_2 - \cdots - r_{k-1} = r_k$ elements. Thus

$$\begin{aligned}\binom{n}{r_1 r_2 \cdots r_k} &= \binom{n}{r_1}\binom{n-r_1}{r_2}\binom{n-r_1-r_2}{r_3}\cdots\binom{n-r_1-\cdots-r_{k-1}}{r_k} \\ &= \frac{n!}{r_1! r_2! \cdots r_k!}\end{aligned}$$

When $k = 2$, $\binom{n}{k, n-k} = \frac{n!}{k!(n-k)!} = \binom{n}{k}$. We can also see this combinatorially: the multinomial coefficient $\binom{n}{k, n-k}$ counts the number of ways to distribute the elements of a set of n distinguishable elements among two subsets, a set of k elements deemed to be *selected* and a set of $n - k$ elements deemed to be *not selected*.

The numbers $\binom{n}{r_1 r_2 \cdots r_k}$ are called *multinomial coefficients* because they appear in the Multinomial Theorem, which was conjectured by Leibnitz and proved by Bernoulli. It gives an expansion for a non-negative integer power of a multinomial $(x_1 + x_2 + \cdots + x_k)$. When $k = 2$ the a slight translation shows that the Multinomial Theorem is the Binomial Theorem.

Theorem 2.4.1 [The Multinomial Theorem] *For $n \geq 0$,*

$$(x_1 + x_2 + \cdots + x_k)^n = \sum_{\substack{r_1 + r_2 + \cdots + r_k = n \\ r_i \geq 0}} \binom{n}{r_1 r_2 \cdots r_k} x_1^{r_1} x_2^{r_1} \cdots x_k^{r_k}.$$

We do an example to familiarize ourselves with the concepts before doing the proof.

$$\begin{aligned}(x_1+x_2+x_3)^2 &= \binom{2}{2,0,0}x_1^2 + \binom{2}{0,2,0}x_2^2 + \binom{2}{0,0,2}x_3^2 \\ &+ \binom{2}{1,1,0}x_1x_2 + \binom{2}{1,0,1}x_1x_3 + \binom{2}{0,1,1}x_2x_3 \\ &= x_1^2+x_2^2+x_3^2+2x_1x_2+2x_1x_3+2x_2x_3\end{aligned}$$

Proof. $(x_1+x_2+\cdots+x_k)^n = (x_1+x_2+\cdots+x_k)(x_1+x_2+\cdots+x_k)\cdots(x_1+x_2+\cdots+x_k)$ (a total of n factors). When the RHS is multiplied out, all terms are of the form $x_1^{r_1}x_2^{r_2}\cdots x_k^{r_k}$ where $r_1+r_2+\cdots+r_k=n$. The number of ways in which such a term can arise equals the number of outcomes of distributing the n factors among the set of r_1 factors that contribute x_1, the set of r_2 factors that contribute x_2, and so on until the set of r_k factors contribute x_k. Thus, the coefficient of $x_1^{r_1}x_2^{r_2}\cdots x_k^{r_k}$ in the product is $\binom{n}{r_1 r_2 \cdots r_k}$. ■

Here are two points to note:

1. the number of terms in the summation in the Multinomial Theorem equals the number of ways to write n as an ordered sum of k non-negative integers. In turn, this equals the number of sequences of n 1's and $k-1$ +'s, which is $\binom{n-(k-1)}{n}$.
2. Since $\binom{n}{r_1 r_2 \cdots r_k}$ equals zero unless $r_i \geq 0,\ 1 \leq i \leq k$ and $r_1+r_2+\cdots r_k = n$, the summation can be assumed to run over all collections of integers $r_1, r_2, \ldots, r_k$. The conditions given just restrict to the terms that make a non-zero contribution to the sum.

Corollary 2.4.2

$$\sum_{\substack{r_1+r_2+\cdots+r_k=n \\ r_i \geq 0}} \binom{n}{r_1 r_2 \cdots r_k} = k^n$$

Proof. By the Multinomial Theorem

$$(x_1+x_2+\cdots+x_k)^n = \sum_{\substack{r_1+r_2+\cdots+r_k=n \\ r_i \geq 0}} \binom{n}{r_1 r_2 \cdots r_k} x_1^{r_1}x_2^{r_1}\cdots x_k^{r_k}.$$

The result follows on letting $x_1 = x_2 = \cdots = x_k = 1$. ■

It is left as an exercise to find a combinatorial proof of Corollary 2.4.2.

The multinomial coefficients satisfy a similar recurrence to the binomial coefficients. With a bit of imagination, this recurrence can be used to build a "Pascal's Pyramid".

Proposition 2.4.3

$$\binom{n}{r_1 r_2 \cdots r_k} = \binom{n}{(r_1-1) r_2 \cdots r_k} + \binom{n}{r_1 (r_2-1) \cdots r_k} + \cdots + \binom{n}{r_1 r_2 \cdots (r_k-1)}.$$

Proof. Exercise. ■

We conclude this section with an example to show that the multinomial coefficients are more than "just" a generalization of the binomial coefficients. Recall the following facts from introductory graph theory:

1. the *degree sequence* of a graph G with vertex set $\{v_1, v_2, \ldots . v_n\}$ is the sequence $d_1, d_2, \ldots, d_n$, where $d_i = deg_G(v_i),\ 1 \le i \le n$;
2. if $d_1, d_2, \ldots, d_n$ is the degree sequence of the graph G, then $d_1 + d_2 + \cdots + d_n = 2|E|$
3. a *tree* is a connected, acyclic graph, or equivalently a connected graph with n vertices and $n-1$ edges;
4. a tree with n vertices has exactly $n-1$ edges (that is, $|E| = n-1$);
5. every tree with $n \ge 2$ vertices has at least two vertices of degree 1 (*leaves*);
6. the graph obtained by deleting a leaf from a tree is a tree;

It follows from points 2 and 4 above that if $d_1, d_2, \ldots, d_n$ is the degree sequence of a tree, then $d_1+d_2+\cdots+d_n = 2n-2$, and consequently $(d_1-1)+(d_2-1)+\cdots+(d_n-1) = (2n-2)-n = n-2$. Points 3, 4 and 6 are used in the proof of the next theorem.

Theorem 2.4.4 *The number of different trees with vertex set* $\{1, 2, \ldots, n\}$ *and degree sequence* $d_1, d_2, \ldots, d_n$ *is the multinomial coefficient*

$$\binom{n-2}{(d_1-1)(d_2-1)\cdots(d_n-1)}$$

Proof. (Prüfer, 1918) The multinomial coefficient $\binom{n-2}{(d_1-1)(d_2-1)\cdots(d_n-1)}$ counts the number of sequences of length $n-2$ comprised of (d_1-1) 1's , (d_2-1) 2's, $\ldots$, (d_n-1) n's. We show that the number of such sequences equals the number of trees with degree sequence $d_1, d_2, \ldots, d_n$ in two steps.

Let T be a tree with vertex set $\{1, 2, \ldots, n\}$ and degree sequence $d_1, d_2, \ldots, d_n$. Construct the sequence $(x_1, x_2, \ldots, x_{n-2})$ as follows. Let i_1 be the least integer such that vertex i_1 has degree 1 in T; the unique vertex adjacent to i_1 is taken to be x_1. We now repeat the process with the tree $T - \{i_1\}$. Let i_2 be the least integer such that vertex i_2 has degree 1 in $T - \{i_1\}$; the unique vertex adjacent to i_2 in $T - \{i_1\}$ is taken to be x_2. The process is repeated until x_{n-2} has been defined and a tree with just two vertices remains. Notice that every vertex v

appears in the sequence once for every edge with which it is incident, except one. That is, v appears in the sequence $d_v - 1$ times. In this way, every tree with vertex set $\{1, 2, \ldots, n\}$ gives rise to a unique sequence comprised of $d_1 - 1$ 1's , $d_2 - 1$ 2's, $\ldots, d_n - 1$ n's, as needed. (In particular, note that the vertices of degree 1 are precisely the vertices that do not appear in the sequence.) It can be shown, for example by induction on n, that different trees give rise to different sequences. Hence our function mapping trees to sequences is 1-1. It follows that the number of sequences is greater than or equal to the number of trees.

The procedure just described is reversible (hence is a 1-1 correspondence). Let $(x_1, x_2, \ldots, x_{n-2})$ be a sequence comprised of $(d_1 - 1)$ 1's , $(d_2 - 1)$ 2's, $\ldots, (d_n - 1)$ n's. Let i_1 be the least integer not in $\{x_1, x_2, \ldots, x_{n-2}\}$. Join i_1 to x_1. Now let i_2 be the least integer not in $\{i_1\} \cup \{x_2, x_3, \ldots, x_{n-2}\}$ and join i_2 to x_2. Repeat this process until the $n-2$ edges $i_1x_1, i_2x_2, \ldots, i_{n-2}x_{n-2}$ have been defined. Finally, join the two vertices i_{n-1} and i_n not in $\{i_1, i_2, \ldots, i_{n-2}\}$. By definition of the process, each vertex is incident with one more edge than the number of times it appears in the sequence $(x_1, x_2, \ldots, x_{n-2})$. Thus the graph constructed has degree sequence $d_1, d_2, \ldots, d_n$. It can be shown, for example by induction on n, that the graph with $n-1$ edges produced by this process is a tree and, further, that different sequences give rise to different trees. Hence our function mapping sequences to trees is 1-1. It follows that the number of trees is greater than or equal to the number of sequences.

Therefore the number of sequences sequences of length $n-2$ comprised of $(d_1 - 1)$ 1's , $(d_2 - 1)$ 2's, $\ldots, (d_n - 1)$ n's equals the number of trees with degree sequence $d_1, d_2, \ldots, d_n$. ■

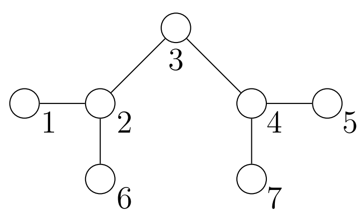


Figure 2.1: Example to illustrate the proof of Theorem 2.4.4

We use the tree in Figure 2.4 to illustrate the proof of Theorem 2.4.4. Applying the procedure in the proof to the tree should give a sequence of length 5 with no 1's, two 2's, one 3, two 4's, and no 5's, 6's or 7's. The vertices of degree one are 1, 5, 6, and 7, so that $i_1 = 1$ and $x_1 = 2$. Deleting vertex 1 leads to $i_2 = 5$ and $x_2 = 4$. Continuing, $i_3 = 6$ and $x_3 = 2$; $i_4 = 2$ and $x_4 = 3$; and finally $i_5 = 3$ and $x_5 = 4$. Hence the procedure produces the sequence $(2, 4, 2, 3, 4)$.

Applying the reverse procedure to the sequence $(2, 4, 2, 3, 4)$ leads to $i_1 = 1$ (the smallest element not in $\{2, 3, 4\}$), $i_2 = 5$ (the smallest element not in $\{1\} \cup \{2, 3, 4\}$), and similarly $i_3 = 6$, $i_4 = 2$ (the smallest element not in $\{1, 5, 6\} \cup \{3, 4\}$) and $i_5 = 3$. This part of the process defines the five edges $12, 54, 62, 23$ and 34. The last edge joins 4 and 7, the two vertices not in $\{1, 5, 6, 2, 3\}$. The tree that has been constructed by carrying out the procedure is the one in Figure 2.4.

Corollary 2.4.5 [Cayley, 1897] *The number of different trees with vertex set* $\{v_1, v_2, \ldots, v_n\}$ *is* n^{n-2}.

Proof. The result follows from Theorem 2.4.4 and Corollary 2.4.2 on summing over all possibilities for the degree sequence. ■

2.5 Exercises

1. Give a combinatorial proof that $1 + 2 + 2^2 + \cdots + 2^{n-1} = 2^n - 1$.

2. Find a closed form for the sum
$$\frac{n+1}{n+2}\binom{n}{n} + \frac{n}{n+1}\binom{n}{n-1} + \cdots + \frac{1}{2}\binom{n}{0}.$$

3. Prove the identity
$$\sum_{k=0}^{m}\binom{m}{k}\binom{n}{r+k} = \binom{m+n}{m+r}$$
using (a) a committee selection argument, and (b) a block-walking argument.

4. Prove that the number of odd terms in the sequence $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}$ is a power of two.

5. Prove that every element of the sequence $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}$ is odd if and only if $n = 2^t - 1$ for some integer $t \geq 0$.

6. Prove that
$$\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}\binom{n}{k} = 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$

7. (a) Give a combinatorial proof of Proposition 2.1.6.
   (b) Generalize Proposition 2.1.6 by providing, with proof, a closed form for the sum
$$1^k\binom{n}{1} + 2^k\binom{n}{2} + 3^k\binom{n}{3} + \cdots + n^k\binom{n}{n}.$$
   (c) Give a combinatorial proof for the identity in (b), unless you already did so there.

8. (a) Give a combinatorial proof of the identity
$$3^n = \sum_{k=0}^{n}\binom{n}{k}2^k.$$
   Hint: Count pairs (X, Y) where $X \subseteq \{1, 2, \ldots, n\}$ and $Y \subseteq X$. Why are there three possibilities for what happens to each element of $\{1, 2, \ldots, n\}$?

(b) Let r be a positive integer. Generalize part (a) to give a combinatorial proof that

$$(r+1)^n = \sum_{k=0}^{n} \binom{n}{k} r^k.$$

Hint: referring to the hint for part (a), note that each subset $Y \subseteq X$ is equivalent to a function $f : X \to \{0, 1\}$.

9. (a) Let g_1, g_2 and g_3 be non-negative integers. Use a combinatorial argument to evaluate

$$\sum_{s_1+s_2+s_3=n} \binom{g_1}{s_1}\binom{g_2}{s_2}\binom{g_3}{s_3}$$

where $s_1, s_2, s_3 \geq 0$.

(b) State and prove the generalization of part (a) to k factors in each summand.

10. We know that the number of paths from $(0, 0)$ to $(m+n, n)$ in the plane model is $\binom{m+n}{n}$.

(a) Find an identify by counting these paths by cases according the first move Up.

(b) Find an identity by counting these paths by cases according to the location after n moves have been made.

11. Generalize the block walking model to paths between the lattice points $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$ where each step must increase some component by exactly one. Carefully describe the model, and determine the number of paths between two given lattice points. (Note that there could be none.)

12. (a) (*The cashier problem.*) One hot summer day, Gary decides to set up a stand selling glasses of ice cold lemonade for \$1 each. Of the $n + t$ people who show up to buy lemonade, n have only a one dollar coin and t have only a two dollar coin. In how many ways can the people arrive so that Gary can always give change when necessary?

(b) Repeat part (a) in the case where Gary starts with a float of $f \geq 0$ one dollar coins.

13. Find a closed form for the sum

$$\binom{n}{0} - 2\binom{n}{1} + 3\binom{n}{2} + \cdots + (-1)^n(n+1)\binom{n}{n}$$

and justify the equality with a combinatorial argument.

14. (a) Use Proposition 2.2.2 to prove Corollary 2.2.3.

(b) Use a block walking argument in the Pascal's Triangle model to prove Corollary 2.2.3. (Hint: count the number of routes that go through (n, k) and sum over all possible values of k.)

15. Find a closed form for the sum

$$\sum_{k=0}^{n} \left(\frac{1}{k+1}\right)^2 \binom{n}{k}.$$

16. Translate the solution to The Ballot Problem into a block walking argument in the Pascal's Triangle Model.

17. Give a proof of Corollary 2.4.2 that involves counting the number of sequences length n of the symbols $x_1, x_2, \ldots x_k$.

18. Prove Proposition 2.4.3.

19. Referring to the proof of Theorem 2.4.4, show that:

    (a) different trees give different sequences;

    (b) the graph produced by each sequence is a tree, and different sequences give rise to different trees.

20. Let k be a positive integer.

    (a) Give two proofs, at least one of which uses a combinatorial argument, that k^n evenly divides $(kn)!$ for all non-negative integers n.

    (b) Prove that $k!$ divides the product of any k consecutive integers.

21. Use a combinatorial argument to show that

$$1\binom{n}{1}^2 + 2\binom{n}{2}^2 + n\binom{n}{n}^2 = n\binom{2n-1}{n-1}.$$

Chapter 3

Advanced Counting Numbers

3.1 Stirling Numbers of the First Kind

The numbers in this section and the next are named after James Stirling (1692-1770). This is the same Stirling as in Stirling's approximation to $n!$:

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n .$$

The (unsigned) *Stirling number of the first kind*, $\left[{n \atop k}\right]$, equals the number of permutations of $n \geq 0$ distinguishable objects that have exactly k cycles in their disjoint cycle representation.

The modifier "unsigned" arises because these numbers are also the coefficients in certain polynomials whose coefficients alternate in sign. When such polynomials are used to define the Stirling numbers of the first kind, the sign is sometimes included. We will consider these polynomials later in this section.

For $n > 0$, it follows from the definition that $\left[{n \atop k}\right] = 0$ if $k \leq 0$ or $k > n$. The question arises about what $\left[{0 \atop 0}\right]$ ought to be. By definition, it is the number of permutations of zero distinguishable objects with no cycles. The permutation of zero objects (the empty permutation) has no cycles, so $\left[{0 \atop 0}\right] = 1$.

It is easy to see that $\left[{n \atop n}\right] = 1$ (remember that disjoint cycles commute) and $\left[{n \atop 1}\right] = n!/n = (n-1)!$, the number of circular permutations of n distinguishable objects. Also, $\left[{n \atop n-1}\right] = \binom{n}{2}$ because there is one 2-cycle (whose elements can be chosen in $\binom{n}{2}$ ways) and $(n-2)$ 1-cycles.

To further illustrate the definition, we show that $\left[{4 \atop 2}\right] = 11$. A permutation of $1, 2, 3, 4$ with two cycles in its disjoint cycle representation has either a 1-cycle and a 3-cycle, or two 2-cycles. Since there are $3!/3 = 2$ cyclic permutations of three distinguishable objects, there are $4 \times 2 = 8$ permutations with a 1-cycle and a 3-cycle: (1)(234); (1)(243); (2)(134); (2)(143); (3)(124); (3)(142); (4)(123); (4)(132). Since disjoint cycles commute, there are $\binom{4}{2}/2 = 3$ permutations with two 2-cycles: (12)(34); (13)(24); (14)(23).

Thus for $k = 0, 1, 2, 3, 4$ we have $\left[{4 \atop k}\right] = 0, 6, 11, 6, 1$, respectively.

An equivalent definition is that $\left[{n \atop k}\right]$ equals the number of ways to seat $n \geq 0$ people at k identical (i.e. indistinguishable) circular tables, none of which are empty, and where arrangements that differ by a rotation of some of the tables are considered to be the same.

Theorem 3.1.1 *For* $n \geq 0$, $\left[{n \atop r}\right] = (n-1)\left[{n-1 \atop r}\right] + \left[{n-1 \atop r-1}\right]$

Proof. Consider a permutation of $1, 2, \ldots, n$ with k cycles. Either n is in a cycle of length one or it is in a cycle of length greater than one.

If n is in a cycle of length one, then deleting that cycle gives a permutation of $1, 2, \ldots, n-1$ with $k-1$ cycles. There are $\left[{n-1 \atop k-1}\right]$ of these.

If n is in a cycle of length greater than one, then deleting n gives a permutation of $1, 2, \ldots, n-1$ with k cycles. Given a permutation of $1, 2, \ldots, n-1$ with k cycles, the element n can be inserted into any cycle of the permutation to follow any of the other $n-1$ elements. All of these reduce as above to the same permutation of $1, 2, \ldots, n-1$ with k cycles, and so the number of permutations where n belongs to a cycle of length greater than one is $(n-1)\left[{n-1 \atop k}\right]$.

Therefore, by the Addition Principle, $\left[{n \atop r}\right] = (n-1)\left[{n-1 \atop r}\right] + \left[{n-1 \atop r-1}\right]$. ∎

Using the fact that $\left[{n \atop 0}\right] = 0$, one can use Theorem 3.1.1 obtain by induction that $\left[{n \atop 1}\right] = (n-1)!$, $\left[{n \atop n}\right] = 1$, and $\left[{n \atop n-1}\right] = (n-1) + \binom{n-1}{2} = \binom{n-1}{1} + \binom{n-1}{2} = \binom{n}{2}$.

Theorem 3.1.1 can be used to obtain a triangle of Stirling numbers of the first kind.

$$\begin{array}{ccccccc} & & & \left[{0 \atop 0}\right] & & & \\ & & \left[{1 \atop 0}\right] & & \left[{1 \atop 1}\right] & & \\ & \left[{2 \atop 0}\right] & & \left[{2 \atop 1}\right] & & \left[{2 \atop 2}\right] & \\ \cdot^{\cdot^{\cdot}} & & \cdot^{\cdot^{\cdot}} & \vdots & \ddots & & \ddots \end{array}$$

The rule for constructing the interior elements of the n-th row is

$$\begin{array}{ccccc} \left[{n-1 \atop k-1}\right] & & \left[{n-1 \atop k}\right] & & \leftarrow \text{Row } n-1 \\ & \searrow \;\; \swarrow & & & \\ & \left[{n \atop k}\right] = \left[{n-1 \atop k-1}\right] + (n-1)\left[{n-1 \atop k}\right] & & & \end{array}$$

The first few rows of the triangle are

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 0 & & 1 & & & \\ & & 0 & & 1 & & 1 & & \\ & 0 & & \underline{2} & & \underline{3} & & 1 & \\ 0 & & 6 & & \underline{11} & & 6 & & 1 \end{array}$$

In general, the n-th row of the triangle is constructed by putting a zero on the left, a 1 on the right, and for each intermediate entry adding the entry above and to the left to $n-1$ times the entry above and to the right. Remembering that the rows are numbered from zero, the underlined entries indicate that $11 = 2 + 3 \times 3$.

As mentioned earlier, the Stirling numbers of the first kind also arise as the coefficients of a certain polynomial. Suppose $x \geq n \geq 1$ items are available. The number of ways to select n of these x items, where order counts and repetitions are not allowed is $x(x-1)(x-2)\cdots(x-(n-1))$. When this expression is expanded as a polynomial in x, from each factor except the first we choose either x or a negative integer. If we choose one fewer x, we choose one more negative integer. Therefore the coefficients of the polynomial alternate in sign. Let $c(n,k)$ denote the absolute value of coefficient of x^k in $x(x-1)(x-2)\cdots(x-(n-1))$, that is, the coefficient without the sign.

$$x(x-1)\cdots(x-(n-1)) = c(n,n)x^n - c(n,n-1)x^{n-1} + c(n,n-2)x^{n-2} - \cdots + (-1)^n c(n,0).$$

The facts listed below follow from the definition of $c(n,k)$.

1. $c(n,k) = 0$ if $k \leq 0$ or $k > n$, as the polynomial $x(x-1)\cdots(x-(n-1))$ has no term x^k. In particular, $c(n,0) = 0$ because the polynomial has no constant term.

2. $c(n,n) = 1$ because the coefficient of x^n is 1.

3. $c(n,1) = (n-1)!$. The coefficient of x^1 is the product of all of the negative integers $(-1)(-2)\cdots(-(n-1)) = (-1)^{n-1}(n-1)!$.

4. $c(n,n-1) = \binom{n}{2}$. To obtain a term of the form x^{n-1} we choose the integer part from only one factor at at time; therefore the coefficient is $(-1)+(-2)+\cdots+(-(n-1)) = -\frac{n(n-1)}{2} = -\binom{n}{2}$.

Hence, for $n \geq 1$ the numbers $c(n,k)$ satisfy the same boundary conditions as the numbers $\left[{n \atop k}\right]$, and agree with them on some other values as well. It is tempting to believe that for $n \geq 1, c(n,k) = \left[{n \atop k}\right]$. Equality will follow if it can be shown that the numbers $c(n,k)$ satisfy the same recurrence as the numbers $\left[{n \atop k}\right]$.

Theorem 3.1.2 *For $n \geq 1$, let $c(n,k)$ denote the absolute value of coefficient of x^k in $x(x-1)(x-2)\cdots(x-(n-1))$. Then $c(n,k) = c(n-1,k-1) + (n-1)c(n-1,k)$.*

Proof. $x(x-1)(x-2)\cdots(x-(n-2)) = c(n-1,n-1)x^{n-1} - c(n-1,n-2)x^{n-2} + \cdots + (-1)^{n-1}c(n-1,0)$. Multiply both sides by $(x-(n-1))$ to get:

$$\begin{aligned} x(x-1)\cdots(x-(n-1)) &= c(n-1,n-1)x^n - c(n-1,n-2)x^{n-1} + \cdots + (-1)^{n-1}c(n,0)x \\ &\quad -(n-1)c(n-1,n-1)x^{n-1} + (n-1)c(n-1,n-2)x^{n-2} - \\ &\quad \cdots + (-1)^n(n-1)c(n-1,0) \end{aligned}$$

Equating coefficients of like terms gives $c(n,n) = c(n-1,n-1) = c(n-1,n-1)+(n-1)c(n-1,n)$ (by definition, the last summand equals zero) and, for $0 \leq k < n$, $(-1)^k c(n,k) = (-1)^{k-1}c(n-1,k-1) - (-1)^k c(n-1,k)$, so that $c(n,k) = c(n-1,k-1) + (n-1)c(n-1,k)$. ■

Corollary 3.1.3 *For $n \geq 1$, $c(n,k) = \left[{n \atop k}\right]$.*

Proof. The two quantities satisfy the same recurrence with the same boundary conditions. ■

3.2 Stirling Numbers of the Second Kind

Recall that a *partition* of a set X is an unordered collection of non-empty, pairwise disjoint subsets of X whose union is X. The subsets in the collection are called *blocks* (or *cells* or *parts*) of the partition.

The *Stirling number of the second kind*, $\left\{{n \atop k}\right\}$, equals the number of partitions of a set of $n \geq 0$ elements into k (non-empty) subsets.

The notation is helpful in distinguishing Stirling numbers of the first and second kind. Stirling numbers of the second kind count sets, so we use the same sort of brackets in $\left\{{n \atop k}\right\}$. Stirling number of the first kind count permutations, so we use the same sort of brackets (as are sometimes used for permutations) in $\left[{n \atop k}\right]$.

As an example, we compute $\left\{{4 \atop 2}\right\}$. There are $\binom{4}{1}$ partitions with a part of size 1 and one of size 3; and $\binom{4}{2}$ with two parts of size 2. Therefore, $\left\{{4 \atop 2}\right\} = \binom{4}{1} + \binom{4}{2} = 7$. The partitions of $\{a,b,c,d\}$ with exactly 2 parts are: $\{a\},\{b,c,d\}$; $\{b\}\{a,c,d\}$; $\{c\},\{a,b,d\}$; $\{d\},\{a,b,c\}$; $\{a,b\},\{c,d\}$; $\{a,c\},\{b,d\}$; $\{a,d\},\{b,c\}$.

It follows from the definition that $\left\{{n \atop 0}\right\} = 0$ when $n > 0$. The question arises as to what $\left\{{0 \atop 0}\right\}$ ought to be. The empty collection of subsets is a partition of $\emptyset$ because every element of $\emptyset$ belongs to a subset in the collection. Hence $\left\{{0 \atop 0}\right\} = 1$. The definition also implies that $\left\{{n \atop k}\right\} = 0$ if $k < 0$ or $k > n$. Further, $\left\{{n \atop 1}\right\} = 1$, $\left\{{n \atop n}\right\} = 1$, and $\left\{{n \atop n-1}\right\} = \binom{n}{2}$ (choose the two elements that belong to the part of size two).

To compute $\left\{{n \atop 2}\right\}$ consider the n-set $X = \{x_1, x_2, \ldots, x_n\}$. The part containing x_n can be any proper subset of $S \subset X$ that contains x_n, and the remaining elements of X comprise the other part in the partition. That is, there is one outcome for x_n (it must belong to the subset) and then $S - \{x_n\}$ can be any proper subset of $X = \{x_n\} = \{x_1, x_2, \ldots, x_{n-1}\}$. Since there are $2^{n-1} - 1$ possibilities for $S - \{x_n\}$, we have $\left\{{n \atop 2}\right\} = 2^{n-1} - 1$. (Another way to compute $\left\{{n \atop 2}\right\}$ is to first count ordered partitions – in which there is a first part and a second part – and then apply the Division Principle.)

The Stirling numbers of the second kind satisfy a similar recurrence to the Stirling numbers of the first kind. Notice that the multiplier of the second term is different. In the recurrence for the Stirling number of the first kind, the multiplier of the second term is $n-1$, the "top" of the second term (the part one would say first if describing the symbol). In the recurrence for the Stirling number of the second kind, the multiplier of the second term is k, the "bottom" of the second term (the part one would say second if describing the symbol). The best way to remember which is which is (the same as) the best way to remember the recurrence: remember the proof. The multiplier of the second term is the number of ways that x_n can be inserted back to get a desired configuration of the elements $x_1, x_2, \ldots, x_n$.

Theorem 3.2.1 *For $n \geq 0$, $\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k \left\{ {n-1 \atop k} \right\}$.*

Proof. The LHS counts the number of partitions of $\{x_1, x_2, \ldots, x_k\}$ into k non-empty subsets. We argue that the RHS counts the same thing by cases organized according to whether x_n belongs to a subset of size one, or to a subset of size greater than one (that is, whether $\{x_n\}$ is one of the subsets).

Consider a partition of $\{x_1, x_2, \ldots, x_n\}$ into k non-empty subsets. If $\{x_n\}$ is a subset, then the remaining subsets are a partition of $\{x_1, x_2, \ldots, x_{n-1}\}$ into $k-1$ subsets. There are $\left\{ {n-1 \atop k-1} \right\}$ of these. If x_n belongs to a subset of size greater than one, then deleting it results in a partition of $\{x_1, x_2, \ldots, x_{n-1}\}$ into k subsets. There are $\left\{ {n-1 \atop k} \right\}$ of these. Given a partition of $\{x_1, x_2, \ldots, x_{n-1}\}$ into k subsets, x_n can be inserted into any one of them to obtain a partition of $\{x_1, x_2, \ldots, x_n\}$ into k subsets. All of these reduce as above to the same partition of $\{x_1, x_2, \ldots, x_{n-1}\}$ into k subsets. Therefore, the number of partitions of $\{x_1, x_2, \ldots, x_n\}$ in which x_n belongs to a part of size greater than one is $k \left\{ {n-1 \atop k} \right\}$.

Therefore, by the Addition Principle $\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k \left\{ {n-1 \atop k} \right\}$. ■

We can use Theorem 3.2.1 to generate a triangle of Stirling numbers of the second kind.

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 0 & & 1 & & & & \\
 & & & 0 & & 1 & & 1 & & & \\
 & & 0 & & 1 & & 3 & & 1 & & \\
 & 0 & & 1 & & \underline{7} & & \underline{6} & & 1 & \\
0 & & 1 & & 15 & & \underline{25} & & 10 & & 1
\end{array}
$$

As for the Stirling numbers of the first kind, the rows are numbered from zero. Number the backwards diagonals (right to left) according to the row in which they begin, so that the backwards diagonal number corresponds to the "bottom" (that is k) of all numbers $\left\{ {n \atop k} \right\}$ that appear on it. To make the next row of the triangle, first first put a 0 on the left, and a 1 on the right. To produce the interior entries, add the number above and to the left to the backwards diagonal number times the number above and to the right (the entry being made lies on the same backwards diagonal as the number above and to the right). The underlined entries indicate that $25 = 7 + 3 \times 6$.

Proposition 3.2.2 $\left\{ {n+1 \atop r} \right\} = \binom{n}{0} \left\{ {0 \atop r-1} \right\} + \binom{n}{1} \left\{ {1 \atop r-1} \right\} + \cdots + \binom{n}{n} \left\{ {n \atop r-1} \right\}$.

Proof. The LHS counts the number of partitions of $\{x_1, x_2, \ldots, x_{n+1}\}$ into r subsets. The RHS counts the same thing by cases according to the size of the subset that contains x_{n+1}. Suppose the subset containing x_{n+1} has $1+(n-k)$ elements. Then the remaining $n+1-(1+(n-k)) = k$ elements among $x_1, x_2, \ldots, x_n$ are partitioned into $r-1$ subsets. There are $\binom{n}{k}$ ways to select these elements and then $\left\{ {k \atop r-1} \right\}$ ways to partition them into $r-1$ subsets. Therefore by AP, the number of partitions of $\{x_1, x_2, \ldots, x_{n+1}\}$ into r subsets is $\binom{n}{0} \left\{ {0 \atop r-1} \right\} + \binom{n}{1} \left\{ {1 \atop r-1} \right\} + \cdots + \binom{n}{n} \left\{ {n \atop r-1} \right\}$. ■

The formula in the proposition may be easier to understand as $\left\{ {n+1 \atop r} \right\} = \binom{n}{n} \left\{ {0 \atop r-1} \right\} + \binom{n}{n-1} \left\{ {1 \atop r-1} \right\} + \cdots + \binom{n}{0} \left\{ {n \atop r-1} \right\}$.

3.3 Bell Numbers

The *Bell number*, B_n, equals the number of partitions of a set with $n \geq 0$ elements.

Note. $B_0 = 1$.

Proposition 3.3.1 $B_n = \left\{ {n \atop 0} \right\} + \left\{ {n \atop 1} \right\} + \cdots + \left\{ {n \atop n} \right\}$.

Proof. The Stirling number of the second kind, $\left\{ {n \atop k} \right\}$, equals the number of partitions of an n-set with exactly k parts. Since $1 \leq k \leq n$, the result follows from the Addition Principle. ■

Proposition 3.3.2 $B_n = \binom{n-1}{0} B_0 + \binom{n-1}{1} B_1 + \cdots + \binom{n-1}{n-1} B_{n-1}$.

Proof. The RHS counts the number of partitions of $\{x_1, x_2, \ldots, x_n\}$ by cases according to the size of the subset containing x_n.

If the subset containing x_n has size $n - k$, $0 \leq k \leq n - 1$, the remaining k elements are partitioned into subsets. There are $\binom{n-1}{k}$ ways to select the k elements not in the same subset as x_n and then B_k ways to partition them into subsets. The result follows by summing over k. ■

Corollary 3.3.3 $B_n = \binom{n-1}{0} B_{n-1} + \binom{n-1}{1} B_{n-2} + \cdots + \binom{n-1}{n-1} B_0$.

Proof. Exercise. ■

3.4 Partitions

A *partition* of an integer n is an unordered collection of (not necessarily distinct) positive integers that sum to n. The number of partitions of the integer n is denoted by $p(n)$.

If follows from the definition that $p(n) = 0$ if $n < 0$. Since the empty sum equals zero, $p(0) = 1$.

There is clearly a 1–1 correspondence between the set of partitions of n and the set of non-negative integer solutions to $1 \cdot x_1 + 2 \cdot x_2 + \cdots + n \cdot x_n = n$: the integer x_i equals the number of times the integer i appears in the partition. Hence the number of partitions of n equals the number of non-negative integer solutions to this equation.

The following are all of the partitions of 4. It is customary to write partitions as sums.

$$\begin{aligned} 4 &= 4 \\ &= 3+1 \\ &= 2+2 \\ &= 2+1+1 \\ &= 1+1+1+1 \end{aligned}$$

Therefore $p(4) = 5$.

Any partition of a positive integer n can be represented by a *Ferrers diagram* (sometimes called a star diagram by authors who use stars instead of dots) composed of n •'s, arranged in left-justified rows corresponding to the parts in non-increasing order.

For example, the partition $17 = 5+5+4+1+1+1$ leads to the Ferrers diagram

```
• • • • •
• • • • •
• • • •
•
•
•
```

The *conjugate partition* of a partition Π is the partition obtained by reading the Ferrer's diagram by columns. There is clearly a 1–1 correspondence between partitions of n and their conjugates.

The conjugate of the partition of 17 shown above is $17 = 6+3+3+3+2$. Its Ferrers diagram is:

```
• • • • • •
• • •
• • •
• • •
• •
```

the transpose of the Ferrers diagram of its conjugate.

Proposition 3.4.1 *The number of partitions of n into at most k parts equals the number of partitions of n into parts each of which is at most k.*

Proof. The conjugate of a partition of n into at most k parts is a partition of n into parts each at most k. Therefore there is a 1–1 correspondence between these two collections. ■

We now sketch a second proof of Proposition 3.4.1. There is a 1–1 correspondence between the set of partitions of n into parts each of which is at most k and the set of non-negative integer solutions to $1 \cdot u_1 + 2 \cdot u_2 + \cdots + k \cdot u_k = n$ ($u_i = \#$ of i's in the partition). There is also a 1–1 correspondence between the set of partitions of n into at most k parts and the set of non-negative integer solutions to $w_1 + w_2 + \cdots + w_k = n$ such that $0 \leq w_1 \leq w_2 \leq \cdots \leq w_k \leq n$. We will show that there is a 1–1 correspondence between the solution sets of these two equations.

Suppose $0 \leq w_1 \leq w_2 \leq \cdots \leq w_k$ and $w_1 + w_2 + \cdots + w_k = n$. Set

$$\begin{aligned} u_k &= w_1 \\ u_{k-1} &= w_2 - w_1 \\ u_{k-2} &= w_3 - w_2 \\ &\vdots \\ u_1 &= w_k - w_{k-1} \end{aligned}$$

Then, working backwards gives,

$$\begin{aligned} w_1 &= u_k \\ w_2 &= u_{k-1} + w_1 \\ &= u_{k-1} + u_k \\ w_3 &= u_{k-2} + w_2 \\ &= u_{k-2} + u_{k-1} + u_k \\ &\vdots \\ w_n &= u_k + u_{k-1} + \cdots + u_1. \end{aligned}$$

Furthermore,

$$\begin{aligned} n &= w_1 + w_2 + \cdots + w_k \\ &= \ u_k \\ &\quad +u_k + u_{k-1} \\ &\quad +u_k + u_{k-1} + u_{k-2} \\ &\quad + \cdots + \\ &\quad + u_k + u_{k-1} + u_{k-2} + \cdots + u_1 \\ &= ku_k + (k-1)u_{k-1} + (k-2)u_{k-2} + \cdots + 1 \cdot u_1 \end{aligned}$$

and $u_1, u_2, \ldots, u_k \geq 0$ as $w_1 \leq w_2 \leq \cdots \leq w_k$. Since the procedure is reversible, there is a 1–1 correspondence between the solution sets of the two equations.

Proposition 3.4.2 *The number of partitions of n into exactly k parts equals the number of partitions of n in which the largest part is k.*

Proof. Exercise. ■

In a partition of n into odd parts, each part is required to be odd. In a partition of n into distinct parts, no two parts can be equal. Euler was the first to show that there are the same number of these two types of partition.

Theorem 3.4.3 [Euler] *Let n be a positive integer. The number of partitions of n into odd parts equals the number of partitions of n into distinct parts.*

Proof. We establish a 1–1 correspondence between the collection of partitions of n into odd parts and the collection of partitions of n into distinct parts.

Suppose a partition Π of n into odd parts is given. For $i = 1, 2, \ldots, n$, let x_i be the number of parts of size i in this partition (so $x_i = 0$ when i is even). Associate with Π the partition

$$n = \sum_{i=1}^{\frac{n}{2}} (2i+1)(2^{a_{i1}} + 2^{a_{i2}} + \cdots + 2^{a_{ik}})$$

where $2^{a_{i1}}(2i+1) + 2^{a_{i2}}(2i+1) + \cdots + 2^{a_{ik}}$ is the sum of powers of two that comprise the base-2 representation of x_{2j+1}. This is a partition of n into distinct parts because parts with the same largest odd divisor $2i+1$ are multiplied by different powers of two. Different partitions of n into odd parts have a different number of parts of some size, and hence lead to different partitions of n into distinct parts

The process is reversible. Suppose a partition of n into distinct parts is given. For each odd integer $2i+1$, let the parts whose largest odd divisor equals $2i+1$ be $2^{a_{i1}}(2i+1), 2^{a_{i2}}(2i+1), \ldots, 2^{a_{ik}}(2i+1)$. Then $a_{ij} \neq a_{i\ell}$ if $j \neq \ell$, so that $2^{a_{i1}}(2i+1) + 2^{a_{i2}}(2i+1) + \cdots + 2^{a_{ik}}$ is the base-2 representation of some number b_{2i+1}. A partition of n into odd parts can be obtained by replacing (each of) these (collections) by b_{2i+1} parts of size $2i+1$. This completes the proof. ■

In the proof of Theprem 3.4.3, the partition into distinct parts arising from a partition into odd parts can be seen to arise from the following procedure: Suppose a partition of n into odd parts is given. If the parts are distinct, then it is a partition of n into distinct parts. Otherwise, add the two largest equal parts together (necessarily producing an even part), and repeat the process until a partition with distinct parts results (the process must terminate since the number of parts is reduced at each stage).

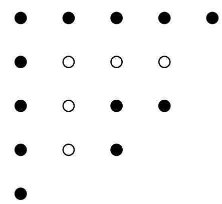
A *self-conjugate partition* of n is one that is its own conjugate.

Proposition 3.4.4 *The number of self-conjugate partitions of n equals the number of partitions of n into distinct odd parts.*

Proof. The Ferrers diagram of a self-conjugate partition of n is symmetric about the forward diagonal of the enclosing square.

Reading the diagram as a sequence of nested elbows (as shown) gives a partition of n into distinct odd parts. Since the procedure is reversible, the correspondence is 1–1. ■

The partition $17 = 9 + 5 + 3$ has distinct odd parts. The Ferrers diagram of corresponding self-conjugate partition is:



The differently filled circles indicate the "nested elbows".

In the following, we obtain a lower bound for $p(n)$. We will an upper bound later, when we study generating functions. Taken together, our bounds show that $p(n)$ is "about" $e^{\sqrt{n}}$. The main idea in the proof of the lower bound is to show that each subset of $X = \{1, 2, \ldots, \lfloor\sqrt{n}\rfloor\}$ can be used to generate a partition of n, and that these partitions are all different.

Theorem 3.4.5 *For $n \geq 2$, $p(n) \geq 2^{\lfloor\sqrt{n}\rfloor} = e^{\lfloor\sqrt{n}\rfloor \ln 2}$.*

Proof. We have

$$\begin{array}{rcrcl}
p(2) & = & 2 & \geq & 2^{\lfloor\sqrt{2}\rfloor} = 2 \\
p(3) & = & 3 & \geq & 2^{\lfloor\sqrt{3}\rfloor} = 2 \\
p(4) & = & 5 & \geq & 2^{\lfloor\sqrt{4}\rfloor} = 4 \\
p(5) & = & 7 & \geq & 2^{\lfloor\sqrt{5}\rfloor} = 4 \\
p(6) & = & 11 & \geq & 2^{\lfloor\sqrt{6}\rfloor} = 4 \\
p(7) & = & 14 & \geq & 2^{\lfloor\sqrt{7}\rfloor} = 4 \\
p(8) & = & 22 & \geq & 2^{\lfloor\sqrt{8}\rfloor} = 4
\end{array}$$

The LHS of the statement is non-decreasing and the RHS only increases at squares. Hence assume $n \geq 9$. Let $X = \{1, 2, \ldots, \lfloor\sqrt{n}\rfloor\}$. We claim that each subset $S \subseteq X$ generates a partition of n. This implies the theorem, since X has $2^{\lfloor\sqrt{n}\rfloor}$ subsets.

For $S \subseteq X$, let $\sigma(S)$ be the sum of the elements in S. Then

$$\sigma(S) \leq \sigma(X) = \frac{\lfloor\sqrt{n}\rfloor(\lfloor\sqrt{n}\rfloor + 1)}{2} \leq \frac{\sqrt{n}(\sqrt{n} + 1)}{2} = \frac{n + \sqrt{n}}{2} < \frac{n + n}{2} = n$$

and the elements of S together with the number $n - \sigma(S)$ is a partition of n.

Furthermore, distinct subsets generate distinct partitions: Let $S_1, S_2 \subseteq X, S_1 \neq S_2$. If $|S_1| \neq |S_2|$ then these subsets generate distinct partitions. Hence, assume $|S_1| = |S_2| = k$.

Since $S_1 \neq S_2$, if these subsets generate the same partition, then $n - \sigma(S_1) \in S_2$ and $n - \sigma(S_2) \in S_1$ (otherwise they are the same set as $n - \sigma(S_1) = n - \sigma(S_2)$ and all other parts are equal). We claim this is impossible as $n - \sigma(S_i) > \lfloor\sqrt{n}\rfloor$.

For $i = 1, 2$

$$\begin{aligned}
n - \sigma(S_i) &\geq \lfloor\sqrt{n}\rfloor^2 - (\sigma(X) - 1) \\
&\quad \text{b/c } |S_1| = |S_2| \text{ and } S_1 \neq S_2 \Rightarrow \text{neither is } X; \text{ thus } \sigma(S_i) < \sigma(X). \\
&= \lfloor\sqrt{n}\rfloor^2 - \left(\frac{\lfloor\sqrt{n}\rfloor(\lfloor\sqrt{n}\rfloor^2 + 1) - 2}{2}\right) \\
&= \frac{\cancel{2}\lfloor\sqrt{n}\rfloor^2 - \cancel{\lfloor\sqrt{n}\rfloor^2} - \lfloor\sqrt{n}\rfloor + 2 + (2\lfloor\sqrt{n}\rfloor - 2\lfloor\sqrt{n}\rfloor)}{2} \\
&= \lfloor\sqrt{n}\rfloor + \frac{\lfloor\sqrt{n}\rfloor^2 - 3\lfloor\sqrt{n}\rfloor + 2}{2} \\
&= \lfloor\sqrt{n}\rfloor + \frac{(\lfloor\sqrt{n}\rfloor - 1)(\lfloor\sqrt{n}\rfloor - 2)}{2} > \lfloor\sqrt{n}\rfloor.
\end{aligned}$$

Since $n \geq 9$, $\sqrt{n} \geq 3$ and therefore $n - \sigma(S_1) \notin S_2$ and different subsets generate different partitions. This completes the proof.

■

3.5 The Twelvefold Way

Our work in the previous sections can be tied together by considering the problems of placing k balls into n boxes under various restrictions. We will consider the 12 possible situations that arise depending on whether the balls are distinguishable, whether the boxes are distinguishable, whether empty boxed are allowed, and whether more than one ball can be placed in a box. Apparently the idea for bringing the counting numbers together in this way originated with Gian-Carlo Rota, and the name "the 12-fold way" was suggested by Joel Spencer.

Suppose we have a collection of k balls and n boxes. The number of outcomes for placing these balls into the boxes is described as follows:

1. If the balls are distinguishable,
   (a) the boxes are distingishable, and there are no other restrictions, then the number of outcomes is n^k, the number of functions from the balls to the boxes;
   (b) the boxes are distinguishable and at most one ball can be placed in any box, then the number of outcomes is $n(n-1)\cdots(n-(k-1))$, the number of 1–1 functions from the balls to the boxes. Notice that this number equals zero if $k > n$;
   (c) the boxes are distinguishable and no box can be left empty, then the number of outcomes is $k!\left\{{k \atop n}\right\}$, the number of functions from the balls onto the boxes. Notice that this number equals zero if $k < n$;
   (d) the boxes are identical, and there are no other restrictions, then the distribution determines a partition of the set of balls, so the number of outcomes is B_k, the k-th Bell number;

(e) the boxes are identical and and at most one ball can be placed in any box, then the number of outcomes is one if $k \leq n$ and zero otherwise;

(f) the boxes are identical and no box can be left empty, then the distribution determines a partition of the set of balls into exactly n parts, so the number of outcomes is $\left\{ {k \atop n} \right\}$.

2. If the balls are identical,

(a) the boxes are distingishable, and there are no other restrictions, then each outcome determines a non-negative integer solution to $x_1 + x_2 + \cdots + x_n = k$, where x_i equals the number of balls in box $i, 1 \leq i \leq n$, so that the number of outcomes is $\binom{k+(n-1)}{k}$;

(b) the boxes are distinguishable and at most one ball can be placed in any box, then the number of outcomes is $\binom{n}{k}$, since placing a ball into a box can be regarded as choosing that box. Notice that this number equals zero if $k > n$;

(c) the boxes are distinguishable and no box can be left empty, then each outcome determines a *positive* integer solution to $x_1 + x_2 + \cdots + x_n = k$, where x_i equals the number of balls in box $i, 1 \leq i \leq n$, so that if $k > 0$ the number of outcomes is $\binom{k-n+(n-1)}{k-n} = \binom{k-1}{k-n}$ (notice that this number equals zero if $n > k$). If $k = 0$ then the number of outcomes is one if $n = 0$ and zero otherwise;

(d) the boxes are identical, and there are no other restrictions, then the distribution determines a partition of the integer k into at most n parts;

(e) the boxes are identical and and at most one ball can be placed in any box, then the number of outcomes is one if $k \leq n$ and zero otherwise;

(f) the boxes are identical and no box can be left empty, then the distribution determines a partition of the integer n into exactly n parts.

3.6 Derangements

Given an arrangement $x_1 x_2 \cdots x_n$ of $n \geq 0$ distinguishable objects, a *derangement* is a permutation π of the objects such that, for $i = 1, 2, \ldots, n$, we have $\pi(x_i) \neq x_i$.

Let d_n be the number of derangements of $x_1 x_2 \cdots x_n$. Then $d_0 = 1$ (why?), $d_1 = 0$, $d_2 = 1$, and so on.

Of the $3! = 6$ permutations of 1 2 3, there are two which are derangements of 1 2 3. These are 2 3 1 and 3 1 2. All of the others have 1 in position, 2 in position 2, or 3 in position 3.

In order to work towards a recurrence for d_n, we consider some derangements of 1 2 3 4 5. In the derangement 2 4 5 1 3 the elements 5 and 3 have swapped places from their original positions. That means the elements in the other places have been deranged (among themselves: the derangement of 1 2 4 is 2 4 1) and the given derangement splits into one of size two that involves 5, and one of size 3. In general there are four choices for the element with which 5 swaps places, so there are $4d_3$ derangements of this type. In the derangement 2 5 4 1 3, the element 5 is in the position originally occupied by 2 but the element 2 is not in the position originally occupied by 5. Hence, replacing the 5 by the element in the last position (the position originally occupied by 5), gives a derangement of 1 2 3 4, namely 2 3 4 1. It can be helpful to think of derangements of this type as being formed from a derangement of 1 2 3 4 by replacing one of the elements by 5 and then putting the element that was replaced at the end of the list. Since 5 can replace any of the four elements in the way just described, there are $4d_4$ derangements of this type. Thus, by the Addition Principle, $d_5 = 4d_4 + 4d_3$. The argument generalizes to become the proof of the next theorem.

Theorem 3.6.1 *Let $n \geq 2$ be an integer. Then $d_n = (n-1)d_{n-1} + (n-1)d_{n-2}$.*

Proof. Consider a derangement of $x_1 x_2 \cdots x_n$. Suppose the element in the last position is x_i. Either x_n is in the i^{th} position, or it isn't.

If x_n is in the i^{th} position, so that x_i and x_n have swapped places, then deleting x_i and x_n (or ignoring them) gives a derangement of the remaining $n-2$ objects. This process is reversible, so there is a 1–1 correspondence between derangements of the given type and derangements of $x_1\ x_2\ \cdots\ x_{i-1}\ x_{i+1}\ \cdots\ x_{n-1}$. Since there are $n-1$ choices for i and d_{n-2} derangements of the remaining elements, there are $(n-1)d_{n-2}$ derangements of this type.

Suppose that x_n is in the k^{th} position, $k \neq i$. Then we get a derangement of $x_1\ x_2\ \cdots\ x_{n-1}$ by replacing x_n with x_i. Since this procedure is reversible, we have a 1–1 correspondence between derangements of the given type and derangements of $x_1\ x_2\ \cdots\ x_{n-1}$. This holds for each of the $n-1$ choices for i, and thus there are $(n-1)d_{n-1}$ derangments of the given type.

Thus, by the Addition Principle, $d_n = (n-1)d_{n-1} + (n-1)d_{n-2}$. ■

This recurrence can be used to obtain a formula for d_n. Consider the ratio $E_n = \frac{d_n}{n!}$. This quantity is the proportion of the permutations of n distinguishable objects that are derangements. From the definition of d_n we have $E_1 = 0$, $E_0 = 1$, and

$$\begin{aligned}
E_n = \frac{d_n}{n!} &= \frac{(n-1)d_{n-1}}{n!} + \frac{(n-1)d_{n-2}}{n!} \\
&= \frac{n-1}{n}\frac{d_{n-1}}{(n-1)!} + \frac{n-1}{n(n-1)}\frac{d_{n-2}}{(n-2)!} \\
&= \left(1 - \frac{1}{n}\right)\frac{d_{n-1}}{(n-1)!} + \left(\frac{1}{n}\right)\frac{d_{n-2}}{(n-2)!} \\
&= \left(1 - \frac{1}{n}\right)E_{n-1} + \left(\frac{1}{n}\right)E_{n-2}
\end{aligned}$$

Hence $E_n - E_{n-1} = \left(-\frac{1}{n}\right)(E_{n-1} - E_{n-2})$, and similarly $E_{n-1} - E_{n-2} = \left(-\frac{1}{n-1}\right)(E_{n-2} - E_{n-3})$; $E_{n-2} - E_{n-3} = \left(-\frac{1}{n-2}\right)(E_{n-3} - E_{n-4})$, and so on until, finally, $E_2 - E_1 = \frac{1}{2}$. We can now iterate and obtain $E_n - E_{n-1} = \left(-\frac{1}{n}\right)(E_{n-1} - E_{n-2}) = \left(-\frac{1}{n}\right)\left(-\frac{1}{n-1}\right)(E_{n-2} - E_{n-3}) = \left(-\frac{1}{n}\right)\left(-\frac{1}{n-1}\right)\cdots\left(-\frac{1}{2}\right)(E_1 - E_0) = \frac{(-1)^n}{n!}$, since n terms have -1's.

Therefore, $E_n = \frac{(-1)^n}{n!} + E_{n-1} = \frac{(-1)^n}{n!} + \frac{(-1)^{n-1}}{(n-1)!} + E_{n-2} = \cdots = \frac{(-1)^n}{n!} + \frac{(-1)^{n-1}}{(n-1)!} + \cdots + \frac{(-1)^2}{2!} + E_1 = \frac{(-1)^n}{n!} + \frac{(-1)^{n-1}}{(n-1)!} + \cdots + \frac{(-1)^2}{2!} + (\frac{(-1)^1}{1!} + \frac{(-1)^0}{0!})$. Since $E_n = \frac{d_n}{n!}$, we get $d_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots \pm \frac{1}{n!}\right)$.

As a "fun fact" we show that, for $n \geq 2$, the integer d_n is the closest integer to $\frac{n!}{e}$. Recall the MacLaurin series for e^x: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$. Therefore, $e^{-1} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$. It follows that $\frac{d_n}{n!} \approx e^{-1}$, or $d_n \approx \frac{n!}{e}$. Now,

$$\begin{aligned} \left|\frac{n!}{e} - d_n\right| &= \left|n!\left(\frac{1}{(n+1)!} - \frac{1}{(n+2)!} + \frac{1}{(n+3)!} - \cdots\right)\right| \\ &= \left|\frac{n!}{(n+1)!}\left(1 - \frac{1}{n+2} + \frac{1}{(n+2)(n+3)} - \cdots\right)\right| \\ &\leq \frac{1}{(n+1)}\left(1 + \frac{1}{(n+2)} + \frac{1}{(n+2)^2} + \cdots\right) \\ &= \frac{1}{(n+1)} \cdot \frac{1}{1 - \frac{1}{(n+2)}} = \frac{n+2}{(n+1)^2} \end{aligned}$$

The final expression is a decreasing function for $n \geq 0$. For $n = 2$ it equals $\frac{4}{9} < \frac{1}{2}$. Since d_n is an integer, the result follows.

3.7 Exercises

1. Use the fact that $\left[{n \atop 0}\right] = 0$ and Theorem 3.1.1 to prove:
   (a) $\left[{n \atop 1}\right] = (n-1)!$;
   (b) $\left[{n \atop n}\right] = 1$;
   (c) $\left[{n \atop n-1}\right] = \binom{n}{2}$.
2. (a) How many ways can $n \geq 0$ people be seated at k identical (i.e. indistinguishable) tables?
   (b) How many ways can $n \geq 0$ people be seated at k distinguishable circular tables so that no table is empty?
   (c) How many ways can $n \geq 0$ people be seated at k distinguishable circular tables?
3. Show that for $n \geq 2$, $\left[{m \atop 2}\right] = (m-1)! \sum_{k=1}^{m-1} \frac{1}{k}$.
4. Show that $n! = \left[{n \atop n}\right] n^n - \left[{n \atop n-1}\right] n^{n-1} + \left[{n \atop n-2}\right] n^{n-2} - \cdots$
5. For $n, t \geq 0$ let $m(n,t)$ be the number of permutations of nt distinguishable objects with exactly t cycles of length n in their disjoint cycle representation. Find a recurrence relation and initial conditions for $m(n,t)$.
6. Prove that $\left[{n+1 \atop r}\right] = \binom{n}{0}\left[{n \atop r-1}\right] 0! + \binom{n}{1}\left[{n-1 \atop r-1}\right] 1! + \cdots + \binom{n}{n}\left[{n-n \atop r-1}\right] (n-1)!$.
7. Let $C_r(n,k)$ be the number of permutations of the elements of an n-set with exactly k cycles, each of size at least r, in its disjoint cycle representation.
   (a) Find boundary values for $C_r(n,k)$.

(b) Give a combinatorial argument to prove that

$$C_r(n+1,k) = nP_r(n,k) + \binom{n}{r-1}(r-1)!C_r(n+1-r,k-1).$$

8. Let $X = \{x_1, x_2, \ldots, x_n\}$, $Y = \{y_1, y_2, \ldots, y_m\}$, and $r \geq 0$. How many functions $f : X \to Y$ have $|f(X)| = r$?

9. Let $O(n,k)$ denote the number of functions from a n-set $\{x_1, x_2, \ldots, x_n\}$ *onto* a k-set $\{y_1, y_2, \ldots, y_k\}$.

   (a) It is implicit in the definition of $O(n,k)$ that $n, k \geq 0$. Explain why $O(n,k) = 0$ if $k > n$. What is $O(n,0)$, including in the case $n = 0$? How about $O(n,1)$?

   (b) Give a combinatorial argument to show that $O(n,k) = kO(n-1,k-1)+kO(n-1,k)$.

   (c) Use parts (a) and (b) above to compute $O(n,n)$.

   (d) Write out the first five rows of an "onto triangle" and describe a rule for constructing each subsequent row.

   (e) Show that $\left\{ n \atop k \right\} = \frac{1}{k!}\left\{ n \atop k \right\}$.

   (f) Use part (e) above and Theorem 3.2.1 to give another proof of the recurrence in part (b).

10. Let m and n be positive integers. Prove that

$$m^n = \sum_{k=1}^{n} \left\{ n \atop k \right\} \frac{m!}{(m-k)!}.$$

11. Let $P_r(n,k)$ be the number of partitions of an n-set into exactly k parts each of size at least r.

    (a) Find boundary values for $P_r(n,k)$.

    (b) Give a combinatorial argument to prove that

$$P_r(n+1,k) = kP_r(n,k) + \binom{n}{r-1}P_r(n+1-r,k-1).$$

12. Let n be a positive integer. A *composition* of n is an ordered sum that equals n. For example, the expressions $1+2$ and $2+1$ are different compositions of 3. Each term in the sum is called a *part* of the composition.

    (a) Let $c(n)$ equal the number of compositions of n. Show that $c(n) = 2^{n-1}$.

    (b) Let $c_e(n)$ denote the number of compositions of n into even parts (i.e. each summand is an even number). Prove $c_e(2k+1) = 0$ and $c_e(2k) = 2^{k-1}(= c(k))$.

13. Prove Proposition 3.4.2.

14. Prove Corollary 3.3.3.

15. Show that the procedure described following the proof of Theorem 3.4.3 produces the same partition of n into distinct parts as is constructed in the proof of the theorem, and then describe the reverse procedure.

16. Prove that the number of partitions of $(a-c)$ into exactly $(b-1)$ parts none of which is larger than c, equals the number of partitions of $(a-b)$ into $(c-1)$ parts none of which is larger than b.

17. Prove that $p(n)$ equals the number of partitions of $2n$ into exactly n parts.

18. Give a combinatorial argument to show that $n! = \binom{n}{0}d_0 + \binom{n}{1}d_1 + \cdots + \binom{n}{n}d_n$.

19. (a) Show that $d_n = nd_{n-1} + (-1)^n$, for $n \geq 1$.
    (b) Use part (a) to prove Theorem 3.6.1.
    (c) Show that the number of permutations π of $\{1, 2, \ldots, n\}$ that fix exactly k elements (that is, for which there are exactly k elements i such that $pi(i) = i$) equals $\binom{n}{k}d_{n-k}$.

20. Given a circular arrangement $(x_1\ x_2\ \cdots\ x_n)$ of $n \geq 0$ distinguishable objects, define a *circular derangement* to be a circular permutation $(x_1\ y_2\ \cdots\ y_n)$ of these objects such that, for $i = 2, 3, \ldots, n$ the element $y_i \neq x_i$. Put differently, if element x_1 is rotated to the "top" of the circle, the no other element is in the same position as before. Let c_n be the number of circular derangements of $(1\ 2\ \cdots\ n)$. Find c_0, c_1, c_2, c_3 and c_4, and then find c_n for all n.

21. Prove that

$$\sum_{\substack{n_1+n_2+\cdots+n_k=n \\ n_i \geq 1}} \binom{n}{n_1\, n_2\, \cdots\, n_k} = O(n,k) = k! \left\{ {n \atop k} \right\},$$

where $O(n, k)$ is as defined in Exercise 9 above.

22. Given $2n$ people of different heights, in how many ways can these people be formed into 2 rows of n each so that everyone in row 1 is taller than the corresponding person in row 2? (*Hint*: one way is find a 1-1 correspondence between these arrangements and another collection of arrangements we have counted.)

23. In this problem we will find the exponential generating function for the sequence $\left[{n \atop n} \right], \left[{n+1 \atop n} \right], \left[{n+2 \atop n} \right], \ldots$ of Stirling numbers of the first kind.
    (a) For $n > 0$, let

$$G_n(x) = \sum_{k=n}^{\infty} \left[{k \atop n} \right] \frac{x^k}{k!}$$

Prove that

$$(1-x)\cdot\frac{d}{dx}G_n(x)=G_{n-1}(x).$$

Conclude from this, and the fact that $G_m(0)=0$, that

$$G_n(x)=\int_0^x \frac{G_{n-1}(y)}{1-y}\,dy.$$

(b) Prove that

$$G_1(x)=x+\frac{1}{2}x^2+\frac{1}{3}x^3+\ldots=-\ln(1-x).$$

Finally, use (a) to prove that

$$G_n(x)=\frac{[-\ln(1-x)]^n}{n!}.$$

24. Prove that the Stirling number of the first kind $\left[{n \atop k}\right]$ is the sum of all products of $n-k$ different integers taken from $\{1,2,\ldots,n-1\}$, and that there are $\binom{n-1}{k-1}$ such products.

25. Prove that the Stirling number of the second kind $\left\{{n \atop k}\right\}$ is the sum of all products of $n-k$ not necessarily distinct integers taken from $\{1,2,\ldots,k\}$, and that there are $\binom{n-1}{k-1}$ such products.

26. Prove that the Stirling number of the second kind $\left\{{n \atop k}\right\}$ has the same parity as the binomial coefficient $\binom{\lceil k/2\rceil+n-k-1}{n-k}$.

27. Let S_n be the number of sequences of integers $a_1,a_2,a_3,\ldots a_n$ such that $1\le a_1\le a_2\le\ldots\le a_n$, and $a_i\le i,\ 1\le i\le n$. What is S_n?

28. The largest square of dots (or stars) in the upper left-hand corner of the Ferrers diagram is called the *Durfee square*.

    (a) Find a 1-1 correspondence between the self-conjugate partitions of n whose Durfee square is $k\times k$ and the partitions of $n-k^2$ into even parts of size at most $2k$.

    (b) Find a generating function for the number of self-conjugate partitions of n whose Durfee square is $k\times k$.

29. Twenty people get on an elevator on the first floor of a building. When the elevator leaves the twelfth floor, it is empty. In how many ways can the people get off of the elevator so that there are at exactly seven floors on which people get off?

Chapter 4

Generating Functions

4.1 Ordinary Power Series Generating Functions

Let $a_0, a_1, a_2, \ldots$ be a sequence of real numbers. The *(ordinary power series) generating function* for the sequence is

$$\begin{aligned} G(x) &= a_0 + a_1x + a_2x^2 + \cdots \\ &= \sum_{n=0}^{\infty} a_n x^n \end{aligned}$$

Usually we're looking for a closed form for $G(x)$. This can help us understand the sequence better. We don't worry about convergence – instead we treat this as formal power series. Algebraically, the set of such series form an integral domain – two series being equal and only if the coefficients of like powers of x are equal.

One answer to the question of why these are called "generating functions" is the following: given a closed form, say $\frac{x}{1-x-x^2}$, then the process of expanding this as a power series by long division can be viewed as generating its coefficients. We have $\frac{x}{1-x-x^2} = 0 + x + x^2 + 2x^3 + 3x^4 + 5x^5 + 8x^6 + 13x^7 + 21x^8 + \cdots$, so that $\frac{x}{1-x-x^2}$ is the generating function for the sequence $0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$.

The sequence "generated" by $\frac{x}{1-x-x^2}$ appears to be the sequence of Fibonacci numbers. We now confirm that it is. Recall that the Fibonacci sequence arises from a problem in the book *Liber abaci* by Leonardo of Pisa (known as Fibonacci, 1175-1250) asks for the number of pairs of rabbits arising from the following scenario:

> *A newly born pair of rabbits is released at UVic. Every pair of rabbits at least two months old will produce one pair of rabbits as offspring every month.*

Notwithstanding that the rabbits are assumed to be born in mixed gender pairs and immortal, let F_n be the number of rabbits on hand at the start of month n. Then $F_0 = 0,\ F_1 = F_2 = 1$

and $F_n = F_{n-1} + F_{n-2}$. Let $G(x)$ be the generating function for the sequence $F_0, F_1, F_2, \ldots$. Then, $G(x) = F_0 + F_1 x + F_2 x^2 + \cdots$, so that $xG(x) = F_0 x + F_1 x^2 + F_2 x^3 + \cdots$ and $x^2 G(x) = F_0 x^2 + F_1 x^3 + F_2 x^4 + \cdots$. It follows that

$$\begin{aligned} G(x) - xG(x) - x^2G(x) &= \cancelto{0}{F_0} + \cancelto{1}{(F_1 - F_0)}x + \cancelto{0}{(F_2 - F_1 - F_0)}x^2 + \cancelto{0}{(F_3 - F_2 - F_1)}x^3 + \cdots \\ &= 0 + 1 \cdot x + 0 \cdot x^2 + 0 \cdot x^3 + \cdots \\ &= x \end{aligned}$$

Therefore, $G(x)(1 - x - x^2) = x$ or $G(x) = \frac{x}{1-x-x^2}$, as desired.

Given a closed form for a generating function $G(x)$, it is often possible to explicitly determine the sequence that is generated. One way is to use knowledge of some standard power series and the method of partial fractions, as is illustrated below. Another way to notice that the coefficient of x^n equals $\frac{1}{n!}$ times $G^{(n)}(0)$, the n-th derivative of $G(x)$ evaluated at $x = 0$ (think about this in the context of the Taylor Series expansion of $G(x)$).

In the case of $G(x) = \frac{x}{1-x-x^2}$, by partial fractions,

$$\frac{x}{1 - x - x^2} = \frac{x}{(1 - \alpha x)(1 - \beta x)} = \frac{A}{1 - \alpha x} + \frac{B}{1 - \beta x}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

Therefore,

$$\begin{aligned} A(1 - \beta x) + B(1 - \alpha x) &= x \\ A + B &= 0 \end{aligned}$$

The second equation gives $B = -A$. With this information, on equating coefficients of like powers of x, the first equation gives $-\beta A - \alpha(-A) = 1$. Solving:

$$\begin{aligned} & -\beta A - \alpha(-A) = 1 \\ \Leftrightarrow\ & (\alpha - \beta)A = 1 \\ \Leftrightarrow\ & A = \frac{1}{\alpha - \beta} = \frac{1}{\sqrt{5}} \end{aligned}$$

Therefore $B = -\frac{1}{\sqrt{5}}$. Now,

$$\frac{1}{1 - \alpha x} = 1 + \alpha x + \alpha^2 x^2 + \cdots$$

and

$$\frac{1}{1 - \beta x} = 1 + \beta x + \beta^2 x^2 + \cdots$$

so

$$\begin{aligned} G(x) &= \frac{1}{\sqrt{5}}(1 + \alpha x + \alpha^2 x^2 + \cdots) - \frac{1}{\sqrt{5}}(1 + \beta x + \beta^2 x^2 + \cdots) \\ &= \frac{1}{\sqrt{5}}((\alpha - \beta)x + (\alpha^2 - \beta^2)x^2 + (\alpha^3 - \beta^3)x^3 + \cdots) \end{aligned}$$

and since F_n is the coefficient of x^n in $G(x)$ we have

$$F_0 = 0 \qquad F_1 = \frac{1}{\sqrt{5}}(\alpha - \beta) = 1 \qquad F_2 = \cdots$$

and

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

Remarkably, this quantity is an integer! Notice that $\left|\frac{1-\sqrt{5}}{2}\right| \approx 0.62$, so

$$\left|\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n\right| < 0.28.$$

Hence, the n-th Fibonacci number, F_n, equals $\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n$ rounded to the nearest integer.

Ordinary power series generating functions are useful in counting unordered selections of objects from a variety of different types. For example, if we have a (really) large collection of identical red balls, and a (really) large collection of identical blue balls, then the number of ways to select 7 balls so that the selection contains an even number of red balls and an odd number of blue balls is the coefficient of x^7 in $(1+x^2+x^4+\cdots)(x+x^3+\cdots)$: a term x^7 arises once for each way of choosing $r \in \{0, 2, 4, \ldots\}$ and $b \in \{1, 3, 5, \ldots\}$ so that $x^7 = x^r x^b = x^{r+b}$. There is nothing special about 7: the number of ways to select n balls so that the selection contains an even number of red balls and an odd number of blue balls is the coefficient of x^n in $(1+x^2+x^4+\cdots)(x+x^3+\cdots)$ The last expression equals $\frac{1}{1-x^2}\frac{x}{1-x^2} = x\left(\frac{1}{1-x^2}\right)^2$. The coefficient of x^n is nonzero only for odd n (which makes sense given the problem under consideration). Say $n = 2k+1 \geq 1$. The coefficient of x^n turns out to equal $\binom{k+(2-1)}{k} = k+1$. (This makes sense, since once the red balls are chosen, there is only one possibility for the number of blue balls to choose.)

The above example suggests that the rules of exponents are a reason underlying why generating functions are useful in counting arguments. This statement will be made more explicit in Lemma 4.1.1 below. Before that, we recall some basic power series that are useful. To get started, recall that

$$\frac{1}{1-y} = 1 + y + y^2 + \cdots .$$

Putting $y = x^k$ gives

$$\begin{aligned} \frac{1}{1-x^k} &= 1 + x^k + (x^k)^2 + (x^k)^3 + \cdots \\ &= 1 + x^k + x^{2k} + x^{3k} + \cdots . \end{aligned}$$

Notice that putting $y = -x^k$ gives an alternating series.

Lemma 4.1.1 *Let $r_1, r_2, \ldots$ be fixed integers. The coefficient of x^n in the series expansion of*

$$\frac{1}{(1-x^{r_1})(1-x^{r_2})(1-x^{r_3})\cdots}$$

equals the number of integer solutions to $n = k_1r_1+k_2r_2+k_3r_3+\cdots$, where $k_i \geq 0$ for $i = 1, 2, \ldots$.

Proof.

$$\frac{1}{(1-x^{r_1})(1-x^{r_2})(1-x^{r_3})\cdots} = (1+x^{r_1}+x^{2r_1}+\cdots)(1+x^{r_2}+x^{2r_2}+\cdots)(1+x^{r_3}+x^{2r_3}+\cdots)\cdots$$

A term x^n arises for each choice of $x^{k_1r_1}$ from the first factor, $x^{k_2r_2}$ from the second factor, and so on, such that $n = k_1r_2 + k_2r_2 + \cdots$ and $k_i \geq 0$ for $i = 1, 2, \ldots$. (Note that only finitely many factors can $k_i > 0$.)

■

A different way to state Lemma 4.1.1 is that the The coefficient of x^n in the series expansion of $\frac{1}{(1-x^{r_1})(1-x^{r_2})(1-x^{r_3})\cdots}$ equals the number of ways to write n as an unordered sum of terms, each of which equals r_i for some i. (These are partitions of n into terms of given size.)

4.2 Catalan Numbers

The counting numbers discussed this section are named after Eugene Catalan (1814-1894). They count a diverse collection of thins. We will discuss a few of them here, and a few more in the exercises. For even more, look at Richard Stanley's book, *Enumerative Combinatorics, Volume 2*.

To parenthesize a product means to insert enough parentheses so that every pair of parentheses encloses a product involving exactly two factors. For example, $x_1x_2x_3x_4$ can be parenthesized as

$$((x_1x_2)(x_3x_4)) \quad (((x_1x_2)x_3)x_4)$$
$$((x_1(x_2x_3))x_4) \quad (x_1((x_2x_3)x_4)) \quad (x_1(x_2(x_3x_4)))$$

Notice that the order of the x's doesn't change; they are really just placeholders.

Let a_n be the number of different outcomes of parenthesizing the product $x_1x_2\cdots x_{n+1}$. The shift of one in the subscript is so that there are n sets of parentheses involved. Then $a_0 = 1$ because the only way to parenthesize x_1 as required is to use no parentheses. Direct computation gives $a_1 = 1$, and $a_2 = 2$. The example above shows that $a_3 = 5$.

We now obtain a (nonlinear) recurrence for a_n. Suppose $n \geq 2$ and consider a parenthesization of $x_1x_2\cdots x_{n+1}$. Then, the outside set of parentheses multiplies two terms. The first is the product of $x_1, x_2, \ldots, x_r$ for some $1 \leq r \leq n$, and the second is the product of the $n-r+1$ terms $x_{r+1}, x_{r+2}, \ldots, x_{n+1}$. There are a_{r-1} outcomes of parenthesizing the first r of the x's , and a_{n-r} outcomes of parenthesizing the last $(n-r+1)$ of them. Thus, the number of outcomes in which the outer parentheses multiply r times $(n-r)$ x's is $a_{r-1}a_{n-r}$. Therefore, by the Addition Principle, $a_n = a_0a_{n-1} + a_1a_{n-2} + \cdots + a_{n-1}a_0$.

To test our recurrence, we compute that $a_0a_1+a_1a_0 = 2 = a_2$, and $a_0a_2+a_1a_1+a_2a_0 = 5 = a_3$.

Theorem 4.2.1 $a_n = \frac{1}{n+1}\binom{2n}{n}$.

Proof. Let $G(x) = a_0 + a_1x + a_2x^2 + \cdots$. Then

$$\begin{aligned} G(x)^2 &= a_0a_0 + (a_0a_1 + a_1a_0)x + (a_0a_2 + a_1a_1 + a_2a_0)x^2 + (a_0a_3 + a_1a_2 + a_2a_1 + a_3a_0)x^3 + \cdots \\ &= a_1 + a_2x + a_3x^2 + a_4x^3 + \cdots \\ &= \frac{G(x) - 1}{x} \end{aligned}$$

or $xG(x)^2 - G(x) + 1 = 0$. Thus, $G(x) = \frac{1\pm\sqrt{1-4x}}{2x}$ or (better) $xG(x) = \frac{1\pm\sqrt{1-4x}}{2}$. We need to decide whether to choose the plus sign or the minus sign,

When $x = 0$, $xG(x) = 0$, so we must choose $xG(x) = \frac{1-\sqrt{1-4x}}{2}$. Now, by Taylor's theorem,

$$(1+x)^{1/2} = 1 + \frac{1}{2}x + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}x^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}x^3 + \cdots$$

A typical coefficient on the RHS is

$$\begin{aligned} &\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\cdots\left(\frac{1}{2}-(k-1)\right)}{k!} \\ =\ & \frac{1}{2^k}\frac{1}{k}(-1)^{k-1}\frac{(1-2)(1-4)\cdots(1-2(k-1))}{(k-1)!}\cdot\frac{2^{k-1}(k-1)!}{2^{k-1}(k-1)!} \\ =\ & \frac{1}{2^{2k-1}}\frac{1}{k}(-1)^{k-1}\frac{(2k-2)!}{(k-1)!(k-1)!} \\ =\ & \frac{1}{2^{2k-1}}\frac{1}{k}(-1)^{k-1}\binom{2k-2}{k-1}. \end{aligned}$$

Therefore,

$$1+x)^{1/2} = 1 + \frac{1}{2}x - \frac{1}{2^3}\cdot\frac{1}{2}\binom{2}{1}x^2 + \frac{1}{2^5}\cdot\frac{1}{3}\binom{4}{2}x^3 - \frac{1}{2^7}\cdot\frac{1}{4}\binom{6}{3}x^4 + \cdots$$

so that

$$\begin{aligned} (1-4x)^{1/2} &= 1 + \frac{1}{2}(-4x) - \frac{1}{2^3}\cdot\frac{1}{2}\binom{2}{1}(-4x)^2 + \cdots \\ &= 1 - 2x - \frac{2}{2}\binom{2}{1}x^2 - \frac{2}{3}\binom{4}{2}x^3 - \frac{2}{4}\binom{6}{3}x^4 - \cdots \end{aligned}$$

Therefore,

$$\begin{aligned} xG(x) &= \frac{1-\sqrt{1-4x}}{2} \\ &= \frac{1}{2}\left[2x + \frac{2}{2}\binom{2}{1}x^2 + \frac{2}{3}\binom{4}{2}x^3 + \cdots\right] \\ &= x + \frac{1}{2}\binom{2}{1}x^2 + \frac{1}{3}\binom{4}{2}x^3 + \frac{1}{4}\binom{6}{3}x^4 + \cdots \end{aligned}$$

Since a_n the coefficient of x^{n+1} in $xG(x)$, we have $a_n = \frac{1}{n+1}\binom{2n}{n}$. ■

The *n-th Catalan number* is the quantity $\frac{1}{n+1}\binom{2n}{n}$. Hence the n-th Catalan number counts the number of ways to parenthesize $x_1 x_2 \ldots x_{n+1}$ so that every pair of parentheses encloses a product involving exactly two factors.

4.2.1 Triangulations of convex n-gons

Let T_n = the number of different triangulations of a labelled convex n-gon ($n \geq 2$). Then $T_3 = 1$, and $T_4 = 2$. Define $T_2 = 1$. Consider an n-gon with vertices $v_1, v_2, \ldots, v_n$. The side $v_1 v_n$ must belong to some triangle in the triangulation. The third vertex of this triangle can be v_k, where $2 \leq k \leq n-1$. If the third vertex is v_k, then the triangle $v_n v_1 v_k$ divides the n-gon into three smaller figures: itself, a k-gon with vertices $v_1, v_2, \ldots v_k$ (note what happens if $k = 2$ or $n-1$), and a $(n-k+1)$-gon with vertices $v_k, v_{k+1}, \ldots, n_n$. The first k-gon has T_k different triangulations, the triangle $v - 1 v_n v_k$ has one, and the $(n-k+1)$-gon has T_{n-k+1}. Therefore the number of triangulations in which the third vertex is v_k equals $T_k T_{n-k+1}$. The addition principle now yields $T_n = T_2 T_{n-1} + T_3 T_{n-2} + \cdots + T_{n-1} T_2,\ n \geq 3$, where $T_2 = T_3 = 1$. We claim that $T_n = a_{n-2}$, and therefore $T_n = \frac{1}{n-1}\binom{2n-4}{n-2}$. To see this, let $t_n = T_{n+2}$, so that the recurrence becomes $t_n = t_0 t_{n-1} + t_1 t_{n-2} + \cdots + t_{n-1} t_0$, with initial conditions $t_0 = t_1 = 1$. (This is the exact same recurrence and initial conditions as for a_n, above.) The result now follows.

4.2.2 Binary trees

Recall that a *rooted tree* is an ordered pair (T, r), where T is a tree and r is with a specially designated vertex of T called the *root*. A *binary tree* is a rooted tree which is either empty, or consists of a right and left subtree, each of which is a binary tree.

For $n \geq 0$, let b_n be the number of binary trees on n vertices. It follows from the definition that $b_0 = b_1 = 1$. Direct computation shows that $b_2 = 2$. Let (T, r) be a binary tree on n vertices. The left subtree is a binary tree on some number k of vertices where $0 \leq k \leq n-1$, and the right subtree is a binary tree on $n-1-k$ vertices. Hence the number of binary trees on n vertices in which the left subtree contains k vertices is $b_k b_{n-1-k}$. By the Addition Principle $b_n = b_0 b_{n-1} + b_1 b_{n-2} + \cdots + b_{n-1} b_0$, with initial conditions $b_0 = b_1 = 1$. Therefore, $b_n = \frac{1}{n+1}\binom{2n}{n}$.

4.2.3 Arranging symbols

Suppose $2n$ people stand in line to buy a cookie (each person wants to buy just one). The cost is 50¢ and n of the people have exactly this amount. The other n have a loonie. If the cashier starts off with no change, how many arrangements of the $2n$ people will allow change to be made whenever necessary? i.e. at each point more people with 50¢ need to have been served than people with loonies. Such an arrangement corresponds to a sequence of n F's and n L's such that, counting from left to right, the number of F's is never less than the number of L's. For the sake of this discussion, let's call these *good arrangements*, and the other ones *bad arrangements*. Notice that the total number of arrangements, including both the good and bad ones, is $\binom{2n}{n}$. We will count the bad arrangements and obtain the number of good arrangements by subtraction.

Suppose we have an bad arrangement. Then there exists a first place where the number of L's is greater than the number of F's. This position is an L, and before that L there are an equal number of F's and L's, say m of each. Thus, the position in question is $2m + 1$.

We form a 1–1 correspondence between bad sequences and sequences of $n + 1$ F's and $n - 1$ L's. Take the first $2m + 1$ positions and change all F's to L's and L's to F's, so that there are now $n + 1$ F's and $n - 1$ L's. In this way, every bad sequence gives rise to a sequence of $n + 1$ F's and $n - 1$ L's. Furthermore, different bad sequences lead to different arrangements of these symbols. Since the procedure is reversible (look for the first time the number of F's is greater than the number of L's, and reverse; this situation is guaranteed to arise because the number of F's is greater than the number of L's.), we have a 1–1 correspondence. Therefore the number of bad sequences is $\binom{2n}{n-1}$. Thus, the number of good sequences is $\binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1}\binom{2n}{n}$.

The problem just discussed is related to the Ballot Problem – the election ends up tied but the first candidate never trails. The solution method above is essentially a reworking of the reflection method we used in solving the Ballot Problem. Since these arrangements are counted by the Catalan numbers, it ought to be possible to show that they satisfy (what we'll call) the Catalan recurrence. Let s_n be the number of good arrangements (as above) of n F's and n L's. Then $s_0 = s_1 = 1$, and $s_2 = 2$. Consider a good arrangement of n F's and n L's. Then the first symbol is an F, and the last symbol is an L (otherwise the arrangement can't be good because at some point the number of L's is greater than the number of F's). Therefore, there is a first position where the number of F's and L's is equal. It must be an even numbered position, say $2k$, where $1 \leq k \leq n$, and the symbol in this position must be an L. Removing the F in position 1 and the L in position $2k$ gives a good arrangement of $k - 1$ F's and $k - 1$ L's in positions 2 through $2k - 1$, and a good arrangement of $n - k$ F's and $n - k$ L's in positions $2k + 1$ through $2n$. (Note that of $k = 1$ then $k - 1 = 0$, and if $k = n$ then $n - k = 0$.) Hence the number of good arrangements in which the first time the number of F's equals the number of L's occurs in position $2k$ is $s_{k-1}s_{n-k}$. Hence, by the Addition Principle, $s_n = s_0 s_{n-1} + s_1 s_{n-2} + \cdots + s_{n-1}s_0$, which is the Catalan recurrence.

4.2.4 1–1 Correspondences

We now have four quantities counted by the n-th Catalan number:

1. the number of different parenthesizations of $x_1x_2\dots x_{n+1}$ so that every pair of parentheses encloses a product involving exactly two factors;

2. the number of triangulations of a convex $(n+2)$-gon;

3. the number of binary trees on n vertices; and

4. the number of sequences of n F's and n L's in which counting from left to right, the number of F's is never less than the number of L's.

In principle it should be possible to find a 1–1 correspondence directly between any pair of these.

As an example, we find a 1–1 correspondence between the set of parenthesizations of $x_1x_2\dots x_{n+1}$ so that every pair of parentheses enclose a product involving exactly two factors and the set of sequences of n F's and n L's in which, counting from left to right, the number of F's is never less than the number of L's.

We first take a short, but related, detour. The rules for precedence of arithmetic operations in our usual (infix) notation are sometimes remembered by the acronym BEDMAS (Brackets, Exponents, Multiply or Divide, Add or Subtract). Our "operand - operator - operand" notation (in *infix* the operator is inside of the operands) requires brackets to write $(2+3)\times 4$ instead of $2\times 4+3\times 4$, and $(2+3)^4$ instead of $2^4+4\cdot 2^3\cdot 3+6\cdot 2^2\cdot 3^2+4\cdot 2^1\cdot 3^3+3^4$, where we have used the Binomial Theorem. Evaluating an infix expression requires scanning back and forth to see which operation should be performed next according to the rules for the order of operations.

Precedence rules and brackets are not required if *postfix* notation (also known as suffix Polish notation, or reverse Polish notation) is used instead. Postfix notation involves writing expressions in the form "operand - operand - operator". The operator acts on the last two operands. For example, in postfix notation $(2+3)\times 4$ becomes $2\ 3\ +\ 4\ \times$ and $(2+3)^4$ becomes $2\ 3\ +\ 4\ \wedge$. It can be shown by induction on the number of operations that every infix expression can be expressed in postfix form.

It is possible to evaluate an expression given in postfix form without having to scan back and forth searching for the next operation according to the precedence rules (because there are no such rules). The computation is carried out by scanning the postfix expression from left to right, and forming a list of numbers according two the two rules:

1. if the next item in the expression is an operand (number), add it to the end of the list;

2. if the next item in the expression is an operator, use it combine the last two operands in the list and replace them by the result.

For example, the postfix expression $2\ 3\ 4\ +\ \times\ 5\ \times\ 6\ -$ is evaluated as follows:

ϵ	the list is initially empty
2	add 2 to the empty list
2 3	add 3 to the list
2 3 4	add 4 to the list
2 7	replace the last two items in the list by their sum
14	replace the last two items in the list by their product
14 5	add 5 to the list
70	replace the last two items in the list by their product
70 6	add 6 to the list
64	replace the last two items in the list by their difference

In this way, the postfix expression $2\ 3\ 4\times\ +\ 5\times\ \ 6\ -$ corresponds to the infix expression $(((2\times(3+4))\times 5)-6)$.

We now form a 1–1 correspondence between the set of parenthesizations of $x_1x_2\ldots x_{n+1}$ and the set of sequences of n F's and n L's in which, counting from left to right, the number of F's is never less than the number of L's. In order to explain the origin of our correspondence, we insert an intermediate step. Given a parenthesization of $x_1x_2\cdots x_{n+1}$, consider the postfix expression corresponding to evaluating the product as indicated by the brackets. This is obtained by deleting all left brackets, and replacing all right brackets by $\times$. (Since each product involves two factors, the positions of the right brackets imply the positions of the left brackets.) The first step in evaluating this expression as above is to add x_1 to the empty list. After this is done, there remain n elements x_i, $2\le i\le n+1$ and n elements $\times$, and the computation involves $2n$ further steps. At each step either the next element x_i is added to the list, or the last two elements on the list are multiplied together. We denote the first operation by F and the second one by L. Hence we have a sequence of n F's and n L's *that is obtained by deleting x_1 and each left bracket, replacing each remaining x by F, and replacing each right bracket by L.* Since multiplication only takes place when there are at least two elements on the list, the number of F's in the sequence is always at least as large as the number of L's (remember that we're recording the steps after x_1 was added to the empty list). In different parenthesizations of $x_1x_2\cdots x_{n+1}$, at some point, two different factors are multiplied together. Hence different parenthesizations of $x_1x_2\cdots x_{n+1}$ lead to different sequences. Further, the process is reversible: given a sequence of F's and L's as above, we insert x_1 at the front, replace the F's by $x_2, x_3, \cdots x_{n_1}$ in that order, replace the L's by right brackets, and finally recover the parenthesization by inserting the left brackets in the unique way that makes each bracketed term a product of two factors. Therefore the correspondence is 1–1.

We illustrate the correspondence with the five different parenthesizations of $x_1x_2\cdots x_5$.

$$\begin{array}{rcl}((x_1x_2)(x_3x_4)) & \leftrightarrow & \text{FLFFLL}\\ (((x_1x_2)x_3)x_4) & \leftrightarrow & \text{FLFLFL}\\ ((x_1(x_2x_3))x_4) & \leftrightarrow & \text{FFLLFL}\\ (x_1((x_2x_3)x_4)) & \leftrightarrow & \text{FFLFLL}\\ (x_1(x_2(x_3x_4))) & \leftrightarrow & \text{FFFLLL}\end{array}$$

4.3 Partitions

This section revisits partitions of the integer n. These are closely connected to generating functions through Lemma 4.1.1. The goals in this section are to note the generating functions for several types of partitions, to give a different proof of Euler's Theorem (Theorem 3.4.3), and to give an upper bound on the number of partitions of n that matches the lower bound from Theorem 3.4.5.

Proposition 4.3.1 *The generating function for $p(n)$ is* $\frac{1}{(1-x)(1-x^2)(1-x^3)\cdots}$

Proof. In Lemma 4.1.1, take $r_i = i$ for $i = 1, 2, \ldots$. ■

Define

- $p_k(n)$ to be the number of partitions of n into parts each less that or equal to k;
- $p_{\text{odd}}(n)$ to be the number of partitions of n into odd integers; and
- $p_{\text{dist}}(n)$ to be the number of partitions of n into distinct parts (i.e. no two parts are equal).

Proposition 4.3.2 *The ordinary power series generating function for $p_k(n)$ is* $\frac{1}{(1-x)(1-x^2)\cdots(1-x^k)}$.

Proof. Exercise. ■

Proposition 4.3.3 *The ordinary power series generating function for $p_{odd}(n)$ is* $\frac{1}{(1-x)(1-x^3)(1-x^5)\cdots}$.

Proof. Exercise. ■

Proposition 4.3.4 *The ordinary power series generating function for $p_{\text{dist}}(n)$ is* $(1+x)(1+x^2)(1+x^3)\cdots$.

Proof. Exercise. ■

We now use generating functions to give a different, much shorter, proof of Theorem 3.4.3.

Theorem 4.3.5 [Euler] *For all n: $p_{odd}(n) = p_{dist}(n)$*

Proof. We know that $1+x = \frac{1-x^2}{1-x}$, $1+x^2 = \frac{1-x^4}{1-x^2}$, $1+x^3 = \frac{1-x^6}{1-x^3}$, and in general $1+x^k = \frac{1-x^{2k}}{1-x^k}$. Therefore

$$(1+x)(1+x^2)(1+x^3)\cdots = \frac{\cancel{1-x^2}}{1-x}\cdot\frac{\cancel{1-x^4}}{\cancel{1-x^2}}\cdot\frac{\cancel{1-x^6}}{1-x^3}\cdot\frac{\cancel{1-x^8}}{\cancel{1-x^4}}\cdots .$$

The LHS is the ordinary power series generating function for $p_{dist}(n)$. After cancelling (the even powers), the RHS is the ordinary power series generating function for $p_{odd}(n)$ ■

We conclude this section by completing the proof that $p(n)$ is "about" $e^{\sqrt{n}}$.

Theorem 4.3.6 *For all $n \geq 1$, $e^{\ln(2)\sqrt{n}} \leq p(n) < e^{3\sqrt{n}}$.*

Proof. The lower bound was established in Theorem 3.4.5. We prove the upper bound by (eventually) arguing that $\ln(p(n)) < 3\sqrt{n}$.

By Proposition 4.3.1, the generating function for $p(n)$ is

$$G(x) = \frac{1}{(1-x)(1-x^2)(1-x^3)\cdots}.$$

By Exercise 10b the power series expansion of $G(x)$ converges for x small enough. Taking logs converts this infinite product into a sum:

$$\ln(G(x)) = -\ln(1-x) - \ln(1-x^2) - \ln(1-x^3) - \cdots$$

Now, the Taylor Series for the natural logarithm gives

$$-\ln(1-y) = y + \frac{y^2}{2} + \frac{y^3}{3} + \cdots ,$$

so that

$$\begin{aligned}
\ln(G(x)) &= \left(x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right) + \left(x^2 + \frac{x^4}{2} + \frac{x^6}{3} + \cdots\right) + \left(x^3 + \frac{x^6}{2} + \frac{x^9}{3} + \cdots\right) + \cdots \\
&= (x + x^2 + x^3 + \cdots) + \left(\frac{x^2}{2} + \frac{x^4}{2} + \frac{x^6}{2} + \cdots\right) + \left(\frac{x^3}{3} + \frac{x^6}{3} + \frac{x^9}{3} + \cdots\right) + \cdots \\
&= \left(\frac{x}{1-x}\right) + \frac{1}{2}\left(\frac{x^2}{1-x^2}\right) + \frac{1}{3}\left(\frac{x^3}{1-x^3}\right) + \cdots
\end{aligned}$$

We know $\frac{x^n}{1-x^n} = \frac{x}{1-x}\cdot\frac{x^{n-1}}{1+x+\cdots+x^{n-1}}$. Hence, consider the expression $\frac{x^n}{1-x^n}$ for $0 < x < 1$. Since $x^{n-1} < x^{n-2} < \cdots < x^2 < x < 1$, we have

$$x^{n-1} < \frac{x^{n-1} + x^{n-2} + \cdots + x + 1}{n}$$

or

$$\frac{x^{n-1}}{1 + x + x^2 + \cdots + x^{n-1}} < \frac{1}{n}.$$

Thus, $\frac{x^n}{1-x^n} = \frac{x}{1-x} \cdot \frac{x^{n-1}}{1+x+\cdots+x^{n-1}} < \frac{1}{n} \cdot \frac{x}{1-x}$, and so we get

$$\begin{aligned} \ln(G(x)) &< \frac{x}{1-x} + \left(\frac{1}{2}\right)^2 \left(\frac{x}{1-x}\right) + \left(\frac{1}{3}\right)^2 \left(\frac{x}{1-x}\right) + \cdots \\ &= \frac{x}{1-x}\left[1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{3}\right)^2 + \cdots\right]. \end{aligned}$$

From calculus,

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots < 1 + \int_1^\infty \frac{1}{x^2} dx = 2.$$

Therefore $\ln(G(x)) < \frac{2x}{1-x}$.

Since $p(n) > 0$ for all n, for positive values of x the function $G(x) = p(0) + p(1)x + p(2)x^2 + p(3)x^3 + \cdots$ is greater than term in the expansion, that is, $G(x) > p(n)x^n$. Since $\ln(x)$ is an increasing function, it follows that

$$\ln(p(n)) < \ln(G(x)) - n\ln(x).$$

For any number $z > 1$ we know $\ln(z) < z - 1$ so

$$-\ln(x) = \ln\frac{1}{x} < \frac{1}{x} - 1 = \frac{1-x}{x}$$

Thus,

$$\ln(p(n)) < 2\left(\frac{x}{1-x}\right) + n\left(\frac{1-x}{x}\right).$$

If we let $x = \frac{\sqrt{n}}{\sqrt{n}+1}$, we get

$$ln(p(n)) < 2\left(\frac{\sqrt{n}/\cancel{(\sqrt{n}+1)}}{1/\cancel{(\sqrt{n}+1)}}\right) + n\left(\frac{1/\cancel{(\sqrt{n}+1)}}{\sqrt{n}/\cancel{(\sqrt{n}+1)}}\right) = 3\sqrt{n}.$$

Therefore, $p(n) < e^{3\sqrt{n}}$. ■

The sum $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is known to equal $\frac{\pi^2}{6}$, so the constant 3 in the theorem can actually be replaced by $1 + \frac{\pi^2}{6} < 2.65$. Using a more detailed analysis, Hardy and Ramanujan showed that

$$p(n) \approx \frac{1}{4\sqrt{3}n} e^{\pi\sqrt{2n/3}}.$$

Rademacher gave an expansion which, when rounded to the nearest integer, is $p(n)$.

4.4 Stirling Numbers of the Second Kind

The purpose of this section is to prove one theorem. The method of proof introduces the idea of making a 1-variable generating function for a 2-parameter family of numbers by holding one of the parameters constant.

Theorem 4.4.1 *The ordinary power series generating function for the sequence* $\left\{ {n \atop n} \right\}, \left\{ {n+1 \atop n} \right\}, \left\{ {n+2 \atop n} \right\}, \cdots$ *is* $\frac{1}{(1-x)(1-2x)\cdots(1-nx)}$.

Proof. Let

$$G_n(x) = \left\{ {n \atop n} \right\} + \left\{ {n+1 \atop n} \right\} x + \left\{ {n+2 \atop n} \right\} x^2 + \cdots = \sum_{r=0}^{\infty} \left\{ {n+r \atop n} \right\} x^r$$

By Theorem 3.2.1, $\left\{ {n+r \atop n} \right\} = \left\{ {n-1+r \atop n-1} \right\} + n \left\{ {n+r-1 \atop n} \right\}$. Hence,

$$\begin{aligned} G_n(x) &= \sum_{r=0}^{\infty} \left(\left\{ {n-1+r \atop n-1} \right\} + n \left\{ {n+r-1 \atop n} \right\} \right) x^r \\ &= \sum_{r=0}^{\infty} \left\{ {n-1+r \atop n-1} \right\} x^r \quad + \quad \sum_{r=0}^{\infty} n \left\{ {n+r-1 \atop n} \right\} x^r \\ &= G_{n-1}(x) \quad + \quad nx \sum_{r=1}^{\infty} \left\{ {n+r-1 \atop n} \right\} x^{r-1} \end{aligned}$$

where the change in the lower limit of r in the last sum is because $\left\{ {n-1 \atop n} \right\} = 0$. Put $k = r - 1$, so that $r = 1$ implies $k = 0$ when $r = 1$ and $r \to \infty$ implies $k \to \infty$. Then,

$$G_n(x) = G_{n-1}(x) \quad + \quad nx \sum_{k=0}^{\infty} \left\{ {n+k \atop n} \right\} x^k = G_{n-1}(x) + nxG(x).$$

Thus $G_n(x)(1 - nx) = G_{n-1}(x)$, or

$$G_n(x) = G_{n-1}(x) \frac{1}{1 - nx}.$$

We now complete the proof by using induction. First, note that

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots = \left\{ {1 \atop 1} \right\} + \left\{ {1+1 \atop 1} \right\} x + \left\{ {1+2 \atop 1} \right\} x^2 + \cdots .$$

Therefore, if

$$G_{n-1}(x) = \frac{1}{(1-x)(1-2x)\cdots(1-(n-1)x)}$$

then by the above,

$$G_n(x) = G_{n-1}(x) \frac{1}{1-nx} = \frac{1}{(1-x)(1-2x)\cdots(1-nx)},$$

as required. The result now follows. ■

4.5 Exponential Generating Functions

The *exponential generating function* for the sequence of real numbers $a_0, a_1, a_2, \ldots$ is

$$G(x) = a_0 + a_1\frac{x^1}{1!} + a_2\frac{x^2}{2!} + a_3\frac{x^3}{3!} + \cdots$$

The exponential generating function for $1, 1, 1, 1, \ldots$ is $1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = e^x$. To get a feeling for working with exponential generating functions, let's look at

$$e^{kx} = (e^x)^k = \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots\right)^k.$$

Consider the RHS multiplied out. A term x^n arises for every collection of non-negative integers $n_1, n_2, \ldots, n_k$ such that $n_1 + n_2 + \cdots + n_k = n$. The coefficient of such a term is $\frac{1}{n_1!\, n_2! \cdots n_k!}$. The coefficient of x^n is therefore

$$\sum_{\substack{n_1 + n_2 + \cdots + n_k = n \\ n_i \geq 0}} \frac{1}{n_1!\, n_2! \cdots n_k!} \cdot \frac{n!}{n!} = \sum_{\substack{n_1 + n_2 + \cdots + n_k = n \\ n_i \geq 0}} \binom{n}{n_1\, n_2 \cdots n_k} = k^n,$$

by Corollary 2.4.2. Therefore, e^{kx} is the exponential generating function for the sequence $k^0, k^1, k^2, \ldots$.

In the argument that the coefficient of x^n in $\left(\frac{1}{1-x}\right)^k$ equals the number of non-negative integer solutions to $n_1 + n_2 + \cdots + n_k = n$, each term involving x^n that arises from the product contributes the same amount (namely one) to the coefficient of x^n in the power series. By contrast, in the above argument, each term contributes $\frac{1}{n_1!\, n_2! \cdots n_k!}$, and these numbers are not all equal.

In analogy to ordinary power series generating functions, exponential generating functions can also help us understand properties of sequences. Because the terms in the product of exponential generating functions do not all contribute the same amount to the coefficients (as illustrated above), they seem to be useful in situations involving selections or distributions that are ordered in some way. Another analogy with ordinary power series generating functions is that differentiation can be used to find terms of the sequence. If $G(x)$ is the exponential generating function for $a_0, a_1, \ldots$, then a_n equals $G^{(n)}(0)$, the n-th derivative of $G(x)$ evaluated at $x = 0$.

4.5.1 Onto Functions

Let $O(n, m)$ be the number of functions from an n-set onto an m-set. Then $O(n, k) = k! \left\{ {n \atop k} \right\}$. The following theorem can be proved by arguing along the same lines as for showing that e^{kx} is the exponential generating function for the sequence $k^0, k^1, k^2, \ldots$. The result of Section 3

Exercise 21, that

$$\sum_{\substack{n_1+n_2+\cdots+n_k=n \\ n_i \geq 1}} \binom{n}{n_1\, n_2\, \cdots\, n_k} = k! \left\{ {n \atop k} \right\},$$

is helpful near the end of the argument.

Theorem 4.5.1 *The exponential generating function for the sequence $O(0,m), O(1,m), O(2,m), \ldots$ is $(e^x-1)^m$.*

Proof. Exercise. ∎

4.5.2 Bell Numbers

Theorem 4.5.2 *The exponential generating function for the sequence of Bell numbers is $e^{(e^x-1)}$.*

Proof. Let

$$\begin{aligned}
G(x) &= \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(\left\{ {n \atop 0} \right\} + \left\{ {n \atop 1} \right\} + \cdots + \left\{ {n \atop n} \right\} \right) \frac{x^n}{n!} \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left\{ {n \atop m} \right\} \frac{x^n}{n!} \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left\{ {n \atop m} \right\} \frac{x^n}{n!} \quad \left(\text{as } \left\{ {n \atop m} \right\} = 0 \text{ if } m > n\right) \\
&= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left\{ {n \atop m} \right\} \frac{x^n}{n!} \\
&= \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{n=0}^{\infty} m! \left\{ {n \atop m} \right\} \frac{x^n}{n!} \\
&= \sum_{m=0}^{\infty} \frac{1}{m!} (e^x - 1)^m = e^{(e^x-1)}
\end{aligned}$$

(because $e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!}$). ∎

4.5.3 Derangements

Let d_n be the number of derangements of n distinguishable objects. From Exercise 19a in Section 3, we have $d_n = nd_{n-1} + (-1)^n$. This fact is helpful in proving the following.

Theorem 4.5.3 *The exponential generating function for the sequence $d_0, d_1, \ldots$ is $e^{-x}\frac{1}{1-x}$.*

Proof. Exercise. ∎

4.6 Exercises

1. Show that the ordinary power series generating function for the sequence $0, 1, 8, 27, \ldots$ is

$$\frac{x\left(x^2+4x+1\right)}{(1-x)^4}.$$

2. Let F_n be the n-th Fibonacci number.

   (a) Prove that $F_{m+n} = F_m F_{n+1} + F_{m-1} F_n$.

   (b) Prove that every n-th Fibonacci number is divisible by F_n.

   (c) Prove that $F_1 + F_2 + \cdots + F_n = F_{n+2} - 1$.

   (d) Prove that $F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}$.

   (e) Prove that $F_1^2 + F_2^2 + \cdots + F_n^2 = F_n F_{n+1}$.

3. Suppose the sequence $s_0, s_1, s_2, \ldots$ is defined by

$$s_0 = 0, s_1 = 1 \quad s_{n+1} = as_n + bs_{n-1}, \quad n \geq 1.$$

   (a) Show that the ordinary power series generating function for this sequence is

$$G(x) = \frac{x}{1-ax-bx^2.}$$

   (b) Assuming that $1 - ax - bx^2$ can be factored as $(1-r_1x)(1-r_2x)$, prove that

$$G(x) = \frac{1}{r_1-r_2}\left(\frac{1}{1-r_1x} - \frac{1}{1-r_2x}\right).$$

   (c) Prove that

$$s_n = \frac{r_1^n - r_2^n}{r_1 - r_2}.$$

4. Let e_n be the number of ways to roll n identical dice and obtain an even sum. Show that the (ordinary power series) generating function for e_n is

$$\frac{1}{2(x-1)^3}\left[\frac{1}{(x-1)^3}+\frac{1}{(x+1)^3}\right].$$

5. Let $R(n,k)$ be the number of ways of selecting a collection of k objects from a set of n distinguishable objects, where repetition is allowed but order does not matter. Show that the ordinary power series generating function for the sequence $R(n,0), R(n,1), R(n,2), \ldots$ is $G_n(x) = \left(\frac{1}{1-x}\right)^m$, and use it to show that $R(n,k) = \binom{k+n-1}{k}$.

6. Let $f(n,k)$ be the number of k-element subsets that can be selected from $\{1,2,\ldots,n\}$ and that do not contain a pair of consecutive elements.

   (a) Give a combinatorial argument to show that $f(n,k) = f(n-2,k-1) + f(n-1,k)$.

   (b) Prove that $f(n,k) = \binom{n-k+1}{k}$.

   (c) Prove that the total number of subsets of $\{1,2,\ldots,n\}$ that do not contain a pair of consecutive integers is F_{n+2}, the $(n+2)$-nd Fibonacci number.

7. Suppose that $2n$ distinguishable points are equally spaced around the circumference of a circle. Determine the number of ways that these points can be paired off using n non-intersecting straight-line chords of the circle.

8. Find a 1-1 correspondence between:

   (a) the set of binary trees on n vertices and the set of parenthesizations of $x_1x_2\ldots x_{n+1}$.

   (b) the set of triangulations of a convex $(n+2)$-gon and the set of binary trees on n vertices.

   (c) the set of sequences of n F's and n L's in which, counting from the left, the number of F's is always at least as large as the number of L's and the set of triangulations of a convex $(n+2)$-gon.

9. Let ℓ_n be the number of lattice paths from $(0,0)$ to (n,n) in the Plane Model that never cross the diagonal. Show that $\ell_n = \frac{2}{n+1}\binom{2n}{n}$.

10. Let n be a positive integer. A *composition* of n is an ordered sum that equals n. For example, $1+2$ and $2+1$ are different compositions of 3. Each term in the sum is called a *part* of the composition.

    (a) Let $c(n)$ equal the number of compositions of n. Show that $c(n) = 2^{n-1}$.

    (b) Use part (a) to show that the the ordinary power series generating function for $p(n)$,

    $$G(x) = \sum_{n=0}^{\infty} p(n)x^n$$

    converges.

(c) Let $c_2(n)$ denote the number of compositions of n into parts which are all even. Prove $c_2(2k+1) = 0$ and $c_2(2k) = 2^{k-1} = c(k)$.

11. Let k be a positive integer, and let $c_k(n)$ be the number of compositions of n in which each part is a multiple of k. (If $k = 1$ this equals $c(n)$. See the previous question for the definition of a composition.) Show that the ordinary power series generating function for $c_k(n)$ is
$$\sum_{n=1}^{\infty} c_k(n)x^n = \frac{x^k}{1-2x^k}.$$
Use this information to find a formula (not just a recurrence relation) for $c_k(n)$. (Your formula will depend on both n and k, and possibly on some relationship between n and k.)

12. Let $r \geq 0$ be an integer, and let a_r be the number of incongruent triangles with sides of integer length and perimeter r. Find the ordinary power series generating function for the sequence $a_0, a_1, a_2, \ldots$.

13. Let $P(n,k)$ be the number of k-permutations of n distinct objects, that is, the number of linear arrangements of k of the n objects.

    (a) Find a recurrence relation and initial conditions for $P(n,k)$.

    (b) Let $G_n(x) = \sum_{k=0}^{\infty} P(n,k)\frac{x^k}{k!}$. Use part (a) to help find a closed form for $G_n(x)$ and use it to determine $P(n,k)$.

14. Let a_n be the number of ways to colour the squares of a $1 \times n$ board with red and white so that at least one square is painted red and an even number of squares are painted white. Show that the exponential generating function for the sequence $a_0, a_1, \ldots$ is
$$G_n(x) = (e^x - 1)\frac{e^x + e^{-x}}{2}$$
(this can be done without finding a recurrence first) and use the RHS to find a formula for a_n.

15. Prove that

    (a) The ordinary power series generating function for $p_k(n)$ is $\frac{1}{(1-x)(1-x^2)\cdots(1-x^k)}$.

    (b) The ordinary power series generating function for $p_{odd}(n)$ is $\frac{1}{(1-x)(1-x^3)(1-x^5)\cdots}$.

    (c) The ordinary power series generating function for $p_k(n)$ is $(1+x)(1+x^2)(1-x^3)\cdots$.

16. Let E(n) and O(n) be the number of partitions of n into an even number of parts, and an odd number of parts, respectively. Let DO(n) be the number of partitions of n into distinct odd parts. Prove that
$$|\mathrm{O}(n) - \mathrm{E}(n)| = \mathrm{DO}(n)$$

17. Let's define the *parts number* of n to be the sum, over all partitions of n, of the number of distinct parts in the partition, and the *ones number* of n to be the sum, over all partitions of n, of the number of 1's in the partition. For example,

$$5 = 5 = 4 + 1 = 3 + 2 = 3 + 1 + 1 = 2 + 2 + 1 = 2 + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 1$$

so the parts number of 5 is

$$1 + 2 + 2 + 2 + 2 + 2 + 1 = 12$$

and the ones number of 5 is

$$0 + 1 + 0 + 2 + 1 + 3 + 5 = 12.$$

Prove that, for all n, the ones number of n equals the parts number of n.

18. Let $r \geq 0$ be an integer, and let a_r be the number of incongruent triangles with sides of integer length and perimeter r. Find the generating function for the sequence $a_0, a_1, a_2, \ldots$.

19. Determine the generating function for the number of partitions of n into parts from $\{a_1, a_2, \ldots, a_k\}$, where a_i is used at most $n_i \geq 0$ times, and $a_i > 0, \quad 1 \leq i \leq k$.

20. Prove Theorem 4.5.1: the exponential generating function for the sequence $O(0, m), O(1, m), O(2, m), \ldots$ is $(e^x - 1)^m$.

21. Prove Theorem 4.5.3: The exponential generating function for the sequence $d_0, d_1, \ldots$, where d_n is the number of derangements of n distinguishable objects, is $e^{-x}\frac{1}{1-x}$.

22. Let p_n be the number of sequences of n numbers 1 and n numbers -1 so that all $2n$ partial sums are non-negative. Determine p_n.

23. A clown stands on the edge of a swimming pool, holding a bag containing n red and n blue balls. He draws the balls out one at a time and throws them away. Each time he draws a red ball he takes one step back. Each time he draws a blue ball he takes one step forward. (All of his steps have the same size.) Determine the probability that the clown stays dry.

Chapter 5

PIE and Generalizations

5.1 The Principle of Inclusion and Exclusion (PIE)

The Principle of Inclusion and Exclusion is a theorem about the cardinality of a union of n finite sets $A_1, A_2, \ldots A_n$. It can be regarded as a generalization of the Addition Principle since the two theorems are identical when the sets are pairwise disjoint.

Theorem 5.1.1 [The Principle of Inclusion and Exclusion] *Let $A_1, A_2, \ldots, A_n$ be finite sets. Then*

$$\begin{aligned}
|A_1 \cup A_2 \cup \cdots \cup A_n| &= |A_1| + |A_2| + \cdots + |A_n| \\
&\quad -(|A_1 \cap A_2| + |A_1 \cap A_3| + \cdots + |A_{n-1} \cap A_n|) \\
&\quad +(|A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + \cdots + |A_{n-2} \cap A_{n-1} \cap A_n|) \\
&\quad +(-1)^{n-1}|A_1 \cap A_2 \cap \cdots \cap A_n| \\
&= \sum_{k=1}^{n} (-1)^{k-1} \sum_{\substack{I \subseteq \{1,2,\ldots,n\} \\ |I| = k}} \left| \bigcap_{i \in I} A_i \right|
\end{aligned}$$

Proof. Let x be an element of the universe. If x does not belong to any of $A_1, A_2, \ldots, A_n$, then it contributes 0 to the LHS and RHS.

Suppose x belongs to $t \geq 1$ of the sets. Then x is counted once on the LHS. We want to show it is counted once on the RHS. The element x is counted

$$\begin{array}{cl}
t & \text{times in } |A_1| + |A_2| + \cdots + |A_n| \\
-\binom{t}{2} & \text{times in } -(|A_1 \cap A_2| + |A_1 \cap A_3| + \cdots + |A_{n-1} \cap A_n|) \\
+\binom{t}{3} & \text{times in } (|A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + \cdots + |A_{n-2} \cap A_{n-1} \cap A_n|) \\
\vdots & \\
(-1)^{t-1}\binom{t}{t} & \text{times in } (-1)^{t-1}(|A_1 \cap A_2 \cap \cdots \cap A_t| + \cdots)
\end{array}$$

and never again. Therefore on the RHS x is counted $\binom{t}{1}-\binom{t}{2}+\binom{t}{3}-\cdots\pm\binom{t}{t} = -(1-1)^t+\binom{t}{0} = 1$ times. Therefore, each element is counted the same number of times on each side. This completes the proof. ■

Notice that there is no requirement in the theorem that the sets come from a finite universe. However, PIE is usually used in conjunction with the Subtraction Principle (counting the complement), and in those situations the universe must be finite as well (otherwise the answer is "infinitely many").

As a first example, we compute the number of integers in $X = \{1, 2, \ldots, 1,000,000\}$ that are perfect squares, perfect cubes, or perfect fourth powers. Let A_s, A_c, and A_f respectively denote the set of integers in X that are perfect squares, perfect cubes, and perfect fourth powers. Then the number we seek is $|A_s \cup A_c \cup A_f|$, which can be computed using PIE. Remember that a number is a perfect k-th power if and only if every exponent in its prime factorization is a multiple of k. Then, for example, a number is a cube and a fourth power if and only if every exponent in its prime factorization is a multiple of $\operatorname{lcm}(3,4) = 12$. Hence,

$$\begin{aligned}
|A_s| &= \lfloor\sqrt{1,000,000}\rfloor = 1000\\
|A_c| &= \lfloor\sqrt[3]{1,000,000}\rfloor = 100\\
|A_f| &= \lfloor\sqrt[4]{1,000,000}\rfloor = 31\\
|A_s \cap A_c| &= \lfloor\sqrt[6]{1,000,000}\rfloor = 10\\
|A_s \cap A_f| &= \lfloor\sqrt[4]{1,000,000}\rfloor = 31\\
|A_c \cap A_f| &= \lfloor\sqrt[12]{1,000,000}\rfloor = 3\\
|A_s \cap A_c \cap A_f| &= \lfloor\sqrt[12]{1,000,000}\rfloor = 3
\end{aligned}$$

Hence, by PIE, $|A_s \cup A_c \cup A_f| = 1000 + 100 + 31 - 10 - 31 - 3 + 3 = 1090$.

As a second example, we obtain the identity

$$0 = \binom{n}{0}\binom{n}{m} - \binom{n}{1}\binom{n-1}{m-1} + \binom{n}{2}\binom{n-2}{m-2} - \cdots \pm \binom{n}{m}\binom{n-m}{0}$$

by counting the number of m-subsets of $\{1, 2, \ldots, n\}$ using PIE. For $1 \leq i \leq n$, let A_i be the collection of m-subsets that contain i. Then $|A_1 \cup A_2 \cup \cdots \cup A_n|$ is the number of m-subsets of $\{1, 2, \ldots, n\}$.If $i_1 \neq i_2 \neq \cdots \neq i_k$ then $|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| = \binom{n-k}{m-k}$. Therefore

$$\begin{aligned}
\binom{n}{m} = \binom{n}{0}\binom{n}{m} &= |A_1 \cup A_2 \cup \cdots \cup A_n|\\
&= \binom{n}{1}\binom{n-1}{m-1} - \binom{n}{2}\binom{n-2}{m-2} + \cdots \pm \binom{n}{m}\binom{n-m}{0}.
\end{aligned}$$

The identity follows after rearranging.

Identities that involve terms of alternating sign can often be proved with PIE (the alternating signs suggest using PIE as a tool). The above example offers several clues as to how to proceed.

The first term, $\binom{n}{0}\binom{n}{m}$ suggested what we should count. The presence of the binomial coefficients $\binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}$ suggested that there should be n sets involved in the setup, and that the size of intersection of any k of them should be independent of which sets are chosen (there are $\binom{n}{k}$ possible intersections of k sets chosen from a collection of n different sets).

5.2 Three Classical PIE Examples

In each of the following examples, PIE is used in conjunction with counting the complement. This brings us to an important rule of thumb. When the order of quantifiers in the definition of what is to be counted is "for all, there exists", the order of quantifier is its negation is "there exists, for all". This is a clue that counting the complement may be involved. When the quantifiers are in the order "there exists, for all", it can be easy to set up sets whose union capture exactly the elements in question (and thus the set of objects that do not satisfy the original definition), by defining one set for each possible object that could satisfy the "there exists" condition.

5.2.1 Onto Functions and Stirling Numbers of the Second Kind

In this subsection we find a formula for the number of functions from $X = \{1, 2, \ldots, n\}$ onto $Y = \{1, 2, \ldots, m\}$. A function $f : X \to Y$ is *onto* if for every $y \in Y$ there exists $x \in X$ such that $f(x) = y$. Hence f is not onto if there exists y in Y such that $f(x) \neq y$ for all $x \in X$.

Here, we are counting functions, so the universe, $\mathcal{U}$ is the set of all functions from X to Y, and $|\mathcal{U}| = m^n$.

For $1 \leq i \leq m$, let A_i be the set of such functions such that $i \neq f(x)$ for any $x \in \{1, 2, \ldots, n\}$. Then $A_1 \cup A_2 \cup \cdots \cup A_m$ is the set of functions that are not onto, so we want $|\mathcal{U}| - |A_1 \cup A_2 \cup \cdots \cup A_m|$.

Since $|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| = (m-k)^n$ and there are $\binom{m}{k}$ intersections of k sets from $A_1, A_2, \ldots, A_m$,

$$|A_1 \cup A_2 \cup \cdots \cup A_m| = \binom{m}{1}(m-1)^n - \binom{m}{2}(m-2)^n + \cdots + (-1)^{m-1}\binom{m}{m}0^n.$$

So the number of onto functions is

$$\begin{aligned} &= \binom{m}{0}m^n - \binom{m}{1}(m-1)^n + \binom{m}{2}(m-2)^n + \cdots + (-1)^m\binom{m}{m}0^n \\ &= \sum_{k=0}^{m}(-1)^k\binom{m}{k}(m-k)^n \end{aligned}$$

We know that $\left\{ {n \atop m} \right\} m!$ equals the number of onto functions as above, so we also obtain a formula for the Stirling numbers of the second kind:

$$\left\{ {n \atop m} \right\} = \frac{1}{m!}\sum_{k=0}^{m}(-1)^k\binom{m}{k}(m-k)^n.$$

5.2.2 Derangements

We have obtained a formula for d_n in Chapter 3. It can be obtained using the PIE. Since the quantifiers in the definition of a derangement are "for all, there exists" (for every i there exists $j \neq i$ such that $\pi(x_i) = x_j$), we proceed by counting the complement. We are counting permutations, so the universe $\mathcal{U}$ is the set of all permutations of the n distinguishable objects in question, say $x_1, x_2, \ldots, x_n$, and $|\mathcal{U}| = n!$.

For $1 \leq i \leq n$ let A_i be the set of permutations π of $x_1 x_2 \cdots x_n$ that have $\pi(x_i) = x_i$. Then $A_1 \cup A_2 \cup \cdots \cup A_n$ is the set of permutations that are not derangements, so we want $|\mathcal{U}| - |A_1 \cup A_2 \cup \cdots \cup A_n|$. Since, if $i_1 \neq i_2 \neq \cdots \neq i_k$, we have

$$|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| = (n-k)!,$$

it follows that

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \binom{n}{1}(n-1)! - \binom{n}{2}(n-2)! + \cdots + (-1)^{n-1}\binom{n}{n}(n-n)!.$$

Therefore,

$$\begin{aligned} d_n &= n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \cdots (-1)^n \binom{n}{n}(n-n)! \\ &= \sum_{k=0}^{n} (-1)^k \binom{n}{k}(n-k)! \end{aligned}$$

Since $\binom{n}{k}(n-k)! = \frac{n!}{k!}$ we get

$$d_n = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!} = n! \sum_{k=0}^{n} (-1)^k \frac{1}{k!}.$$

The summation on the RHS is the first $n+1$ terms in the power series for e^{-1} so, as was seen before, $\frac{d_n}{n!} \approx \frac{1}{e}$.

5.2.3 Euler's ϕ Function

Recall that if n is a positive integer, then $\phi(n)$ is the number of positive integers $\leq n$ and relatively prime to n. By direct counting, $\phi(1) = 1, \phi(2) = 1, \phi(3) = 2, \phi(4) = 2, \phi(5) = 4, \phi(6) = 2, \phi(10) = 4$.

In number theory class, we obtain the formula

$$\phi(n) = n \prod_{p|n} (1 - \frac{1}{p})$$

in two steps. First, if p is prime and $k \geq 1$, then the numbers not relatively prime to p^k are $p, 2p, \ldots, p^{k-1}p$, so that $\phi(p^k) = (p^k - p^{k-1})$. Next, it is shown that ϕ is *multiplicative*, meaning that $\phi(1) = 1$ and if a and b are relatively prime, then $\phi(ab) = \phi(a)\phi(b)$. The beauty of multiplicative functions is that their value on prime powers completely determines their value on all positive integers: if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where each exponent is at least one, then apply the definition of multiplicative k times to obtain

$$\begin{aligned} \phi(n) &= \phi(p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}) \\ &= \phi(p_1^{\alpha_1})\phi(p_2^{\alpha_2}) \cdots \phi(p_k^{\alpha_k}) \\ &= (p_1^{\alpha_1} - p_1^{\alpha_1} - 1)(p_2^{\alpha_2} - (p_2^{\alpha_2 - 1}) \cdots (p_k^{\alpha_k} - (p_k^{\alpha_k - 1}) \\ &= n \prod_{p|n} (1 - \frac{1}{p}) \end{aligned}$$

We can use PIE to get a formula for $\phi(n)$. If $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ then any number ℓ not relatively prime to n is divisible by one of $p_1, p_2, \ldots, p_k$ because if $d = \gcd(\ell, n)$ then $d = p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k}$, where for $i = 1, 2, \ldots, k$ we have $0 \leq \beta_i \leq \alpha_i$.

For $1 \leq i \leq k$, let A_i be the set of integers between 1 and n that are divisible by p_i. Then $|A_1 \cup A_2 \cup \cdots \cup A_k|$ is the number of integers between 1 and n that are not relatively prime to n, so $\phi(n) = n - |A_1 \cup A_2 \cup \cdots \cup A_k|$.

If $i_1 \neq i_2 \neq \cdots \neq i_t$, then

$$|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_t}| = \frac{n}{p_{i_1} p_{i_2} \cdots p_{i_t}},$$

so that

$$\begin{aligned} |A_1 \cup A_2 \cup \cdots \cup A_k| &= \frac{n}{p_1} + \frac{n}{p_2} + \cdots + \frac{n}{p_k} \\ & -\frac{n}{p_1 p_2} - \frac{n}{p_1 p_3} - \cdots \frac{n}{p_{k-1} p_k} \\ & +\frac{n}{p_1 p_2 p_3} + \cdots + (-1)^{k-1} \frac{n}{p_1 p_2 \cdots p_k}. \end{aligned}$$

Therefore,

$$\begin{aligned} \phi(n) &= n - \frac{n}{p_1} - \frac{n}{p_2} - \cdots + \frac{n}{p_1 p_2} + \frac{n}{p_1 p_3} + \cdots + (-1)^k \frac{n}{p_1 p_2 \cdots p_k} \\ &= n \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right) \\ &= n \prod_{p|n} \left(1 - \frac{1}{p}\right). \end{aligned}$$

5.3 In Exactly m Sets or In At Least m Sets

The Principle of Inclusion and Exclusion gives a formula for the number of elements that belong to at least one member of a collection of n sets. The purpose of this section is to generalize the principle in two ways.

If $A_1, A_2, \ldots, A_n$ are finite sets, we define

$$\begin{aligned} S_k &= |A_1 \cap A_2 \cap \cdots \cap A_k| + |A_1 \cap A_2 \cap \cdots \cap A_{k-1} \cap A_{k+1}| \\ &\quad + \cdots + |A_{n-k+1} \cap A_{n-k+2} \cap \cdots \cap A_n| \\ &= \sum_{\substack{I \subseteq \{1,2,\ldots,n\} \\ |I|=k}} \left| \bigcap_{i \in I} A_i \right| \end{aligned}$$

The Principle of Inclusion and Exclusion is the statement that

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = S_1 - S_2 + S_3 - \cdots + (-1)^{n-1} S_n.$$

Theorem 5.3.1 *Let $A_1, A_2, \ldots, A_n$ be finite sets. The number of elements belonging to exactly m of these sets is*

$$N_m = S_m - \binom{m+1}{m} S_{m+1} + \binom{m+2}{m} S_{m+2} - \cdots + (-1)^{n-m} \binom{n}{m} S_n.$$

Proof. No element belonging to fewer than m sets is counted on the RHS. All elements belonging to exactly m sets are counted once in S_m and not in any other term, so they are counted once on the RHS. We must show that an element x which belongs to $r > m$ sets is counted 0 times on the RHS. Such an element x is counted:

$$\begin{array}{ll} \binom{r}{m} & \text{times in } S_m \\ \binom{r}{m+1} & \text{times in } S_{m+1} \\ \binom{r}{m+2} & \text{times in } S_{m+2} \\ \vdots & \quad \vdots \\ \binom{r}{r} & \text{times in } S_r \end{array}$$

and never again. Therefore, on the RHS, the element x is counted

$$\binom{r}{m} - \binom{m+1}{m}\binom{r}{m+1} + \binom{m+2}{m}\binom{r}{m+2} - \cdots + (-1)^{r-m}\binom{r}{m}\binom{r}{r}$$

times.

By Exercise 8 in Chapter 1 we know $\binom{k}{m}\binom{r}{k} = \binom{r}{m}\binom{r-m}{k-m}$. Substituting, we find that x is counted

$$\begin{aligned} &\binom{r}{m} - \binom{r}{m}\binom{r-m}{1} + \binom{r}{m}\binom{r-m}{2} - \cdots + (-1)^{r-m}\binom{r}{m}\binom{r-m}{r-m} \\ &= \binom{r}{m}\left[1 - \binom{r-m}{1} + \binom{r-m}{2} - \cdots + (-1)^{r-m}\binom{r-m}{r-m}\right] \\ &= \binom{r}{m}(1 + (-1))^{r-m} = 0 \end{aligned}$$

times, as required. ■

As an example of the use of Theorem 5.3.1 we count distributions of k labelled balls into n labelled boxes that leave exactly 3 empty boxes. This equals the number of functions f from a k-set to an n-set such that the range has size $n-3$, so the answer should be $\binom{n}{3}O(k,n-3)$. There is nothing special about the integer 3 in anything that follows.

For $i=1,2,\ldots,n$ let A_i be the set of distributions in which box i is empty. Then (as before) if $i_1 \neq i_2 \neq \cdots \neq i_t$, then $|A_{i_1} \cap A_{i_2} \cap \cdots A_{i_t}| = (n-t)^k$. Therefore, $S_t = \binom{n}{t}(n-t)^k$ and the number of distributions is

$$\begin{aligned} N_3 &= S_3 - \binom{4}{3}S_4 + \binom{5}{3}S_5 - \cdots + (-1)^{n-3}\binom{n}{3}S_n \\ &= \sum_{i=3}^{n}(-1)^{i+1}\binom{i}{3}\binom{n}{i}(n-i)^k. \end{aligned}$$

With the substitution $\binom{i}{3}\binom{n}{i} = \binom{n}{3}\binom{n-3}{i-3}$ (as before), we obtain

$$\begin{aligned} N_3 &= \sum_{i=3}^{n}(-1)^{i+1}\binom{i}{3}\binom{n}{i}(n-i)^k \\ &= \binom{n}{3}\sum_{i=3}^{n}(-1)^{i+1}\binom{n-3}{i-3}(n-i)^k \\ &= \binom{n}{3}\sum_{j=0}^{n-3}(-1)^{j}\binom{n-3}{j}((n-3)-j)^k \\ &= \binom{n}{3}O(k,n-3) \end{aligned}$$

In order to obtain a formula for N_m^*, the number of elements belonging to at least m of the finite sets $A_1, A_2, \ldots, A_n$, we note that "in at least m" means "in exactly m" or "in at least $m+1$". Hence, $N_m^* = N_m + N_{m+1}^*$. Equivalently, $N_{m+1}^* = N_m^* - N_m$.

Theorem 5.3.2 *Let $A_1, A_2, \ldots, A_n$ be finite sets. The number of elements belonging to at least m of these sets is $N_m^* = S_m - \binom{m}{m-1}S_{m+1} + \binom{m+1}{m-1}S_{m+2} - \cdots + (-1)^{n-m}\binom{n-1}{m-1}S_n$.*

Proof. The proof is by induction on m. When $m=1$ the statement is

$$\begin{aligned} N_1^* &= S_1 - \binom{1}{0}S_2 + \binom{2}{0}S_3 - \cdots + (-1)^{n-1}\binom{n-1}{0}S_n \\ &= S_1 - S_2 + S_3 - \cdots + (-1)^{n-1}S_n, \end{aligned}$$

which is the Principle of Inclusion and Exclusion.

Assume, for some $m \geq 1$, that

$$N_m^* = S_m - \binom{m}{m-1} S_{m+1} + \binom{m+1}{m-1} S_{m+2} - \cdots (-1)^{n-m} \binom{n-1}{m-1} S_n.$$

Then

$$\begin{aligned} N_{m+1}^* &= N_m^* - N_m \\ &= S_m - \binom{m}{m-1} S_{m+1} + \binom{m+1}{m-1} S_{m+2} - \cdots + (-1)^{n-m} \binom{n-1}{m-1} S_n \\ &\quad -S_m + \binom{m+1}{m} S_{m+1} - \binom{m+2}{m} S_{m+2} + \cdots + (-1)^{n-m+1} \binom{n}{m} S_n \\ &= \left[\binom{m+1}{m} - \binom{m}{m-1} \right] S_{m+1} - \left[\binom{m+2}{m} - \binom{m+1}{m-1} \right] S_{m+2} + \cdots + \\ &\quad (-1)^{n-m+1} \left[\binom{n}{m} - \binom{n-1}{m-1} \right] S_n \\ &= \binom{m}{m} S_{m+1} - \binom{m+1}{m} S_{m+2} - \binom{m+2}{m} S_{m+3} + \cdots (-1)^{n-m+1} \binom{n-1}{m} S_n, \end{aligned}$$

where we have used Pascal's identity in to obtain the last equality. The result follows by induction. ■

We continue the same example as above and count the number of distributions in which at least three of the boxes are empty. There is an impulse to say that the answer is $\binom{n}{3}(n-3)^k$, but this overcounts distributions in which more than three boxes are empty. One answer is

$$\sum_{i=3}^{n} \binom{n}{i} O(k, n-i).$$

We could substitute the formula for $O(k, n-i)$ obtained in Subsection 5.2.1 and obtain an expression involving a double summation. Instead, we compute N_3^*. Substituting $S_t = \binom{n}{t}(n-t)^k$ into the expression for N_3^* in Theorem 5.3.2 gives

$$N_3^* = \sum_{i=3}^{n} (-1)^{i+1} \binom{i-1}{2} \binom{n}{i} (n-i)^k.$$

5.4 Rook Polynomials (Arrangements with Forbidden Positions)

The"rook" in the section title is the chess piece. Recall that rook on an empty chessboard can be moved any number of contiguous squares horizontally or vertically, but not in any other direction. It does not matter whether the chessboard is square, only that it is constructed from squares that either share an edge, or do not meet at all. We consider two squares to meet only when they share an edge, so squares that touch at a corner are considered to not meet. Two rooks are called *non-attacking* if they are in different rows and different columns of the chessboard, or if they belong to different pieces of a disconnected chessboard. The idea of non-attacking rooks on chessboards that may not be square is useful in considering problems involving counting permutations with restrictions on where some elements can be placed (some elements are forbidden).

Consider the problem of counting the permutations of a, b, c, d, e such that

a is not in position 1 or 5
b is not in position 2 or 3
c is not in position 3 or 4
e is not in position 5

and d is not restricted.

This can be represented pictorially as shown in Figure 5.1. where the darkened squares

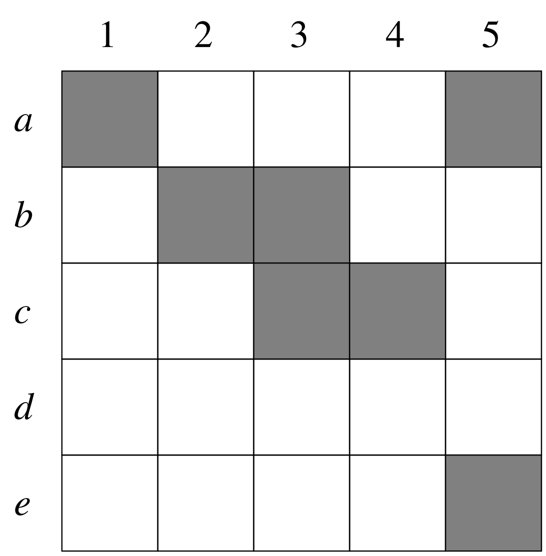


Figure 5.1: The board of forbidden positions

represent the forbidden positions. For $x \in \{a, b, c, e\}$, define A_x to be the set of permutations where x is in a forbidden position. Then the number we want is $5! - |A_a \cup A_b \cup A_c \cup A_e|$. The size of the union can be computed with PIE.

Now, $|A_a| = 2\times 4! = |A_b| = |A_c|$ and $|A_e| = 1\times 4!$ and therefore $S_1 = |A_a|+|A_b|+|A_c|+|A_e| = 7 \times 4!$, since there are 7 ways to select a forbidden square.

Each term $|A_x \cap A_y| = 3!\times$ the number of ways to put x and y in "compatible" forbidden positions. Terms $|A_x \cap A_y \cap A_z|$ are computed similarly. Notice that the number of ways to put elements x, y, and z in "compatible" forbidden positions equals the number of ways to place three non-attacking rooks on the chessboard consisting of the darkened squares in the corresponding rows. (In what follows a "board" means a board consisting of darkened squares.)

If k rooks are non-attacking then they remain so when rows and columns are permuted. In our example, we can permute rows and columns so that the entire array (and board of forbidden positions) becomes the one shown in Figure 5.2. After this has been done, it becomes apparent

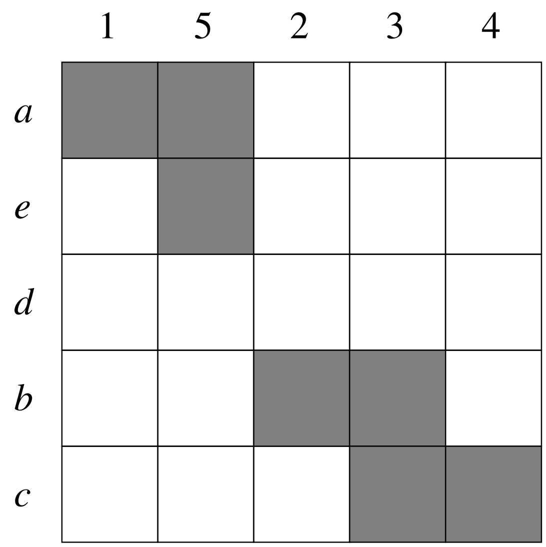


Figure 5.2: The rearranged board of forbidden positions

that our board B of forbidden positions can be decomposed into disjoint (involving different rows and columns) sub-boards. Let B_1 be the darkened squares in rows a and e and columns 1 and 5, and let B_2 be the darkened squares in rows b and c and rows 2, 3, and 4.

Let $r_k(B)$ be the number of ways to place k non-attacking rooks on the board B (of darkened squares).

Theorem 5.4.1 *The number of ways to arrange n distinct objects when there are forbidden positions is*

$$n! - r_1(B)(n-1)! + r_2(B)(n-2)! - \cdots (-1)^n r_n(B)0!$$

where B is the board of forbidden positions.

Proof. Use the Principle of Inclusion and Exclusion. ■

This theorem makes it of interest to compute the numbers $r_k(B)$. Our goal in the remainder of this section is to show how to do that. Ultimately, the method developed involves generating functions. The RHS of the formula in the lemma below is suggestive of a product of power series may be involved at some point. In turn, that is a hint that generating functions may be useful at some point

Lemma 5.4.2 *If B is a board of darkened squares that decomposes into disjoint sub-boards B_1 and B_2, then*

$$r_k(B) = r_0(B_1)r_k(B_2) + r_1(B_1)r_{k-1}(B_2) + \cdots + r_k(B_1)r_0(B_2).$$

Proof. There must be t rooks on B_1 and $k-t$ on B_2 for some t, $0 \le t \le k$. ■

The values $r_k(B)$ can be computed directly, or using Lemma 5.4.2 if B decomposes. Instead of stopping here, we describe an equivalent (and easier) method using generating functions.

The *rook polynomial* $R(x, B)$ of the board B of darkened squares is

$$R(x, B) = r_0(B) + r_1(B)x + r_2(B)x^2 + \cdots$$

Notice that, in the Lemma, the formula for $r_k(B)$ is exactly the formula for the coefficient of x^k in the product $R(x, B_1)R(x, B_2)$.

Theorem 5.4.3 *If B is a board of darkened squares that decomposes into disjoint sub-boards B_1 and B_2, then $R(x, B) = R(x, B_1)R(x, B_2)$.*

Proof. Exercise. ■

Looking back at our example problem, we see that

$$R(x, B_1) = 1 + 3x + x^2 \quad \text{and}$$
$$R(x, B_2) = 1 + 4x + 3x^2$$

Therefore,

$$\begin{aligned} R(x, B) &= (1 + 3x + x^2)(1 + 4x + 3x^2) \\ &= 1 + 7x + 16x^2 + 13x^3 + 3x^4 \end{aligned}$$

and the number of permutations is $5! - 7 \cdot 4! + 16 \cdot 3! - 13 \cdot 2! + 3 \cdot 1! - 0 \cdot 0! = 25$.

We now summarize the theory that has been developed so far. Given a problem involving arrangements with forbidden positions,

- make an array where the forbidden positions are displayed as darkened squares,
- try to rearrange the rows/columns so that the board of darkened squares can be decomposed into disjoint sub-boards B_1 and B_2,
- determine the $R(x, B_i)$'s and use them to find $R(x, B)$, and

- use the coefficients of $R(x, B)$ in the theorem to solve the problem.

As an example of this process, we compute the number of ways to send six birthday cards $C_1, C_2, \ldots, C_6$ to six people $P_1, P_2, \ldots, P_6$ if:

P_1	would not like C_2 or C_4
P_2	would not like C_1 or C_5
P_3	would like any of the cards
P_4	would not like C_1 or C_5
P_5	would not like C_4
P_6	would not like C_6

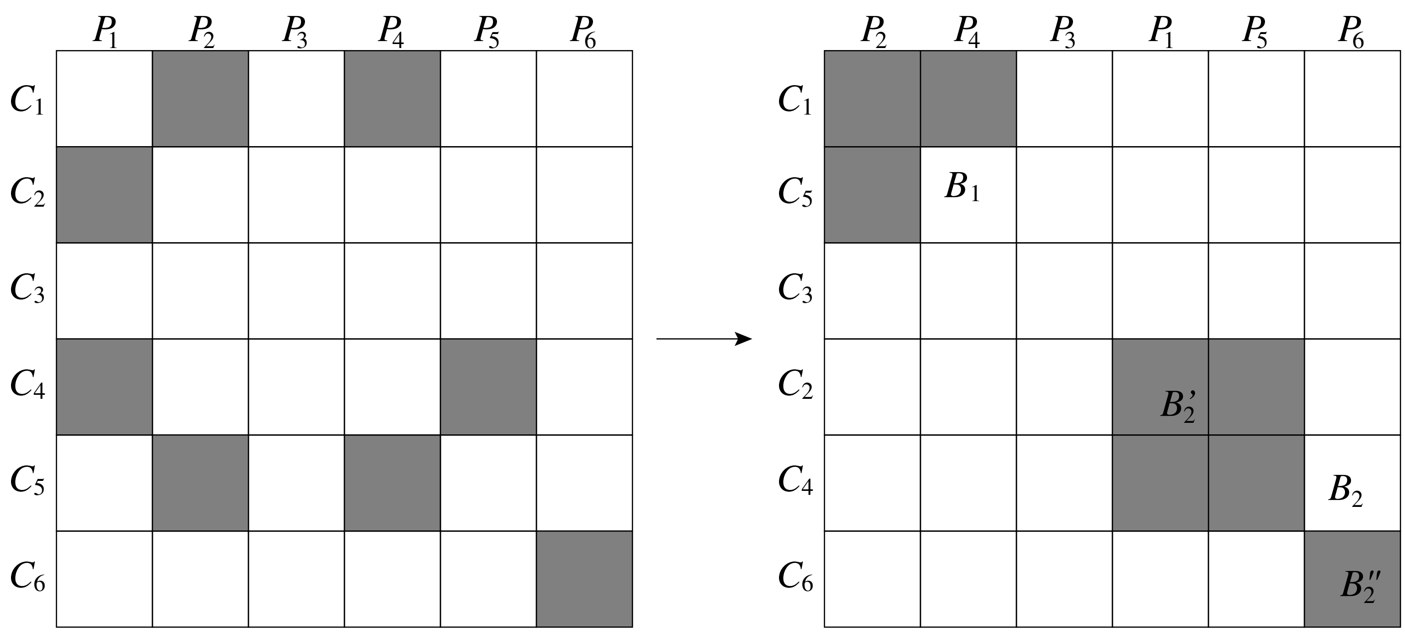


Figure 5.3: Forbidden configurations in the cards example

By permuting rows and columns, the board decomposes into B_1 and B_2 as shown. Notice that B_2 also decomposes into B_2' and B_2''. We can directly compute $R(x, B_1) = 1 + 3x + x^2$.

Since B_2 also decomposes,

$$\begin{aligned} R(x, B_2') &= 1 + 4x + 2x^2 \\ R(x, B_2'') &= 1 + x \end{aligned}$$

Therefore,

$$\begin{aligned} R(x, B_2) &= (1 + x)(1 + 4x + 2x^2) \\ &= 1 + 4x + 2x^2 + x + 4x^2 + 2x^3 \\ &= 1 + 5x + 6x^2 + 2x^3 \end{aligned}$$

Hence, by Theorem 5.4.3,

$$\begin{aligned} R(x,B) &= R(x,B_1)R(x,B_2) \\ &= (1+3x+x^2)(1+5x+6x^2+2x^3) \\ &= 1+8x+22x^2+25x^3+12x^4+2x^5. \end{aligned}$$

The number of ways to send the cards is therefore

$$6! - 8\cdot 5! + 22\cdot 4! - 25\cdot 3! + 12\cdot 2! - 2\cdot 1! + 0\cdot 0! = 160.$$

Up to this point, it has always been possible to rearrange the rows and columns of the board of forbidden positions so that it decomposes. It remains to figure out what to do if it does not decompose. Our approach is to develop a recursion based on the two possible scenarios that could arise for any fixed square S: either there is a rook on S or there is no rook on S.

Let B be a board of darkened squares and S any one of the squares on B. Let B_S be the board obtained from B by deleting S and B_S^* the board obtained by deleting S and all squares in the same row and/or column as S.

Lemma 5.4.4 $r_k(B) = r_k(B_S) + r_{k-1}(B_S^*)$.

Proof. If the square S is not used, then the k rooks are placed on B_S. On the other hand, if the square S is used then there is no other rook in its row or column, so $k-1$ rooks are placed on B_S^*. ∎

It is important to notice that the board B_S^* decomposes. By applying the lemma repeatedly, if necessary, $r_k(B)$ can be computed.

Theorem 5.4.5 *Let B be a board of darkened squares and let S be one of the squares. Then*

$$R(x,B) = R(x,B_S) + xR(x,B_S^*)$$

Proof.

$$\begin{aligned} R(x,B) = \sum_{k=0}^{\infty} r_k(B)x^k &= \cancelto{1}{r_0(B)} + \sum_{k=1}^{\infty}\left[r_k(B_S) + r_{k-1}(B_S^*)\right]x^k \\ &= \sum_{k=0}^{\infty} r_k(B_S)x^k + x\sum_{k=1}^{\infty} r_{k-1}(B_S^*)x^{k-1} \\ &= R(x,B_S) + xR(x,B_S^*). \end{aligned}$$

∎

Suppose in the previous example, P_4 would now be ok with card C_1 but not C_2.

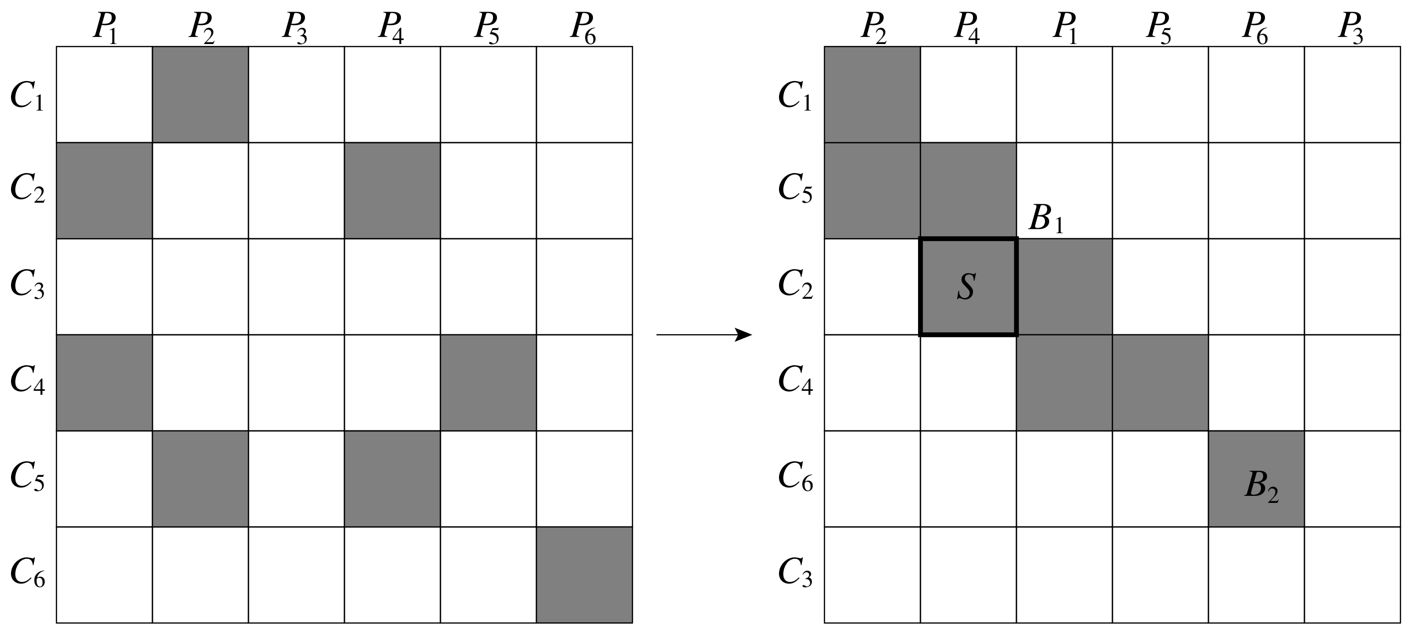


Figure 5.4: Forbidden configurations in the second cards problem

After permuting rows and columns as indicated, our board decomposes into B_1 and B_2 as shown in Figure 5.4. It is easy to see that $R(x, B) = R(x, B_1)(1 + x)$. We need to use Theorem 5.4.5 to compute $R(x, B_1)$. A reasonable goal is to choose S to break up B_1 as much as possible. An astute choice is $S = (C_2, P_4)$. Then, both $(B_1)S$ and $(B_1)^*_S$ decompose. We have $R(x, (B_1)_S) = (1 + 3x + x^2)^2$ and $R(x, (B_1)^*_S) = (1 + 2x)^2$.

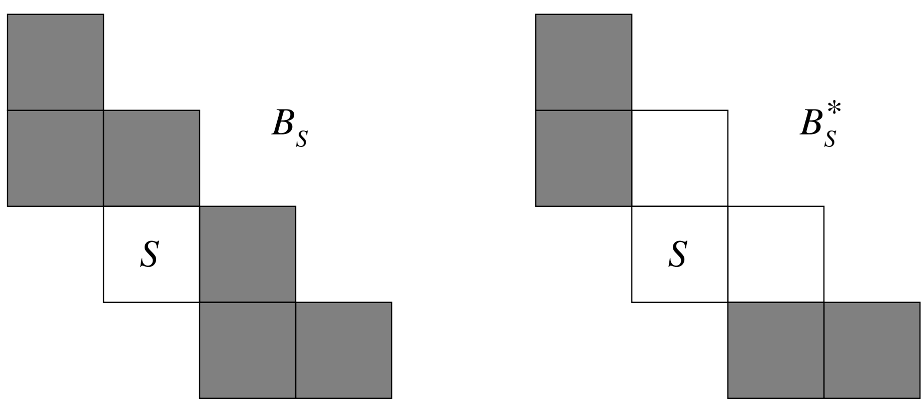


Figure 5.5: Decomposing B_1 after carefully choosing S

Therefore

$$
\begin{aligned}
R(x, B_1) &= (1 + 3x + x^2)^2 + x((1 + 2x)^2 \\
&= 1 + 7x + 15x^2 + 10x^3 + x^4
\end{aligned}
$$

Consequently,

$$\begin{aligned} R(x,B) &= (1+7x+15x^2+10x^3+x^4)(1+x) \\ &= 1+8x+22x^2+25x^3+11x^4+x^5 \end{aligned}$$

Therefore, the number of ways to send the cards is

$$6! - 8\cdot 5! + 22\cdot 4! - 25\cdot 3! + 11\cdot 2! - 1\cdot 1! + 0 = 159.$$

5.5 Exercises

1. Determine the number n term sequences of 0's and 1's that contain neither the subsequence 010 nor the subsequence 101.

2. Give a combinatorial proof that $n! = \binom{n}{0}d_n + \binom{n}{n-1}d_{n-1} + \cdots + \binom{n}{n}d_0$.

3. How many integers n between 0 and $1,000,000$ are such that the sum of the digits of n is at most 46? Is it possible obtain an expression in the case that 46 is replaces by an arbitrary positive integer t?

4. Use the Principle of Inclusion and Exclusion to prove that if $m \le r \le n$,
$$\binom{n-m}{n-r} = \sum_{i=0}^{n}(-1)^i\binom{m}{i}\binom{n-i}{r}.$$

5. Suppose there are n people who all have the same show size. Suppose further that they toss their $2n$ shoes into a big pile, and then every person selects a left shoe and a right shoe at random. Determine the number outcomes in which:
   (a) No person selects both of his shoes.
   (b) No person has a matching pair of shoes.
   (c) At least two people select their own left shoe, and exactly one person selects his own right shoe.

6. Prove Theorem 5.4.1 and Theorem 5.4.3.

7. In how many ways can the letters in CORRESPONDENTS be arranged so that:
   (a) There are exactly three pairs of consecutive identical letters.
   (b) There are at least three pairs of consecutive identical letters.

8. Suppose that $A_1, A_2, \ldots, A_n$ are subsets of a finite universe $mathcalU$. Prove that $N_0 + N_1 + \cdots + N_n = |\mathcal{U}|$.

9. Prove that
$$\binom{n}{m} = \sum_{k=0}^{m} \binom{k}{m} nks^{n-k}.$$
(Hint: use the results in Section 5.3.)

10. Show that the number of permutations of $1, 2, \ldots, n$ in which none of the patterns $12, 23, \ldots, (n-1)n$ occurs is $d_{n-1} + d_n$.

11. There are n people sitting at a circular table. Prove that the number of rearrangements of these people such that no person has the same person on her right as before equals
$$d_{n-1} - d_{n-2} + d_{n-3} - \cdots \pm d_1.$$

12. How many arrangements of $a, a, a, b, b, b, c, c, c$ have no consecutive identical letters? (*Caution*: this is tricky.)

13. Give a combinatorial proof that
$$0 = \binom{n}{0}\binom{n}{m} - \binom{n}{1}\binom{n-1}{m-1} + \cdots + (-1)^m \binom{n}{m}\binom{n-m}{0}.$$

14. When n balls, numbered $1, 2, \ldots, n$ are taken in succession from a container, a *rencontre* occurs if the mth ball withdrawn is numbered m. Find the probability of getting (i) no rencontres; (ii) exactly one rencontre; (iii) at least one rencontre; (iv) r rencontres, where $1 \le r \le n$. Also, approximate the answers to the preceeding questions.

15. Find the number of arrangements of the digits 112233 with the property that no digit appears in one of its original positions.

16. (a) Let B_n be an $n \times n$ chessboard, in which all squares are "darkened". For each integer $k \ge 0$, find $r_k(B_n)$, and hence find $R(x, B_n)$.
    (b) Find the number of arrangements of 111222333 in which no digit appears in one of its original positions.
    (c) Find, if possible, an expression for the number of arrangements of $112233\cdots kk$ in which no digit appears in one of its original positions. (The Multinomial Theorem may be useful.)

17. Find two different chessboards with the same rook polynomial, and the property that neither can be obtained from the other by row and column permutations.

18. Suppose five officials O_1, O_2, O_3, O_4, O_5 are to be assigned five different cars, and Escort, a Lexus, a Nissan, a Taurus, and a Volvo. Official O_1 will not drive an Escort or a Nissan, O_2 will not drive a Taurus, O_3 will not drive a Lexus or a Volvo, O_4 will not drive a Lexus, and O_5 will not drive an Escort or a Nissan. If a feasible arrangement of cars is chosen at random, what is the probability that:
    (a) O_1 gets the Volvo?

(b) O_2 or O_5 gets the Volvo? (Hint: model this constraint with an altered board.)

19. Use the method of rook polynomials to compute d_n, the number of derangements of n different objects.

20. Seven dwarfs $D_1, D_2, D_3, D_4, D_5, D_6, D_7$ each must be assigned to one of seven jobs J_1, J_2, J_3, J_4, J_5, J_6, J_7 in a mine. D_1 cannot do J_2 or J_3; D_2 cannot do J_1 or J_5; D_4 cannot do J_3 or J_6; D_5 cannot do J_2 or J_7; D_7 cannot do J_4. D_3 and D_6 can do all of the jobs. How many ways are there to assign the dwarfs to different jobs?

21. Consider all permutations of $1, 2, \ldots, n$ in which i appears in neither position i nor $i+1$ (n not in n or 1). Such a permutation is called a *ménage*. (**Note:** A ménage corresponds to a circular seating of n married couples in which the sexes alternate and no married couple sits together. Such an arrangement is possible only if $n \geq 3$.) Let $F_n(x)$ be the rook polynomial for the board of forbidden positions in a ménage. Let $F_n^*(x)$ be the rook polynomial when n may appear in position 1, and let $F_n^0(x)$ be the rook polynomial when both 1 and n may appear in position 1.

   (a) Show that $F_n^*(x) = xF_{n-1}^*(x) + F_n^0(x)$, $F_n^0(x) = xF_{n-1}^0(x) + F_{n-1}^*(x)$, and $F_n(x) = xF_{n-1}^*(x) + F_n^*(x)$.

   (b) Using the initial conditions that $F_1^*(x) = 1 + x$, $\quad F_1^0(x) = 1$, show that

$$F_n^*(x) = \sum_{k=0}^{n} \binom{2n-k}{k} x^k \quad \text{and} \quad F_n^0(x) = \sum_{k=0}^{n-1} \binom{2n-k-1}{k} x^k.$$

   (c) Find $F_n(x)$, and M_n, the number of ménages (on n objects).

   (d) Show that the ménage numbers satisfy the following recursion formula.

$$(n-2)M_n = n(n-2)M_{n-1} + nM_{n-2} \pm 4$$

   When is the $+$ used, and when is the $-$ used?

Chapter 6

Ramsey Theory

6.1 The Pigeonhole Principle

In its most basic version, the Pigeonhole Principle (also known as the Dirichlet Drawer Principle) is the assertion that if more than n pigeons live in n pigeonholes, then some pigeonhole has at least two pigeons living in it. The "proof" is to write down the contrapositive: if there is at most one pigeon living in each pigeonhole, then the number of pigeons is at most n, the number of pigeonholes. The same type of argument can be used to prove a variety of more general versions of the Pigeonhole Principle, as well as other assertions like "*If $x_1, x_2, \ldots, x_n$ are real numbers, then there exists i such that $x_i \leq (x_1 + x_2 + \cdots + x_n)/n$*", this is, one number in any collection is at most the average.

We will adopt as our version of the Pigeonhole Principle (PhP) the statement that if the number of pigeons is more than t times the number of pigeonholes, then some pigeonhole has at least $t + 1$ pigeons living in it.

Theorem 6.1.1 [The Pigeonhole Principle] *If k pigeons occupy n pigeonholes then some pigeonhole houses at least $1 + \left\lfloor \frac{k-1}{n} \right\rfloor$ pigeons.*

Proof. Otherwise, each pigeonhole houses at most $\left\lfloor \frac{k-1}{n} \right\rfloor$ pigeons. The total number of pigeons is therefore at most

$$n \left\lfloor \frac{k-1}{n} \right\rfloor \leq n \frac{(k-1)}{n} = k - 1,$$

a contradiction. ■

A similar argument can be used to prove the following statement, which we will see later is a special case of Ramsey's Theorem.

Proposition 6.1.2 *Let $p_1, p_2, \ldots, p_n$ be positive integers. If at least $p_1 + p_2 + \cdots + p_n - n + 1$ objects are placed into n labelled boxes, then there is an integer i such that box i contains at least p_i objects.*

Proof. Exercise. ■

The Pigeonhole Principle can be used to prove a variety of surprising results. Many of the elementary ones are covered in lower level classes. Usually the key' is figuring out how the Principle is applied. The main idea is that there is some collection of objects and you'd like to show that some number of them are related in some way. These objects are the pigeons. The pigeonholes are something that is defined so that when objects lie in the same pigeonhole, then they are related in the desired way. Hopefully there are proportionally fewer pigeonholes than there are pigeons, so that sufficiently many pigeons are required to be in some pigeonhole.

A slightly different way to apply the Pigeonhole Principle is to imagine that the pigeons already live in the pigeonholes in some well-defined way, and then assert that if "enough" pigeons are selected, then the desired number of them are chosen from the same pigeonhole.

For example, suppose we want to show that if $n+1$ different numbers are selected from $\{1, 2, \ldots, 2n-1\}$, then some two of these numbers sum to $2n$, and this result is best possible in the sense that it is possible to select n numbers so that no two of them sum to $2n$. The pigeons are the $2n-1$ numbers, and the pigeonholes are the n sets $\{1, 2n-1\}, \{2, 2n-2\}, \ldots, \{n-1, n+1\}, \{n\}$. These are the pairs of numbers that sum to $2n$, plus the number n which does not belong to such a pair. Since there are n sets in this collection and $n+1$ numbers are selected, the Pigeonhole Principle guarantees that two of them are selected from the same subset. By definition of the subsets, these two numbers sum to $2n$. To see that $n+1$ is the smallest possible value for which the statement is true, consider the selection consisting of the n numbers $1, 2, \ldots, n$.

Sometimes it is necessary to some work before applying the Principle. As an example, suppose that over a 44 day period, Gary trains for triathlons at least once per day, and a total of 70 times in all. We claim that that there is a period of consecutive days during which he trains exactly 17 times. There is a solution that uses the Pigeonhole Principle, but at this point it is not clear what the pigeons and pigeonholes should be. For $i = 1, 2, \ldots, 44$, let x_i be the number of times Gary trains up to the end of day i. Then $1 \leq x_1 < x_2 < x_3 < \cdots < x_{44} = 70$. We need to find subscripts i and j such that $x_i + 17 = x_j$. This implies that Gary trains exactly 17 times in the period of days $i+1, i+2, \ldots, j$. Therefore, we want one of $x_1, x_2, \ldots, x_{44}$ to be equal to one of $x_1 + 17, x_2 + 17, \ldots, x_{44} + 17$. Using the inequality for the x_i s it follows that

$$18 \leq x_1 + 17 < x_2 + 17 < \cdots < x_{44} + 17 = 87.$$

Thus, the 88 numbers $x_1, x_2, \ldots, x_{44}, x_1+17, x_2+17, \ldots, x_{44}+17$ can take on at most 87 different values. The 88 numbers are the pigeons and the 87 possible values are the pigeonholes. By the Pigeonhole Principle, some two of them must be equal. The inequalities imply that one of $x_1, x_2, \ldots, x_{44}$ must equal one of $x_1 + 17, x_2 + 17, \ldots, x_{44} + 17$, which is what we wanted.

Recall that sequence $x_1, x_2, \ldots, x_k$ or real numbers is *monotone increasing* if $x_1 \leq x_2 \leq \cdots \leq x_k$, and *monotone decreasing* if $x_1 \geq x_2 \geq \cdots \geq x_k$. The adjective *strictly* is added in front of "monotone" if the inequalities are required to be strict.

Proposition 6.1.3 [Erdös and Szekeres, 1935] *In any sequence of $mn+1$ numbers there is a monotone increasing subsequence of length $m+1$ or a monotone decreasing subsequence of length $n+1$.*

Proof. Let the sequence be $x_1, x_2, \ldots, x_{mn+1}$. For $i = 1, 2, \ldots, mn+1$ let l_i be the length of the longest monotone increasing subsequence that starts at x_i. If some $l_i \geq m+1$, the proof is complete. Suppose then that $l_i \leq m$ for all i. By the pigeonhole principle, some $n+1$ of the l_i's are equal, say $l_{i_1} = l_{i_2} = \cdots = l_{i_{n+1}}$ where $i_1 < i_2 < \cdots < i_{n+1}$. We claim that $x_{i_1}, x_{i_2}, \ldots, x_{i_{n+1}}$ is a monotone decreasing sequence of length $n+1$. If $x_{i_r} \geq x_{i_s}$ then $l_{i_r} \geq l_{i_s} + 1$, a contradiction. This completes the proof. ■

For example, let $m = n = 3$ and look at the sequence: $1, 7, \underline{2}, 4, 6, \underline{3}, 2, \underline{4}, 3, 1$. The underlined entries in this sequence of $3 \times 3 + 1 = 10$ terms indicate an increasing subsequence of length three.

It turns out that the quantity $mn+1$ is best possible in the sense that there exist sequences of mn numbers that have neither a monotone increasing subsequence of length $m+1$ or a monotone decreasing subsequence of length $n+1$. Such a sequence of length nine is $1, 5, 9, 2, 6, 7, 3, 4, 8$.

Proposition 6.1.4 *Let $\alpha > 0$ be an irrational number. There are infinitely many different rational numbers $\frac{p}{q}$ such that*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}.$$

Proof. Write $\{x\}$ for $x - \lfloor x \rfloor$, the fractional part of x. We claim that for any natural number n there is a rational number $\frac{p}{q}$ with $q \leq n$ such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{nq} < \frac{1}{q^2}.$$

Consider the $n+1$ numbers $\{i\alpha\}, i = 1, 2, \ldots, n+1$. Place these numbers into the n pigeonholes $\left(\frac{j}{n}, \frac{j+1}{n}\right)$, for $j = 0, 1, \ldots, n-1$. Since α is irrational, none of the numbers $\{i\alpha\}$ is an endpoint of such an interval. By the PhP some interval contains two of the numbers, say $\{i_1\alpha\}$ and $\{i_2\alpha\}$, which therefore differ by less than $\frac{1}{n}$. WLOG $i_1 > i_2$. Now,

$$\begin{aligned}(i_1 - i_2)\alpha &= \lfloor i_1\alpha \rfloor + \{i_1\alpha\} - \lfloor i_2\alpha \rfloor + \{i_2\alpha\} \\ &= \{i_1\alpha\} - \{i_2\alpha\} + (\lfloor i_1\alpha \rfloor - \lfloor i_2\alpha \rfloor)\end{aligned}$$

Therefore,

$$(i_1 - i_2)\alpha - (\lfloor i_1\alpha \rfloor - \lfloor i_2\alpha \rfloor) = \{i_1\alpha\} - \{i_2\alpha\} < \frac{1}{n}.$$

Let $q = i_1 - i_2$ and $p = \lfloor i_1\alpha \rfloor - \lfloor i_2\alpha \rfloor$. Then $0 < q \leq n$ and $|q\alpha - p| < \frac{1}{n}$, so that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{nq}\frac{1}{q^2}.$$

Since α is irrational, $\alpha \neq \frac{p}{q}$. Therefore, there exists n_1 with $\left|\alpha - \frac{p}{q}\right| > \frac{1}{n_1}$. Repeating the argument with n_1 in place of n gives another solution $\frac{p_1}{q_1}$, and $\frac{p_1}{q_1} \neq \frac{p}{q}$ because $\left|\alpha - \frac{p_1}{q_1}\right| < \frac{1}{nq_1} \leq \frac{1}{n} < \left|\alpha - \frac{p}{q}\right|$. Continuing this process yields infinitely many such rational approximations. ■

6.2 Two-Colour Graph Ramsey Numbers

The following familiar proposition will help get us started. The proof illustrates a technique that we will use repeatedly. In the statement we are assuming that the relations "is a friend of" and"is a stranger to" are symmetric. That is, if I am a friend of yours, then you are a friend of mine, and if I am a stranger to you, then you are a stranger to me.

Proposition 6.2.1 *In any group of six or more people there is either a group of three mutual friends or three mutual strangers.*

Proof. Let $P_1, P_2, \ldots, P_6$ be six of the people. Consider P_1. By the PhP either three of the remaining people are friends of P_1, or strangers to P_1.

Suppose three of the remaining people are friends of P_1. If any two of these people are friends, then there is a group of three mutual friends: these two and P_1. Otherwise, the three are a group of three mutual strangers.

The case where three of the remaining people are strangers to P_1 is similar. ■

The number six in the proposition is best possible. The configuration of acquaintances among five people in which the pairs of mutual friends are $P_1P_2, P_2P_3, P_3P_4, P_4P_5, P_5P_1$ contains neither a group of three mutual friends nor a group of three mutual strangers.

The statement in Proposition 6.2.1 can be rephrased in terms of graphs. To introduce the graph model, suppose the six people $P_1, P_2, \ldots, P_6$ are the vertices of the complete graph K_6. Colour the edge P_iP_j red if P_i and P_j are mutual friends, and blue if they are mutual strangers. A group of three mutual friends corresponds to collection of three vertices such that the three edges joining them to each other are all coloured red, that is, a complete subgraph on three vertices whose edges are all red (a *red* K_3). A group of three mutual strangers corresponds to collection of three vertices such that the three edges joining them to each other are all coloured blue, that is, a complete subgraph on three vertices whose edges are all blue (a *blue* K_3). The proposition can then be rephrased as *For* $n \geq 6$*, in any colouring of the edges of* K_n *with red and blue, there is either a red* K_3 *or a blue* K_3.

A slightly different point of view is to use the colours "black" and "invisible" instead of red and blue. The proposition then asserts that, for any graph G on $n \geq 6$ vertices, either K_3 is a subgraph of G, or K_3 is a subgraph of $\overline{G}$, where $\overline{G}$ is the graph on the same vertices as G but with edge set $\overline{E(G)}$.

Formally, a 2-*colouring*, or just *colouring*, of the edges of K_n is an assignment of either *red* or *blue* to each of the $\binom{n}{2}$ edges of K_n, that is, it is a function from $E(K_n)$ to $\{\text{red}, \text{blue}\}$. For positive integers a and b, we say that such a colouring has a *red* K_a if there is a complete subgraph on a vertices all of whose edges are coloured red, and it has a *blue* K_b if there is a complete subgraph on b vertices all of whose edges are coloured blue.

Proposition 6.2.2 *In any 2-colouring of the edges of* K_{10} *there is either a red* K_3 *or a blue* K_4.

Proof. Consider a vertex v. Partition the nine remaining vertices into two sets: $R = \{x : vx \text{ is red}\}$ and $B = \{x : vx \text{ is blue}\}$.

If $|R| \geq 4$, then either two vertices in R are joined by a red edge, in which case there is a red K_3, or any two vertices in R are joined by a blue edge, in which case there is a blue K_4.

Assume, then, that $|R| \leq 3$. Thus, $|B| \geq 6$. By the previous result, the $K_{|B|}$ whose vertices are in B either contains a red K_3 or a blue K_3. In the former case there is a red K_3, and in the latter case, the blue K_3 together with vertex v forms a blue K_4. This completes the proof. ■

The same argument with red and blue interchanged proves:

Proposition 6.2.3 *In any 2-colouring of $E(K_{10})$ there is either a red K_4 or a blue K_3.*

We let $R(a, b)$ be the smallest integer, if it exists, such that if $n \geq R(a, b)$ then in any 2-colouring of $E(K_n)$ there is either a red K_a or a blue K_b.

We have proved $R(3, 3) = 6$, $R(3, 4) \leq 10$ and $R(4, 3) \leq 10$. It is not clear that $R(a, b)$ should exist at all. To prove that $R(a, b) = k$ one must: (i) prove that $R(a, b) \leq k$; and (ii) prove that $R(a, b) > k - 1$ (usually by giving an example of a 2-colouring of $E(K_{k-1})$ with no red K_a and no blue K_b).

Proposition 6.2.4 *Let a and b be positive integers. Then*

(a) $R(a, b) = R(b, a)$

(b) $R(a, 2) = a$

(c) $R(a, 1) = 1$

Proof. Statement (a) is true by symmetry of red and blue in the definition. To see (b), note that a 2-edge-colouring of $E(K_a)$ either has a red K_2 (a red edge) or all edges are blue. Therefore $R(a, 2) \leq a$. A K_{a-1} with all edges blue demonstrates that $R(a, 2) > a - 1$. Finally, statement (c) is true because a complete graph on 1 vertex has no edges, and therefore any 2-colouring of its edges has a blue K_1, and a complete graph on zero vertices has neither a red K_1 nor a blue K_1. ■

It turns out that the *Ramsey numbers* $R(a, b)$ exist, and much more is true. To show that each one exists, it suffices to find an upper bound. Since the positive integers are well-ordered, once we know there exists n_0 such that for all $n \geq n_0$ any 2-colouring of $E(K_n)$ there is either a red K_a or a blue K_b, then we know there is a least such integer.

Theorem 6.2.5 *For all $a \geq 2$ and $b \geq 2$, $R(a, b)$ is finite and*

$$R(a, b) \leq R(a - 1, b) + R(a, b - 1)$$

Proof. By induction on $a+b$. By our previous work the statement is true for $a+b \leq 6$. Suppose it holds whenever $a+b \leq k$ and consider the case $a+b=k+1$.

By the induction hypothesis, both $R(a-1,b)$ and $R(a,b-1)$ exist (and are finite). Let $N = R(a-1,b)+R(a,b-1)$. Consider a 2-colouring of $E(K_N)$. Let x be a vertex and partition the remaining $N-1$ vertices into $R = \{y : xy \text{ is red}\}$ and $B = \{y : xy \text{ is blue}\}$. By the PhP, either $|R| \geq R(a-1,b)$ or $|B| \geq R(a,b-1)$.

If $|R| \geq R(a-1,b)$ the subgraph with vertex set R has a red K_{a-1} or a blue K_b. Similarly, if $|B| \geq R(a,b-1)$. In the former case, x and the vertices of the red K_{a-1} form a red K_a , and in the latter case there is a blue K_b.

The result now follows by induction. ∎

By the theorem, $R(3,4) \leq R(2,4) + R(3,3) = 4+6 = 10$. Is $R(3,4) = 10$? It turns out that it isn't. Hence the numbers produced by the recurrence in the theorem only give an upper bound of $R(a,b)$. Eventually we will show that $R(3,4) = 0$. Doing that requires two things. First, we need to show that $R(a,b) \leq 9$. The theorem below makes that possible. And then we need to prove that $R(3,4) > 8$ by describing a graph G on 8 vertices such that G does not contain K_3 and $\overline{G}$ does not contain K_4.

Theorem 6.2.6 *If $R(a-1,b)$ and $R(a,b-1)$ are both even and greater than two, then*

$$R(a,b) \leq R(a-1,b) + R(a,b-1) - 1$$

Proof. Let $N = R(a-1,b) + R(a,b-1)$, and consider a 2-edge-colouring of $E(K_{N-1})$. For $i = 1,2,\ldots,N-1$, let d_i be the number of red edges incident with vertex i. Since a red edge ij contributes one to d_i and one to d_j, the sum $d_1 + d_2 + \cdots + d_{N-1}$ is even. Since $N-1$ is odd, at least one d_i is even. WLOG d_1 is even. Let $R = \{x|1x\text{is red}\}$ and $B = \{x|1x\text{is blue}\}$. Since $|R|$ is even, so is $|B|$. Now, either $|R| \geq R(a-1,b)-1$ or $|B| \geq R(a,b-1)$. But $R(a-1,b)-1$ is odd, so either $|R| \geq R(a-1,b)$ or $|B| \geq R(a,b-1)$. The result now follows as before (in the previous proof). ∎

Corollary 6.2.7 $R(3,4) \leq 9$

The recurrence in Theorem 6.2.5 is reminiscent of certain recurrences for binomial coefficients, for example Section 1.4, Exercise 4. This suggests a bound for $R(a,b)$ involving binomial coefficients may be possible.

Corollary 6.2.8 *If $a,b \geq 2$ then $R(a,b) \leq \binom{a+b-2}{a-1}$.*

Proof. By induction on $a+b$. If $a+b=4$ then $R(2,2) = 2 = \binom{2+2-2}{2-1}$. Suppose the statement is true whenever $a+b \leq k$ and consider the case $a+b=k+1$.

If $a = 2$ we know $R(2, b) = b = \binom{2+b-2}{2-1}$, and if $b = 2$ we know $R(a, 2) = a = \binom{a+2-2}{a-1}$. Suppose $a \geq 3$ and $b \geq 3$. Then, by the induction hypothesis,

$$R(a-1, b) \leq \binom{a-1+b-2}{a-1-1}$$

and

$$R(a, b-1) \leq \binom{a+b-1-2}{a-1}.$$

By the Theorem,

$$\begin{aligned} R(a,b) &\leq R(a-1,b) + R(a,b-1) \\ &\leq \binom{a+b-3}{a-2} + \binom{a+b-3}{a-1} = \binom{a+b-2}{a-1} \end{aligned}$$

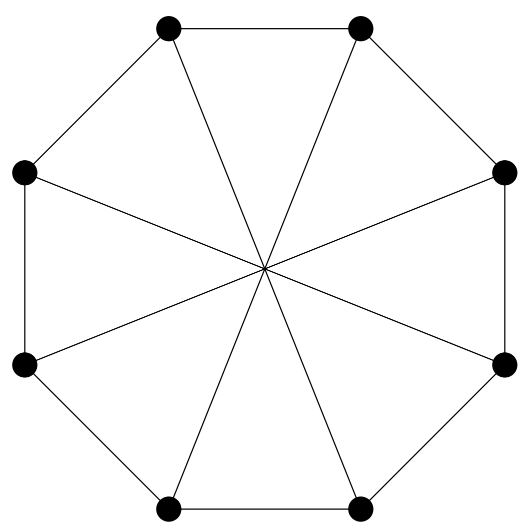
This completes the proof. ■

Proposition 6.2.9 *(a)* $R(3, 4) = 9$

(b) $R(3, 5) = 14$

(c) $R(4, 4) = 18$

Proof. (a) We already have $R(3, 4) \leq 9$. To see that $R(3, 4) > 8$, consider the graph in Figure 6.2. It has vertex set $\mathbb{Z}_8$ and $xy \in E$ if and only if $x - y \in \{\pm 1, 4\}$. This graph has no black K_3



and no white K_4 (check this!).

(b) To see $R(3, 5) = 14$ note that $R(3, 5) \leq R(2, 5) + R(3, 4) = 5 + 9 = 14$ and $R(3, 5) > 13$ because of the graph with vertex set $\mathbb{Z}_{13}$ and $xy \in E$ if and only if $x - y \in \{\pm 1, \pm 5\}$.

(c) To see that $R(4,4) = 18$ note that $R(4,4) \leq R(3,4) + R(4,3) = 9+9 = 18$ and $R(4,4) > 17$ because of the graph with vertex set $\mathbb{Z}_{17}$ and $xy \in E$ if and only if $x - y \in \{\pm 1, \pm 2, \pm 4, \pm 8\}$.

Not many Ramsay numbers are known. Other than what we have mentioned, it is known that

$$\begin{array}{ll} R(3,6) = 18 & R(4,5) = 25 \\ R(3,7) = 23 & 36 \leq R(4,6) \leq 41 \\ R(3,8) = 28. & \end{array}$$

Lots of other bounds and values are available in the dynamic survey in the Electronic Journal of Combinatorics (http://www.combintorics.org).

So far our lower bounds have all been found by finding a 2-colouring of the edges of K_n that has neither a red K_a nor a blue K_b. It turns out that a general lower bound is possible when $a = b$. The proof is an example of the *probabilistic method.* The idea is that if the sample space (i.e. the universe) is finite, and the probability of some event occurring can be shown to be less than 1, then the complementary event must occur. Here, we show that if n is not too large, then the probability of a 2-colouring of the edges of K_n containing a red K_a or a blue K_b is less than 1, hence there must be a 2-colouring that has neither. It is interesting to note that most, maybe all, lower bounds for $R(a,b)$ obtained by constructive arguments are weaker than those given in the theorem below and it corollary.

Theorem 6.2.10 [Erdös, 1947] *If $k \geq 2$, then $R(k,k) \geq 2^{k/2}$*

Proof. Since $R(2,2) = 2$, we may assume that $k \geq 3$. Let $\mathcal{G}_n$ be the set of complete graphs on vertex set $V = \{v_1, v_2, \ldots, v_n\}$ with edges coloured red and blue. Then $|\mathcal{G}_n| = 2^{\binom{n}{2}}$.

The number of elements of $\mathcal{G}_n$ that have a particular set of k vertices forming a red K_k is $2^{\binom{n}{2} - \binom{k}{2}}$.

Therefore, the number of elements of $\mathcal{G}_n$ with a red K_k is at most $\binom{n}{k} 2^{\binom{n}{2} - \binom{k}{2}}$ (note that there is lots of double-counting here), and so the probability that a randomly-chosen element of $\mathcal{G}_n$ has a red K_k is at most

$$\frac{\binom{n}{k} 2^{\binom{n}{2} - \binom{k}{2}}}{2^{\binom{n}{2}}} = \binom{n}{k} 2^{-\binom{k}{2}} < \frac{n^k 2^{-\binom{k}{2}}}{k!}$$

Suppose now that $n < 2^{k/2}$. Then, by induction on k, the RHS is less than

$$\frac{2^{k^2/2} 2^{-\binom{k}{2}}}{k!} = \frac{2^{k/2}}{k!} < \frac{1}{2}.$$

Since $\mathcal{G}_n$ is finite, this means that fewer than half the elements in $\mathcal{G}_n$ contain a red K_k. The same argument shows that fewer than half the elements in $\mathcal{G}_n$ contain a blue K_k.

Therefore for any $n < 2^{k/2}$, there is a graph in $\mathcal{G}_n$ that contains neither a red K_k nor a blue K_k; that is, $R(k,k) \geq 2^{k/2}$. ■

Corollary 6.2.11 *If* $m = \min\{a, b\}$ *then* $R(a, b) \geq 2^{m/2}$.

The value of a for which the greatest number of Ramsey numbers are known is $a = 3$. This is undoubtedly due to the simple structure of K_3. By Corollary 6.2.8, we know that $R(3, q) \leq \binom{3+q-2}{2} = \binom{q+1}{2} = \frac{q^2+q}{2}$. It turns out that a better upper bound is possible.

Theorem 6.2.12 [Behzad and Chartrand] $R(3, q) \leq \frac{q^2+3}{2}$.

Proof. By induction on q. For $q = 2$ we have $3 = R(3, 2) \leq \frac{2^2+3}{2} = 3\frac{1}{2}$. Suppose, for some $q \geq 2$, that the statement is true for $q - 1$, and consider $R(3, q)$. We know that

$$\begin{aligned} R(3, q) &\leq R(2, q) + R(3, q-1) \\ &= q + R(3, q-1) \\ &= \frac{q^2 + 2q + 3}{2} \end{aligned}$$

by the Induction Hypothesis.

If q is odd, say $q = 2k + 1$, then this gives $R(3, 2k + 1) \leq 2k^2 + 2k + \frac{5}{2}$. Since $R(3, 2k + 1)$ is an integer, $R(3, 2k + 1) \leq 2k^2 + 2k + 2 = \frac{(2k+1)^2+3}{2}$.

If q is even, say $q = 2k$, then we have $R(3, 2k) \leq 2k^2 + 2 = \frac{4k^2+3}{2} + \frac{1}{2}$, which isn't quite good enough. We need to lower the bound by $\frac{1}{2}$, that is, we need to show that $R(3, 2k) < 2k^2 + 2$.

Suppose not, i.e., that $R(3, 2k) = 2k^2 + 2$. Then there is a colouring of $E(K_{2k^2+1})$ with no red K_3 and no blue K_{2k}.

Since there is no red K_3 and no blue K_{2k}, no vertex is incident with $2k$ or more red edges and so each vertex is incident with at most $2k - 1$ red edges. But $2k^2 + 1$ is odd, so not all vertices can be incident with $2k - 1$ red edges ($2k - 1$ is odd and the number of vertices of odd degree is even). Therefore some vertex x is incident with at most $2k - 2$ red edges.

Consider the graph obtained by deleting x and every vertex joined to it by a red edge. The number of vertices that remain is at least

$$\begin{aligned} 2k^2 + 1 - 1 - (2k - 2) &= 2k^2 - 2k + 2 \\ &= \frac{(2k-1)^2 + 3}{2} \\ &\geq R(3, 2k - 1) \end{aligned}$$

by the induction hypothesis.

Thus the remaining graph has a red K_3, or a blue K_{2k-1}. In the former case, we are done. In the latter case, adding x results in a blue K_{2k}, a contradiction. Therefore $R(3, 2k) < 2k^2 + 2$, and the result for q even follows.

The result now follows by induction. ∎

6.3 Multicolour Graph Ramsey Numbers

Instead of using two colours, more generally, we can consider colouring the edges of K_n with several colours and asking whether there must be a complete graph of a certain size with all edges of the same colour (*a monochromatic complete graph*). The proofs of the basic results are very similar to the corresponding results for two-colour graph Ramsey numbers.

Let $t \geq 2$ and $k_1, k_2, \ldots, k_t$ be integers each ≥ 1. We define

$$R(k_1, k_2, \ldots, k_t)$$

to be the smallest integer such that if $n \geq R(k_1, k_2, \ldots, k_t)$ then any colouring of $E(K_n)$ with t colours has the property that there exists $i, 1 \leq i \leq t$ such that there is a K_{k_i} all of whose edges have colour i.

Proposition 6.3.1 *If some $k_i = 1$, then $R(k_1, k_2, \ldots, k_t) = 1$.*

Proof. Exercise. ■

Theorem 6.3.2 *Each number $R(k_1, k_2, \ldots, k_t)$ is finite. If each $k_i \geq 2$, then*

$$\begin{aligned} R(k_1, k_2, \ldots, k_t) \leq\ & R(k_1 - 1, k_2, \ldots, k_t) + R(k_1, k_2 - 1, \ldots, k_t) + \cdots \\ & + R(k_1, k_2, \ldots, k_t - 1) - t + 2 \end{aligned}$$

Proof. Exercise ■

Corollary 6.3.3 $R(k_1, k_2, \ldots, k_t) \leq \binom{k_1+k_2+\cdots+k_t}{k_1\ k_2\ \cdots\ k_t}$

Proof. Exercise. ■

Only one of these *multicolour graph Ramsey numbers* is known when there are more than two colours involved. It is $R(3, 3, 3)$, which equals 17.

Theorem 6.3.4 $R(3, 3, 3) = 17$.

Proof. Consider a colouring of K_{17} with red, blue, and green. Choose a vertex x_1. By the PhP, some six of the 16 edges incident with x_1 have the same colour, say red. Suppose $x_1x_2, x_1x_3, \ldots, x_1x_7$ are red.

Now consider the (induced) subgraph with vertex set $\{x_2, x_3, \ldots, x_7\}$. If it has a red edge, then there is a red K_3. Otherwise all edges are blue and green. Since $R(3, 3) = 6$, there exists a blue K_3 or a green K_3 and therefore $R(3, 3, 3) \leq 17$.

We need to show that $R(3,3,3) > 16$. We describe a colouring of $E(K_{16})$ using red, green, and blue and having no monochromatic K_3. Let $V(K_{16}) = \mathbb{Z}_4 \times \mathbb{Z}_4$. Define the sets

$$\begin{aligned} E_R &= \{(1,1),(3,3),(1,2),(3,2),(0,2)\} \\ E_B &= \{(2,2),(0,1),(0,3),(3,0),(1,0)\} \\ E_G &= \{(2,1),(2,3),(1,3),(3,1),(0,2)\} \end{aligned}$$

Each set $S \in \{E_R, E_B, E_G\}$ has the following two properties:

(a) $(a,b) \in S \Leftrightarrow (-a,-b) \in S$ (closed w.r.t. inverses)

(b) $(a,b),(c,d) \in S \Rightarrow (a+c, b+d) \notin S$ (sum free)

where arithmetic is in $\mathbb{Z}_4$.

Colour the edge $(a,b)(c,d)$ with colour $X \Leftrightarrow (a-c, b-d) \in E_X$. This is a valid colouring because of (a).

Suppose there exists a monochromatic K_3. Then there are $(x_1,x_2),(y_1,y_2),(z_1,z_2) \in \mathbb{Z}_4 \times \mathbb{Z}_4$ and $S \in \{E_R, E_B, E_G\}$ such that $\alpha = (x_1 - y_1, x_2 - y_2), \beta = (y_1 - z_1, y_2 - z_2)$, and $\gamma = (x_1 - z_1, x_2 - z_2) \in S$. But $\alpha + \beta = \gamma$, which contradicts (b), so there is no monochromatic triangle. ■

6.4 Hypergraphs and Ramsey's Theorem

A *r-uniform hypergraph* is a pair $\mathcal{G} = (V,E)$, where V is a set of objects called *vertices* and E is a set of r-subsets of V called *edges* (or *hyperedges*).

There is a more general definition of hypergraph that allows for edges to contain different numbers of vertices, and for more than one edge to be incident with the same subset of vertices, but we will not pursue that here. Our goal in introducing hypergraphs is to connect the graph Ramsey theory from the previous two sections to Ramsey's general theorem in a way that they look similar. Hypergraphs were not mentioned in Ramsey's original formulation, but the theorem can be viewed as being about them.

Every graph is a 2-uniform hypergraph. The *Fano plane* is an example of a 3-uniform hypergraph. It has vertex set $V = \{1,2,\ldots,7\}$ and edge set $\{\{1,2,3\},\{1,4,7\},\{1,5,6\},\{2,4,6\}, \{2,5,7\}, \{3,4,5\},\{3,6,7\}\}$. The Fano plane is represented pictorially in Figure 6.1. The edges are represented by straight lines and the interior cycle. This drawing of the Fano plane is attractive because it illustrates the classic theorem from geometry that *in any triangle the angle bisectors are concurrent and the point where they intersect is the centre of the inscribed circle of the triangle.*

Another example of a 3-uniform hypergraph is the affine plane (or order 3) shown in Figure 6.2. Each "line" (horizontal, vertical, diagonal) is an edge.

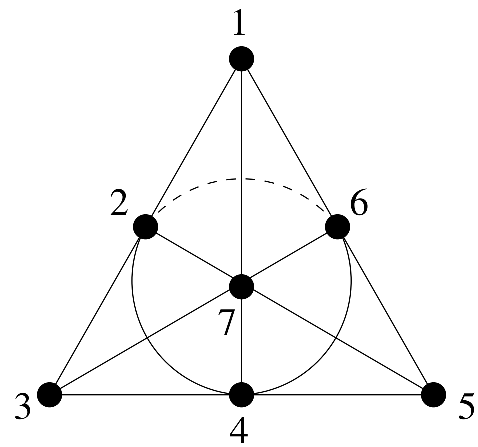


Figure 6.1: The Fano plane

An r-uniform hypergraph $\mathcal{G} = (V(G), E(G))$ is *complete* if every r-subset of V is an edge. Thus, a complete graph is a complete 2-uniform hypergraph. We use K_n^r to denote the complete r-uniform hypergraph on n vertices. That is, the hypergraph on vertex set $V = \{1, 2, \ldots, n\}$ and for which the edges consist of the collection of all $\binom{n}{r}$ r-subsets of V.

A *colouring of the edges of* K_n^r *with* t *colours* is a function $c : E(K_n^r) \to \{1, 2, \ldots, t\}$. That is, an assignment of one of the colours $1, 2, \ldots, t$ to each edge of K_n^r. When $r = 2$, this corresponds to an edge-colouring if the complete graph K_n.

Suppose that the edges of K_n^r are coloured with t colours. If $1 \leq k \leq n$ and $1 \leq i \leq t$, then a K_k^r *of colour* i consists of a subset $X \subseteq V(K_n^r)$ with $|X| = k$ such that every edge of the complete r-uniform hypergraph with vertex set X is assigned colour i (in the given colouring of the edges of K_n^r).

Informally, Ramsey's Theorem says that if you colour the edges of an r-uniform hypergraph with t colours then, provided $|V|$ is large enough, then one of the colour sets will contain all of the edges of an r-uniform hypergraph of a given size.

Theorem 6.4.1 [Ramsey's Theorem] *Let $k_1, k_2, \ldots, k_t$ and r be positive integers with $t \geq 2$. There exists a smallest positive integer $N(k_1, k_2, \ldots, k_t; r)$ such that if $n \geq N(k_1, k_2, \ldots, k_t; r)$ and the edges of the complete r-uniform hypergraph K_n^r are coloured with t colours, then there exists i, $1 \leq i \leq t$, and a k_i-subset of $X \subset V$ which is the vertex set of a $K_{k_i}^r$ of colour i.*

The numbers $N(k_1, k_2, \ldots, k_t; r)$ in Ramsey's Theorem are called *Ramsey Numbers.* Stated differently, Ramsey's Theorem says that if n is large enough and the $\binom{n}{r}$ r-subsets of an n-set S are coloured with t colours, then for some i there is a k_i-subset of S all of whose r-subsets are of colour i.

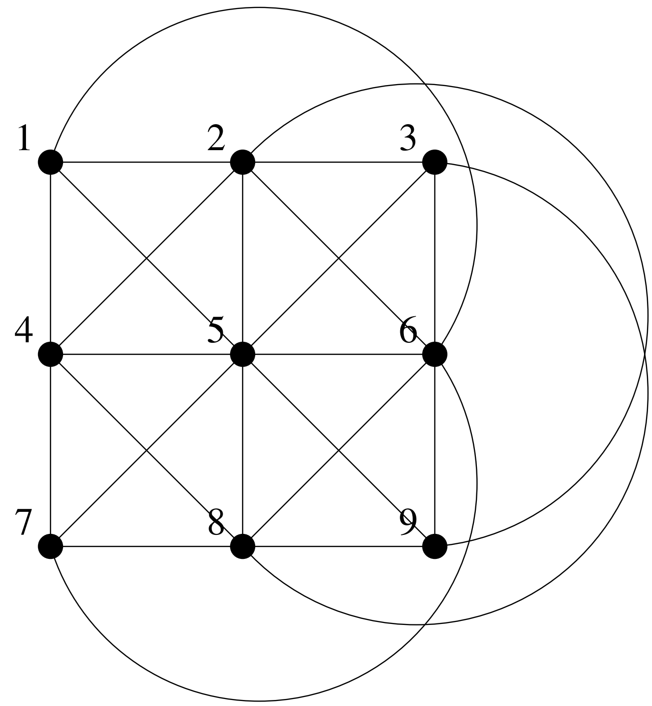


Figure 6.2: The affine plane of order 3

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For example, let $n = 6$. Then K_6^3 has $\binom{6}{3} = \frac{6 \cdot 5 \cdot 4}{6} = 20$ edges. Consider the following 2-colouring of $E(K_6^3)$:

Colour 1	Colour 2
{1,2,3}	{1,2,5}
{1,2,4}	{1,3,4}
{1,2,6}	{1,3,5}
{1,4,5}	{1,3,6}
{1,4,6}	* {2,4,5}
{1,5,6}	* {2,4,6}
{2,3,4}	* {2,5,6}
{3,4,5}	{3,4,6}
{2,3,5}	{3,5,6}
{2,3,6}	* {4,5,6}

A K_4^3 of colour 2 is indicated by the *'s.

Proposition 6.4.2 *Let $k_1, k_2, \ldots, k_t$ and r be positive integers with $t \geq 2$. If there exists i, $1 \leq i \leq t$, such that $k_i < r$, then $N(k_1, k_2, \ldots, k_t; r) == \min\{k_1, k_2, \ldots, k_t\}$.*

Proof. Exercise. ■

Proposition 6.4.3 *Let $t \geq 2$ be an integer. Then*

(a) $N(k_1, k_2, \ldots, k_t; 1) = k_1 + k_2 + \cdots + k_t - t + 1$

(b) $N(k, r; r) = k$

Proof. (a) Let $n \geq k_1 + k_2 + \cdots + k_t - t + 1$. Consider a colouring of the 1-element subsets of $\{x_1, x_2, \ldots, x_n\}$ (the edges of K_n^1) with colours $1, 2, \ldots, t$. By the Pigeonhole Principle, there exists i such that at least k_i of these 1-subsets have colour i. Thus there is a K_{k_i} of colour i.

To see that $N(k_1, k_2, \ldots, k_t; 1) > k_1 + k_2 + \cdots + k_t - t = (k_1 - 1) + (k_2 - 1) + \cdots + (k_t - 1) = n_1$, let $V = \{v_1, v_2, \ldots, v_{n_1}\}$ and colour $k_i - 1$ 1-subsets with colour $i, 1 \leq i \leq t$. This colouring of the edges of $K_{n_1}^1$ with colours $1, 2, \ldots, t$ has no $K_{k_i}^1$ of colour i for any i.

(b) To see that $N(k, r; r) > k - 1$ note that a K_{k-1}^r with all edges of colour 1 has neither a K_k^r of colour 1 nor a K_r^r of colour 2.

To see that $N(k, r; r) \leq k$, note that any colouring of the edges of K_k^r with colours 1 and 2 has either an edge of colour 2 (and hence a K_r^r of colour 2), or all edges have colour 1 (and there is a K_k^r of colour 1). ■

Statement (a) of Proposition 6.4.3 is the special case $r = 1$ of Ramsey's Theorem. It is also recognizable as the Pigeonhole Principle (compare Proposition 6.1.2).

In the previous sections we have proved Ramsey's Theorem in the special case $r = 2$. When $r = 2$, the number $N(k_1, k_2, \ldots, k_t; 2) = R(k_1, k_2, \ldots, k_t)$. We showed that if $n \geq R(k_1, k_2, \ldots, k_t)$ and the edges of K_n (the 2-subsets of S) are coloured with t colours, then there exists i such that there is a K_{k_i} all of whose edges have colour i.

We now prove Ramsey's Theorem for arbitrary edge sizes r in the case $t = 2$. The proof of the general theorem is by induction on t, so this is needed as the base case of the induction. The argument is in some sense similar to the one we used to prove the corresponding result for two-colour graph Ramsey numbers, and in some sense different because it involves a double induction. As before, the argument gives a finite upper bound for the number in question. Since the positive integers are well-ordered, it follows that there exists a least integer with the required property.

Theorem 6.4.4 *Let r, k_1, k_2 be integers. There exists a least integer $N(k_1, k_2; r)$ such that if $n \geq N(k_1, k_2; r)$ then any colouring of K_n^r using red and blue has either a red $K_{k_1}^r$ or a blue $K_{k_2}^r$.*

Proof. By Proposition 6.4.2 we can assume that $k_1, k_2 \geq r \geq 1$. The proof is by induction on r. The base case, $r = 1$, follows from Proposition 6.4.3. Assume, for some $r \geq 2$, that the statement holds for $r - 1$. The proof for r (the induction step) is by induction on $k_1 + k_2$.

If $k_1 + k_2 = 2r$, we have $N(k_1, k_2; r) = N(r, r; r) = r$ by Proposition 6.4.3, and we're done. Suppose, for some $k \geq 2r + 1$, that the statement holds whenever $k_1 + k_2 = k - 1$, and consider the case when $k_1 + k_2 = k$. By Proposition 6.4.3., we can assume $k_1, k_2 > r$.

Let $n_1 = N(k_1 - 1, k_2; r)$ and $n_2 = N(k_1, k_2 - 1; r)$. Let $n \geq N(n_1, n_2; r-1) + 1$ and consider a colouring of K_n^r using red and blue. Choose a vertex $x \in V = V(K_n^r)$. Colour the edges of the complete $(r-1)$-uniform hypergraph with vertex set $V \backslash \{x\}$ by colouring an edge Y of this hypergraph the same colour as the edge $Y \cup \{x\}$ in our colouring of $E(K_n^r)$.

Since $n - 1 \geq N(n_1, n_2; r-1)$ our K_{n-1}^{r-1} has either a red $K_{n_1}^{r-1}$ or a blue $K_{n_2}^{r-1}$. Without loss of generality the former case occurs.

Since $n_1 = N(k_1 - 1, k_2; r)$, in our hypergraph K_n^r these n_1 vertices span a $K_{n_1}^r$ that, by the I.H., has a red $K_{k_1-1}^r$, call it $\mathcal{G}$, or a blue $K_{k_2}^r$. In the latter case, we are done. Suppose the former case occurs. By our work on $K_{n_1}^{r-1}$ we know that any edge containing x and $r-1$ vertices from $K_{n_1}^{r-1}$ is coloured red. Thus $V(\mathcal{G}) \cup \{x\}$ is the vertex set of a red $K_{k_1}^r$.

The result follows by induction. ■

As an example of the colouring of K_{n-1}^{r-1} used in the proof, suppose $r = 3$ and $n = 5$. Assume $V = \{1, 2, 3, 4, 5\}$. Start with a 2-colouring of $E(K_5^3)$:

Colour 1	Colour 2
{1,2,3}	{1,2,4}
{1,2,5}	{1,3,4}
{1,3,5}	{1,4,5}
{2,3,4}	{2,3,5}
{2,4,5}	{3,4,5}

Pick $x \in V$, say $x = 5$. If K_4^2 has vertex set $\{1, 2, 3, 4\}$, the edges are $\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}$, $\{2, 4\}, \{3, 4\}$ and they are coloured 1, 1, 2, 2, 1, 2, respectively.

Proof of Ramsey's Theorem. By Proposition 6.4.2 we can assume that $k_1, k_2, \ldots, k_t \geq r \geq 1$. Let $r \geq 1$ be given. The proof is by induction on t. We will show that $N(k_1, k_2, \ldots, k_t; r) \leq N(k_1, k_2, \ldots, k_{t-2}, N(k_{t-1}, k_t; r); r)$.

We have already proved the result when $t = 2$. Suppose, for some $t \geq 3$, that it is true for $t - 1$. Let $k_1, k_2, \ldots, k_t$ be given, with each $k_i \geq r$.

Let $k'_{t-1} = N(k_{t-1}, k_t; r)$, and $n \geq N(k_1, k_2, \ldots, k_{t-2}, k'_{t-1}; r)$. Consider a colouring of K_n^r with colours $1, 2, \ldots, t$. Recolour the edges coloured $t - 1$ and t with a new colour $(t-1)'$. By the induction hypothesis, either there exists $i < t - 1$ such that the new colouring has a $K_{k_i}^r$ of colour i, or there is a $K_{k'_{t-1}}^r$ of colour $(t-1)'$. In the former case we are done, so suppose the latter case occurs.

Restore the edges of $K_{k'_{t-1}}^r$ to their original colours, so that the edges of this hypergraph are coloured with two colours, $t - 1$ and t. Since $k'_{t-1} = N(k_{t-1}, k_t; r)$, there is either a $K_{k_{t-1}}^r$ of colour $t - 1$ or a $K_{k_t}^r$ of colour t. This completes the proof. ■

6.5 Ramsey-like Theorems

In this section we give a few examples of theorems which can be considered as part of the general body of knowledge called *Ramsey Theory*. Several of these also fall into Combinatorial Number Theory.

6.5.1 Convex m-gons

A collection of points in the plane are in *general position* if no three points are collinear.

A n-gon (or more generally, a region) in the plane is *convex* if the line segment joining any two interior points lies entirely within the n-gon (or region).

We will work up to a general version of the following proposition.

Proposition 6.5.1 *If five points in the plane are in general position, then four of the points are the vertices of a convex quadrilateral.*

Proof. Let the smallest convex polygon containing all five points be an m-gon (i.e. the convex hull). If $m = 4$ or 5, there is nothing to prove.

The only other possibility is that $m = 3$. Then there is a triangle formed by three of the five points, say A, B, and C, that contains the other two points, say D and E, in its interior. The line determined by D and E will divide the triangle into two parts. One of these parts will contain two vertices. These two vertices together with D and E form a convex quadrilateral. ■

Lemma 6.5.2 *Suppose n points in the plane are in general position. If every quadrilateral formed from these n points is convex, then the n points are the vertices of a convex n-gon.*

Proof. Suppose not. Consider the smallest convex polygon that contains the n points (the convex hull, again). At least one of the n points is in the interior of this polygon, say P. Choose a vertex x of the polygon and triangulate it by drawing line segments from x to all other vertices. The point P lies in the interior of one of these triangles, say x, y, z. But then x, y, z, P do not form a convex quadrilateral, a contradiction. ■

Theorem 6.5.3 [Erdös and Szekeres, 1935] *For any integer $m \geq 3$ there exists a least integer $N(m)$ such that whenever $n \geq N(m)$ points in the plane are in general position, some m of the points are the vertices of a convex m-gon.*

Proof. Let $n \geq N(5, m; 4)$, and X be any set of n points in general position. Colour the 4-subsets of X with red and blue such that a 4-subset is coloured blue if and only if it comprises the vertices of a convex quadrilateral.

By Ramsey's Theorem, either there exists a 5-subset of X for which every 4-subset is red or an m-subset of X for which every 4-subset is blue. But the first case is impossible by Proposition 6.5.1 (there must be a convex quadrilateral; hence not all 4-subsets are red). Thus there is an m-subset for which every 4-subset is blue. By Lemma 6.5.2, these m points are the vertices of a convex m-gon. ■

6.5.2 Arithmetic Progressions

Recall that an *arithmetic progression of length* k is a sequence of k integers $a, a+d, a+2d, \ldots, a+(k-1)d$.

Proposition 6.5.4 *If the integers* $1, 2, \ldots, 9$ *are partitioned into two sets, then at least one of the sets contains an arithmetic progression (AP) of length three.*

Proof. Suppose not, and let P, Q be a partition of $\{1, 2, \ldots, 9\}$ such that neither set contains an AP of length three. Without loss of generality, $5 \in P$. Thus, 1 and 9 can't both be in P. There are three cases to consider:

Case 1. $1 \in P$ and $9 \notin P (9 \in Q)$.
$1 \& 5 \in P \Rightarrow 3 \in Q$.
$3 \& 9 \in Q \Rightarrow 6 \in P$.
$5 \& 6 \in P \Rightarrow 4 \in Q, 7 \in Q$.
$3 \& 4 \in Q \Rightarrow 2 \in P$.
$7 \& 9 \in Q \Rightarrow 8 \in P \Rightarrow 2, 5, 8 \in P$, a contradiction.

Case 2 $9 \in P$ and $1 \notin P (1 \in Q)$.
As above, replacing x by $10 - x$.

Case 3. $1 \& 9 \in Q$. Suppose first that $7 \in P$. Since $5 \in P$ we must have $3, 6 \in Q$. Then $3, 6, 9 \in Q$, a contradiction. Now suppose $7 \in Q$. Then $8 \in P$ (because of 9) and $4 \in P$ (because of 1). Since $4, 5 \in P$, $3 \in Q$ and $1, 3 \in Q \Rightarrow 2 \in P$. Therefore $2, 5, 8 \in P$, a contradiction. ■

The proposition above is a special case of a general theorem due to van der Waerden. An r-colouring of $\{1, 2, \ldots, n\}$ is an ordered collection $(C_1, C_2, \ldots, C_r)$ of pairwise disjoint sets such that $C_1 \cup C_2 \cup \cdots \cup C_r = \{1, 2, \ldots, n\}$. The set C_i can be thought of as *the set of numbers assigned colour* i. Given an r-colouring of $\{1, 2, \ldots, n\}$, a *monochoromatic arithmetic progression* is an arithmetic progression for which all elements belong to the same set C_i.

Theorem 6.5.5 [van der Waerden's Theorem] *For any positive integers* k *and* r*, there exists a least positive integer* $W(k, r)$ *such that for any* r*-colouring of* $\{1, 2, \ldots, W(k, r)\}$ *there is a monochromatic arithmetic progression of length* k*.*

6.6 Exercises

1. Prove Proposition 6.1.2.

2. Prove that if $x_1, x_2, \ldots, x_n$ are real numbers, then there exists i such that $x_i \geq (x_1 + x_2 + \cdots + x_n)/n$.

3. Let m and n be positive integers, both of which are at least two. Give an example of a sequence of mn different integers with neither an increasing subsequence of length $n+1$ nor a decreasing subsequence of length $m+1$.

4. Prove that if $n+1$ integers are selected from among $\{1, 2, \ldots, 2n\}$, then the selection includes integers a and b such that $gcd(a, b) = 1$. Is the number $n+1$ best possible?

5. Find conditions on the parameters d, t, and e so that the following problem has a solution, and use them to obtain a large family of Pigeonhole Principle problems of the following type. *Over a d day period, Peter will walk the dog at least once per day, and a total of t times in all. Prove that there is a period of consecutive days in which he walks the dog exactly e times.*

6. Let $n \geq 1$. Show that if $2n$ hockey teams are entered in a round-robin tourament (meaning that every team plays every other team exactly once) in which ties are not allowed then, if no team loses all of its games, some two teams finish the tournament with the same number of wins and some two teams finish with the same number of losses.

7. Suppose
$$0 < x_1 < x_2 < \cdots < x_{mn+1}$$
are $mn+1$ integers. Prove that there is either a set of $m+1$ of them, no one of which divides any other, or a set of $n+1$ of them such that each one divides the following one (when the integers in the set are put into ascending order).

8. Show that of the edges of K_7 are coloured with *red* and *blue*, there are at least three K_3's with all edges of the same colour. (Note that some K_3's may have all edges red and others may have all edges blue. The K_3's need not be edge-disjoint.)

9. Complete the proof of Proposition 6.2.9 by showing
   (a) The graph with vertex set $\mathbb{Z}_{13}$ and $xy \in E$ if and only if $x - y \in \{\pm 1, \pm 5\}$ has no K_3 and its complement has no K_5.
   (b) Neither the graph with vertex set $\mathbb{Z}_{17}$ and $xy \in E$ if and only if $x - y \in \{\pm 1, \pm 2, \pm 4, \pm 8\}$ nor its complement has a K_4.

10. Prove Proposition 6.3.1.

11. Suppose $k_1, k_2, \ldots, k_t$ are integers which are each at least three. Explain why
$$R(2, 2, \ldots, 2, k_1, k_2, \ldots, k_t) = R(k_1, k_2, \ldots, k_t).$$

12. Give the induction step of a proof that each number $R(k_1, k_2, \ldots, k_t)$ is finite by showing, for $t > 2$, that $R(k_1, k_2, \ldots, k_t) \leq R(k_1, k_2, \ldots, k_{t-2}, R(k_{t-1}, k_t))$. Do not directly apply Ramsey's Theorem.

13. (a) Prove that if some $k_i = 1$, then $R(k_1, k_2, \ldots, k_t) = 1$.

    (b) Let $k_1, k_2, \ldots, k_t$ be integers, each at least two. Without appealing to Ramsey's Theorem, prove that the multicolour graph Ramsey number satisfies the inequality

$$R(k_1, k_2, \ldots, k_t) \leq R(k_1 - 1, k_2, \ldots, k_t) + R(k_1, k_2 - 1, \ldots, k_t) + \cdots$$

$$+R(k_1, k_2, \ldots, k_t - 1) - t + 2.$$

    (c) Show that

$$R(k_1, k_2, \ldots, k_t) \leq \binom{k_1 + k_2 + \cdots k_t}{k_1\ k_2\ \ldots\ k_t}.$$

14. For $k \geq 2$, define $R(3k)$ to equal $R(3, 3, \ldots, 3)$ (k threes).

    (a) Prove that, for $k \geq 3$, $R(3k) \leq kR(3k-3) - k + 2$.

    (b) Prove that for $k \geq 3$, $R(3k) \leq 1 + k! \sum_{i=0}^{k} \frac{1}{i!} \leq 1 + \lfloor k!e \rfloor$.

15. Prove Proposition 6.4.2

16. Prove Schur's Theorem: For every natural number r there exists a least integer N, depending on r, such that if the integers $\{1, 2, \ldots, N\}$ are coloured with r colours then the set of integers of some colour contains a solution to the equation $x + y = z$. (Hint: consider a suitable graph in which the edge xy is coloured with colour $|x - y|$.)

17. Use Schur's Theorem (Exercise 16) to prove that the congruence $x^m + y^m \equiv z^m \pmod{p}$ has non-zero solutions for all sufficiently large primes p.

18. Let $G_1, G_2, \ldots, G_m$ be simple graphs. Define $r(G_1, G_2, \ldots, G_m)$ to be the smallest integer n such that every colouring of the edges of K_n with m colours contains a subgraph isomorphic to G_i whose edges all have colour i.

    (a) Explain why $r(G_1, G_2, \ldots, G_m) \leq R(k_1, k_2, \ldots, k_m)$, where $k_i = |V_i|,\ 1 \leq i \leq m$.

    (b) Suppose G is a path on four vertices and H is a cycle on four vertices. Prove that $r(G, G) = 5$, $r(G, H) = 5$, and $r(H, H) = 6$.

19. Prove that in any partition of $\{1, 2, 4, \ldots, 2^8\}$ into two sets, one of the sets contains a geometric progression of length three (i.e., a, ad, ad^2).

Chapter 7

Polya Theory

Circular permutations are an example of a collection of objects that are counted up to equivalence according to some sort of symmetry (namely rotation). The main results in this chapter give methods for doing the same sort of thing when "equivalence" has a more general meaning.

Informally, by a group G acting on a set S we will mean that there is a group G that somehow permutes the elements of the set S. For example, suppose S is the set of unit squares each of whose edges is coloured red or blue. Then $|S| = 2^4$. If G is the group of eight symmetries of the square (rigid motions that map the square back to itself):

$$\begin{aligned}
&I = \textit{identity},\\
&r_{90} = \textit{rotate 90}, \; r_{180} = \textit{rotate 180}, \; r_{270} = \textit{rotate 270},\\
&f = \textit{flip about vertical axis},\\
&f \circ r_{90}, f \circ r_{180}, f \circ r_{270}.
\end{aligned}$$

Since function composition is applied right to left, the rotations occur before the flips. The group G induces a mapping of S to itself. Some elements of S are mapped back to themselves by elements of G: for example, any coloured square whose two vertical sides are the same colour is fixed by f, a flip of the square about its vertical axis. Some elements of S are not fixed by G: for example, the coloured square whose side edge is red and whose other three sides are blue is mapped to the similarly coloured square whose right hand vertical edge is red by r_{90}, clockwise rotation of 90 degrees.

7.1 Orbits and Stabilizers

Let G be a group that permutes the elements of a set S, that is, if $g \in G$ and $s \in S$, then $g(s) \in S$. Two elements $s_1, s_2 \in S$ are called *equivalent* if there exists $g \in G$ such that $g(s_1) = s_2$. Equivalence, defined in this way, is an equivalence relation on S and therefore partitions S into disjoint equivalence classes called *orbits*. The orbit of an element $s \in S$ is denoted by O_s.

We will continue to use as our working example the set S of 16 unit squares each of whose edges is coloured red or blue, and the group G of eight symmetries of the square (as listed previously).

There is a subtlety in the wording that is worth noticing. The group G permutes the elements of S, but we may not be thinking of G as a permutations group on S, that is, as a subgroup of the symmetric group on $|S|$ elements. In the unit squares example, G is a group of permutations of the sides of the squares, not of the coloured squares. The action of G on the unit square sends some coloured squares to others.

Let G be a group that permutes the elements of a set S. If $s \in S$, then the *stabilizer* of s is the set

$$St_s = \{g \in G : g(s) = s\}.$$

That is, the stabilizer of s is the set of elements of G that leave s fixed.

Continuing with our working example, listing the edge colours of the squares in cyclic order clockwise from the top, the orbits associated with G are

{RRRR}
{BBBB}
{RBBB, BRBB, BBRB, BBBR}
{BRRR, RBRR, RRBR, RRRB}
{RBRB, BRBR}
{BBRR, RBBR, RRBB, BRRB}.

Every element of S has a stabilizer. For example,

$$\begin{aligned} St_{\text{RRRR}} &= G \\ St_{\text{RRRB}} &= \{I, f \circ r_{180}\} \\ St_{\text{BBRR}} &= \{I, f \circ r_{270}\} \\ St_{\text{BRBR}} &= \{I, f, r_{180}, f \circ r_{180}\}. \end{aligned}$$

Notice two things. First, in each case the stabilizer of an element is a subgroup of G. Second, and somewhat more importantly, for each element $s \in S$ in the example we have $|O_s| \cdot |St_s| = |G|$. These two facts are true in general.

Proposition 7.1.1 *Let G be a group that permutes the elements of a set S. For any $s \in S$, the set St_s is a subgroup of G.*

Proof. Let $s \in S$. The identity element $I \in St_s$ since I fixes s. If p_1 and p_2 fix s, then so does $p_1 \circ p_2$. Suppose p fixes s. Then $p^{-1}(s) = p^{-1}(p(s)) = (p^{-1} \circ p)(s) = I(s) = s$, so that p^{-1} fixes the element s. Therefore, St_s is a subgroup of G. ■

By Lagrange's Theorem, if G is a finite group and H is a subgroup of G, then $|H|$ divides $|G|$. This limits the possible sizes of the stabilizer of an element. In particular, if G has a prime number of elements, then the stabiilizer of an element either consists of only the identity, or is all of G.

Theorem 7.1.2 [Orbit-Stabilizer Theorem] *Let G be a group that permutes the elements of a set S. Then, for any $s \in S$,*

$$|O_s| \cdot |St_s| = |G|$$

Proof. Let $s \in S$, and $O_s = \{s_1, s_2, \ldots, s_t\}$. By the definition of O_s there exists a set $P = \{p_1, p_2, \ldots, p_t\} = P$ of elements of G such that p_i sends s to $s_i, 1 \leq i \leq t$.

We claim that every element of G can be written in exactly one way as the composition of an element of P and an element of St_s. Let $g \in G$ Suppose g sends s to s_i. Since $p_i \in P$ also sends s to s_i, we have that p_i^{-1} sends s_i to s and $p_i^{-1} \circ g$ sends s to s. Therefore $(p_i^{-1} \circ g) \in St_s$, and $g = p_i \circ (p_i^{-1} \circ g)$ is the composition of an element of P and an element of St_s.

To see that the representation is unique, assume $g_a, g_b \in St_s$, $p_i, p_j \in P$, and $p_i \circ g_a = p_j \circ g_b$. The LHS sends s to s_i and the RHS sends s to a_j. If these are equal, then $i = j$ and, by definition of P, $p_i = p_j$. Therefore

$$s_a = p_i^{-1} \circ (p_i \circ s_a) = p_i^{-1} \circ (p_j \circ s_b) = p_i^{-1} \circ (p_i \circ s_b) = s_b.$$

Hence the representation is unique.

Therefore $|G|$ is the number of ways of selecting an element from St_s and an element from P, which equals $|St_s| \cdot |O_s|$. ■

7.2 Burnside's Lemma

The following is known as Burnside's Lemma because, in 1911, he was the first to state it explicitly. Apparently it appears implicitly in work of Redfield from 1896. It gives an expression for the number of inequivalent elements in a set acted on by a group G.

For any element $g \in G$, let I_g be the set of elements of S that g leaves fixed, that is,

$$I_g = \{s \in S : g(s) = s\}.$$

Put differently, I_g is the set of elements of $s \in S$ for which g belongs to St_s.

Lemma 7.2.1 [Burnside's Lemma] *Let G be a group that permutes the elements of a set S. Then, the number of different orbits is*

$$\frac{1}{|G|} \sum_{g \in G} |I_g|$$

Proof. The number of orbits is

$$\begin{aligned}
\sum_{O_s \subseteq S} 1 &= \sum_{O_s \subseteq S} \frac{|O_s| \cdot |St_s|}{|G|} \\
&= \frac{1}{|G|} \sum_{O_s \subseteq S} |O_s| \cdot |St_s| \\
&= \frac{1}{|G|} \sum_{O_s \subseteq S} |St_s| \sum_{t \in O_s} 1 \\
&= \frac{1}{|G|} \sum_{s \in S} |St_s|
\end{aligned}$$

For $g \in G$, and $s \in S$, let

$$\delta_{g,s} = \begin{cases} 1 & \text{if } g(s) = s \quad (\text{i.e. } s \in I_g) \\ 0 & \text{else} \end{cases}$$

Then $|St_s| = \sum_{g \in G} \delta_{g,s}$ and $|I_g| = \sum_{s \in S} \delta_{g,s}$, and so

$$\begin{aligned}
\frac{1}{|G|} \sum_{s \in S} |St_s| &= \frac{1}{|G|} \sum_{s \in S} \sum_{g \in G} \delta_{g,s} \\
&= \frac{1}{|G|} \sum_{g \in G} \sum_{s \in S} \delta_{g,s} \\
&= \frac{1}{|G|} \sum_{g \in G} |I_g|.
\end{aligned}$$

■

Continuing with our working example, we compute the numbers $|I_g|$ by identifying which collections of sides must receive the same colour if the coloured square is fixed by the motion.

- All coloured squares are fixed by the identity, so $|I_I| = 2^4$;
- r_{90} fixes coloured squares where consecutive sides are the same colour, so $|I_{r_{90}}| = 2^1$;
- r_{180} fixes coloured squares where opposite sides are the same colour, so $|I_{r_{180}}| = 2^2$;
- r_{270} fixes coloured squares where consecutive sides are the same colour, so $|I_{r_{270}}| = 2^1$;
- f fixes coloured squares where the vertical sides are the same colour, so $|I_f| = 2^3$;
- $f \circ r_{90}$ fixes coloured squares where the top and left side are the same colour, and the bottom and right side are the same colour, so $|I_{f \circ r_{90}}| = 2^2$;

- $f \circ r_{180}$ fixes all coloured squares where the top and bottom sides are the same colour, so $|I_{f\circ r_{180}}| = 2^3$;
- $f \circ r_{270}$ fixes all coloured squares where the top and right side are the same colour, and the bottom and left side are the same colour, so $|I_{f\circ r_{270}}| = 2^2$.

Thus, by Burnside's Lemma, the number of orbits is

$$\frac{1}{8}\left[16+2+4+2+8+4+8+4\right] = \frac{48}{8} = 6,$$

which agrees with our earlier direct calculation. For completeness, we list the sets I_g.

$$\begin{aligned}
I_I &= S \\
I_{r_{90}} &= \{RRRR, BBBB\} \\
I_{r_{180}} &= \{RRRR, BBBB, RBRB, BRBR\} \\
I_{r_{270}} &= \{RRRR, BBBB\} \\
I_f &= \{RRRR, BBBB, RBRB, RBBB, BBRB, RRBR, BRRR, BRBR\} \\
I_{f\circ r_{90}} &= \{RRRR, RBBR, BRRB, BBBB\} \\
I_{f\circ r_{180}} &= \{RRRR, RBRB, BBRB, RBBB.BBBB, BRBR, RRBR, BRRR\} \\
I_{f\circ r_{270}} &= \{RRRR, RRBB, BBRR, BBBB\}.
\end{aligned}$$

Notice that the same arguments as above apply when the number of colours is different from 2, so for example if three colours are used on the sides of the square, then the number of coloured squares that are different with respect to symmetry under G is

$$\frac{1}{8}\left[3^4+3+3^2+3+3^3+3^2+3^3+3^2\right] = \frac{168}{8} = 21.$$

As another example, we count the number of ways the four squares of a 2×2 checkerboard can be painted with red blue, and green. The total number of painted configurations is $3^4 = 81$ if the board is fixed in place. Suppose, however, that we want the number of different configurations if those that differ by a rotation of the board are considered to be the same. The set S is the set of 81 painted configurations. The group G that permutes the elements of S is is the group of rotations of the board, $G = \{I, r_{90}, r_{180}, r_{270}\}$. We want the number of different orbits of the elements of S as permuted by G, so we use Burnside's Lemma. We need to compute the number of elements fixed by each element of G.

- The identity fixes all 81 elements of S, so $|I_I| = 81$;
- r_{90} fixes all elements where squares that share a side are the same colour, so $|I_{r_{90}}| = 3$;
- r_{270} fixes all elements where squares that share a side are the same colour, so $|I_{r_{270}}| = 3$;

- r_{180} fixes all elements where squares on the same diagonal are the same colour, so $|I_{r_180}| = 3^2$.

Therefore by Burnside's Lemma, the number of orbits is $\frac{1}{4}(81 + 3 + 3 + 9) = 24$.

7.3 Conjugacy Classes

As it stands, direct application of Burnside's Lemma requires a summation over a group. This is not difficult to do directly when the group has a small number of elements. When the group is large it can be more cumbersome, so we look for ways to simplify the computation. The idea of conjugate elements in a group is helpful.

Let G be a group, and $x, y \in G$. We say that *x is a conjugate of y* if there exists $z \in G$ such that $x = z \cdot y \cdot z^{-1}$. The "conjugacy" relation on G is formally defined by $x \sim y$ if and only if x is a conjugate of y.

Proposition 7.3.1 *Let G be a group. Conjugacy is an equivalence relation on G.*

Proof. For any $x \in G, x = exe^{-1}$, where e is the identity element of G. Hence conjugacy is reflexive.

Suppose x is a conjugate of y. Then there exists $z \in G$ such that $x = zyz^{-1}$. Therefore $y = z^{-1}xz = uxu^{-1}$, where $u = z^{-1}$, and y is a conjugate of x. Hence conjugacy is symmetric.

To see that conjugacy is transitive, suppose x is a conjugate of y and y is a conjugate of z. Then there exist $u, v \in G$ such that $x = uyu^{-1}$ and $y = vzv^{-1}$. Therefore, $x = uyu^{-1} = uvzv^{-1}u^{-1} = (uv)z(uv)^{-1}$, and so x is a conjugate of z. ■

We know from algebra that every finite group is isomorphic to a group of permutations. We use S_n to denote the *symmetric group of order n*: the set of all permutations of n different objects $1, 2, \ldots, n$ together with the operation of function composition. because of what follows, it will ultimately be helpful to regard our symmetry groups as groups of permutations. For example, the group of symmetries of the coloured squares can be regarded as a group of permutations of the sides of the squares.

Let g and h be two elements of a group of permutations. We say that g and h have the same cycle structure if they have the same number of cycles of each length in their disjoint cycle representations.

Proposition 7.3.2 *Two elements of S_n are conjugates if and only if they have the same cycle structure.*

Proof. Exercise. ■

We can therefore obtain the conjugacy classes of permutations in S_n by organizing according to the lengths of cycles in disjoint cycle notation.

For example, (1)(23)(4567) and (12)(3456)(7) are each the product of a 1-cycle, a 2-cycle, and a 4-cycle. By Proposition 7.3.2, these are conjugates. In general, the conjugacy class of p is the set of permutations that have the same number of cycles of each length as p.

The possible conjugacy classes for elements of S_4 are

- a 4-cycle $\quad 4 = 4$
- a 3-cycle and a 1-cycle $\quad 4 = 3 + 1$
- two 2-cycles $\quad 4 = 2 + 2$
- a 2-cycle and two 1-cycles $\quad 4 = 2 + 1 + 1$
- four 1-cycles $\quad 4 = 1 + 1 + 1 + 1$

Proposition 7.3.3 *The number of conjugacy classes in S_n is $p(n)$, the number of partitions of n.*

Proof. Exercise. ■

One reason why conjugacy and cycle structure are important in summing over a group acting on a set is that conjugate permutations fix the same number of elements. Hence instead of summing over all of G, one can sum over the conjugacy classes.

Proposition 7.3.4 *Let G be a group that permutes the elements of a set S. If g and h are conjugates in G, then $|I_g| = |I_h|$.*

Proof. Suppose $s \in S$ is fixed by h, and let $g = f \circ h \circ f^{-1}$. Then g fixes $f(s)$ because

$$g(f(s)) = f(h(f^{-1}(f(s)))) = f(h(s)) = f(s).$$

Since f permutes the elements of S, there is a 1–1 correspondence between the set of elements fixed by h and the set of elements fixed by g. Thus $|I_g| = |I_h|$. ■

In order to make use of the theory of conjugacy developed so far, we need to find the size of each conjugacy class. The conjugacy class of S_n with λ_1 1-cycles, λ_2 2-cycles, ..., and λ_n n-cycles is denoted by the formal symbol $1^{\lambda_1}2^{\lambda_2}\cdots n^{\lambda_n}$. For example, (135)(27)(4)(6) is in class $1^2 2^1 3^1 4^0 5^0 6^0 7^0$. Since we are counting permutations of n symbols, $1 \cdot \lambda_1 + 2 \cdot \lambda_2 + \cdots + n \cdot \lambda_n = n$.

Theorem 7.3.5 [Cauchy Formula] *The number of permutations in S_n in the conjugacy class $1^{\lambda_1}2^{\lambda_2}\cdots n^{\lambda_n}$ equals*

$$\frac{n!}{\lambda_1!\lambda_2!\cdots\lambda_n! 1^{\lambda_1}2^{\lambda_2}\cdots n^{\lambda_n}}$$

(where $1^{\lambda_1}2^{\lambda_2}\cdots n^{\lambda_n}$ is a number).

Proof. All permutations in class $1^{\lambda_1}2^{\lambda_2}\cdots n^{\lambda_n}$ fit the picture

$$\underbrace{(x)(x)\cdots(x)}_{\lambda_1}\ \underbrace{(xx)(xx)\cdots(xx)}_{\lambda_2}\ \cdots\ \underbrace{(xx\cdots x)}_{\lambda_n}$$

The x's can be filled in with $1,2,\ldots,n$ in $n!$ ways, but this counts every k-cycle k times, and every collection of λ_k k-cycles

$$\lambda_k!\ \underbrace{k\cdot k\cdots k}_{\lambda} = \lambda_k!\,k^{\lambda}$$

times, since there are $\lambda_k!$ ways of permuting the k-cycles (and, as just noted, k orders for each). ■

For example, the number of permutations in S_5 in the class $1^3 2^1$ is $\frac{5!}{3!1!1^3 2^1} = 10$.

7.4 Equivalence Classes of Functions

To start this section, we count some of circular necklaces with n beads and c colours. Number the beads $1,2,\ldots,n$ in clockwise order, so that our group is a collection of some of the permutations of these numbers. The point of doing these examples is to indicate the role of the cycle structure of permutations.

First suppose that $n=6$ and equivalence is up to rotation. Then our group G consists of the permutations

$$\begin{aligned} I &= (1)(2)(3)(4)(5)(6) \\ r &= (123456) \\ r^2 &= (135)(246) \\ r^3 &= (14)(25)(36) \\ r^4 &= (153)(264) \\ r^5 &= (165432). \end{aligned}$$

In order for a necklace to be fixed by a permutation, all beads belonging to the same cycle must be the same colour. This illustrates why the cycle structure of permutations is important, since if our permutation p has t cycles in its disjoint cycle representation, $|I_p| = c^t$. Hence, by Burnisde's Lemma, the number of necklaces is

$$\frac{1}{6}\left[c^6 + 2c + 2c^2 + c^3\right].$$

Now suppose that $n=p$, a prime number and equivalence is (still) up to rotation. Recall that the order of a permutation π (the least positive integer t such that π^t is the identity permutation) is the least common multiple of the cycle lengths in its disjoint cycle representation. Since the collection of powers of π forms a subgroup of the group of rotations, Lagrange's Theorem implies that t divides p, so that $t=1$ or $t=p$. The case $t=1$ occurs only for the identity permutation.

Therefore each non-identity permutation fixes only necklaces where all beads are the same colour, so that, by Burnside's Lemma, the number of necklaces is

$$\frac{1}{p}\left[c^p + (p-1)c\right].$$

The above can be viewed as a combinatorial proof that $p|(c^p + (p-1)c)$ for any positive integer c. Another way to see this is to recognize that $c^p \equiv c \pmod{p}$ by Fermat's (Little) Theorem, and so

$$c^p + (p-1)c \equiv pc \equiv 0 \pmod{p}.$$

Finally, suppose that $n = 6$ and equivalence is up to rotation or flipping (so that the group of symmetries is the dihedral group D_6). Then, in addition to $I, r, r^2, \ldots, r^5$ as listed above, we have the six elements

$$\begin{aligned}
f &= (12)(36)(45) \\
r \circ f &= (13)(2)(46)(5) \\
r^2 \circ f &= (14)(23)(56) \\
r^3 \circ f &= (15)(24)(3)(6) \\
r^4 \circ f &= (16)(25)(34) \\
r^5 \circ f &= (1)(26)(35)(4).
\end{aligned}$$

(This time the group is presented so that the flips occur first; it doesn't matter, it is the same group.) In order for a permutation to fix a necklace, beads belonging to the same cycle must be the same colour. Hence, by Burnside's Lemma, the number of necklaces is

$$\frac{1}{12}\left[c^6 + 2c + 2c^2 + 4c^3 + 3c^4\right].$$

As a check that the sum inside the brackets has the correct number of terms, note that the sum of the coefficients must be the number of elements in the group (here, 12).

Let's abstract from the above examples. There is a collection D of n different items (above: the positions on the necklace), each of which is to be assigned an object from some set R of possibilities (above: a bead of some colour). Really, these are functions from the set of items to the set of possibilities, that is functions $f : D \to R$. There is a group G of permutations of the n elements of the domain D (the collection of n different items). The group G permutes the set of functions $f : D \to R$. We say *two functions f_1 and f_2 are equivalent if there is a permutation π of the elements of the domain so that $f_1 \circ \pi = f_2$.*

These considerations lead to a special case of Polya's Theorem. Like Burnside's Lemma, it had been discovered earlier by someone else.

Theorem 7.4.1 [Polya-Redfield] *Let D and R be a finite sets, and G a group of permutations of the elements of D. The number of equivalence classes of functions $f : D \to R$ permuted by G is*

$$\frac{1}{|G|}\sum_{g\in G} m^{\lambda(g)}$$

where $\lambda(g)$ is the number of cycles in g, and $m = |R|$.

Proof. By Burnside's Lemma, the number of equivalence classes (i.e. orbits) is

$$\frac{1}{|G|}\sum_{g\in G}|I_g|.$$

We need to compute $|I_g|$. In order for a function to be fixed by g, all elements of each cycle of g must receive the same image. Since g has $\lambda(g)$ cycles, this can be done in $m^{\lambda(g)}$ ways. Thus, $|I_g| = m^{\lambda(g)}$, and so

$$\frac{1}{|G|}\sum_{g\in G}|I_g| = \frac{1}{|G|}\sum_{g\in G}m^{\lambda(g)}.$$

■

If we think of the set D as the a set of objects, and the set R as a set of colours, then functions from D to R are assignments of the colours to the objects, that is *colourings* of the elements of D. In what follows, we will stay with the colourings terminology.

7.5 Polya's Theorem: The General Case

We now turn to the general case of Polya's Theorem. Instead of simply counting the number of distinct orbits as in Burnside's Lemma, we may also want to know how many objects there are in each orbit. Polya's Theorem describes a generating function that provides this information.

Let $D = \{d_1, d_2, \ldots, d_n\}$ be a set of n objects that we want to colour with colours from the set $R = \{1, 2, \ldots, m\}$.

For $r = 1, 2, \ldots, m$ assign to colour r the weight $w(r)$. For each colouring C of the objects in D, we define the *weight of* C to be the product of the weights of the colours assigned to the elements of D by C.

For example, suppose $D = \{d_1, d_2, d_3\}$ and $R = \{1, 2\}$. If $C : d_1 \to 1, d_2 \to 2, d_3 \to 2$ is $w(1)w(2)w(2)$, then the weight of C is $w(1)w(2)w(2)$.

If S is a set of colourings of the elements, the *inventory of* S, denoted $inv(S)$, is the sum of the weights of the colourings in S. That is, $mathitinv(S)$ is the generating function for the number of colourings of each type. Depending on the choice of weights, $inv(S)$ could be a generating function in several variables. Making certain specific choices for the weights $w(i)$ can help solve counting problems. The name "inventory" is somehow suggestive: an inventory (of merchandise, say) is a listing of all possible items and how many of each are on hand.

Theorem 7.5.1 *The inventory of the set of all possible colourings of* D *by* R *is* $[w(1) + w(2) + \cdots + w(m)]^n$.

Proof. When the expression is multiplied out, a typical term is a product of n not necessarily distinct factors $w(i_1)w(i_2)\cdots w(i_n)$. This is the weight of the colouring in which d_k gets colour $i_k, 1 \le k \le n$. The expansion contains all such possibilities, once each, so it is the sum of the weights of all possible colourings. ■

The next theorem looks confusing at first. It is helpful to keep in mind the necklaces example. For any permutation of the bead positions, the disjoint cycles representation of the permutation gives a partition of the positions. Any necklace fixed by the permutation must have the property that the same colour of bead must be assigned to all elements of each sets in such a partition.

Theorem 7.5.2 *Suppose $D_1, D_2, \ldots, D_k$ is a partition of D and let S be the set of all colourings of the elements of D by colours in the set $R = \{1, 2, \ldots, m\}$ that assign the same colour to two elements of D if they belong to the same subset. Then*

$$\begin{aligned} inv(S) &= \left[w(1)^{|D_1|} + w(2)^{|D_1|} + \cdots + w(m)^{|D_1|}\right] \\ &\times \left[w(1)^{|D_2|} + w(2)^{|D_2|} + \cdots + w(m)^{|D_2|}\right] \\ &\times \cdots \times \left[w(1)^{|D_k|} + w(2)^{|D_k|} + \cdots + w(m)^{|D_k|}\right] \end{aligned}$$

Proof. When this expression is multiplied out, a typical term is of the form

$$w(i_1)^{|D_1|} w(i_2)^{|D_2|} \cdots w(i_k)^{|D_k|}$$

for k not necessarily distinct numbers $i_1, i_2, \ldots, i_k$. This is the weight of the colouring that assigns colour i_1 to all elements of D_1, colour i_2 to all elements of D_2, etc. The expansion contains all such possibilities, once each, so it equals $inv(S)$. ■

Suppose the weights are all equal to 1. Then the expression in the theorem counts the number of colourings that assign the same colour to all elements of each D_i. This is $(1 + 1 + \cdots + 1)(1 + 1 + \cdots + 1) \cdots (1 + 1 + \cdots + 1) = m^k$ (which is what we get by counting directly).

Suppose that five distinguishable dice numbered one through five are cast. For $i = 1, 2, \ldots, 5$, let d_i be the number that turns up on die number i. We shall count the number of rolls in which $d_1 = d_2 = d_3$, $d_4 = d_5$ and the total is 19. We imagine that the five dice are coloured using the colours in $R = \{1, 2, 3, 4, 5, 6\}$ (colour die i with the number that turns up). Assign weight x^r to colour r. (For example, $d_1 = 3, d_2 = 4, d_3 = 1, d_4 = 1, d_5 = 6$ has weight $x^3 x^4 x^1 x^1 x^6 = x^{3+4+1+1+6} = x^{16}$.) Thus the weight of a colouring is x to the power of the sum of the numbers that turn up. Let $D_1 = \{1, 2, 3\}$ and $D_2 = \{4, 5\}$ be a partition of the dice. By the conditions given, the rolls we are counting correspond to colourings of $D = D_1 \cup D_2$ that assign the same colour to all elements of D_1, and the same colour to all elements of D_2. By Theorem 7.5.2, the inventory of all such rolls is

$$\begin{aligned} & ((x^1)^3 + (x^2)^3 + (x^3)^3 + (x^4)^3 + (x^5)^3 + (x^6)^3) \\ & \times((x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 + (x^5)^2 + (x^6)^2) \\ = \ & (x^3 + x^6 + x^9 + x^{12} + x^{15} + x^{18})(x^2 + x^4 + x^6 + x^8 + x^{10} + x^{12}) \end{aligned}$$

We need the number of elements of weight 19, which equals the coefficient of x^{19} in the product. We get an x^{19} from x^9x^{10} and $x^{15}x^4$, so the coefficient of x^{19} is 2, and this is the answer.

The inventory of a set S is a generating function that tells us how many colourings there are of each type. Our goal is to count equivalence classes of colourings under the action of a group G of permutations of the objects being coloured. Next, we introduce the cycle index polynomial: a generating function for the cycle structure of the permutations in G. These two generating functions (the inventory, and the cycle index polynomial) are brought together in Polya's Theorem.

Let G be a group of permutations of a set D, and let $x_1, x_2, \ldots, x_n$ be indeterminate symbols. For each $g \in G$, form the product

$$p(g) = x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}$$

where b_i is the number of cycles of length i induced by g. The *cycle index polynomial* is defined to be

$$\mathbb{Z}_G(x_1, x_2, \ldots, x_n) = \frac{1}{|G|} \sum_{g \in G} p(g).$$

Notice that each summand in the cycle index polynomial, that is, each term $p(g)$, has the property that

$$1 \cdot b_1 + 2 \cdot b_2 + \cdots + n \cdot b_n = n.$$

For example, let D be the set of sides of an equilateral triangle and let G be the group of rotations of the triangle. Then $G = \{(1)(2)(3), (123), (132)\}$.

$$\left.\begin{array}{lcl} p((1)(2)(3)) & = & x_1^3 \\ p((123)) & = & x_3^1 \\ p((132)) & = & x_3^1 \end{array}\right\} \quad \therefore\ \mathbb{Z}_G(x_1, x_2, x_3) = \frac{1}{3}(x_1^3 + 2x_3)$$

Note that if we let each $x_i = m$ we get the number of orbits of colourings of the sides with m colours. (Compare this with the Polya-Redfield Theorem proved in the previous section.)

The special case of Polya's Theorem that we proved before can be stated as follows.

Theorem 7.5.3 [Polya-Redfield] *Let G be a group of permutations of the elements of a set D, and let S be the set of colourings of the elements of D with colours from $R = \{1, 2, \ldots, m\}$. Then, the number of orbits of colourings of elements of S permuted by G is $\mathbb{Z}_G(m, m, m, \ldots, m)$.*

The following is a generalization of Burnside's Lemma. If all of the weights are equal to 1, it coincides with Burnside's Lemma. Our interest in this theorem is mostly because it is used in the proof of Polya's Theorem.

Theorem 7.5.4 [Weighted version of Burnside's Lemma] *Let G be a group acting on a set S whose elements are weighted by the function w. Suppose that in each orbit O all elements have the same weight, $w(O)$. Then*

$$\sum_{O_x \subseteq S} w(O_x) = \frac{1}{|G|} \sum_{g \in G} \overline{w}(g)$$

where $\overline{w}(g)$ is the sum of the weights of the elements fixed by g.

Proof. For each element x, the number of times $w(x)$ is added into the sum $\sum_{g\in G}\overline{w}(g)$ is the number of elements of g that leave x fixed, or $|St_x| = |G|/|O_x|$, so

$$\frac{1}{|G|}\sum_{g\in G}\overline{w}(g) = \sum_{x\in S}\frac{w(x)}{|O_x|}.$$

Since all elements in the same orbit have the same weight, this sum is

$$\sum_{O_x\subseteq S}\frac{w(x)}{|O_x|}\cdot|O_x|$$

as required. ■

Roughly speaking, Polya's theorem says how the inventory of equivalence classes of colourings of the elements of a set D, under the action of a group of permutations on D, can be obtained from the cycle index polynomial.

Theorem 7.5.5 [Polya's Theorem] *Let S be the set of colourings of the elements of $D = \{d_1, d_2, \ldots, d_n\}$ with colours from the set $R = \{1, 2, \ldots, m\}$, and let G be a group of permutations of the elements of D. Then*

$$inv(S) = \mathbb{Z}_G\left(\sum_{r\in R}w(r), \sum_{r\in R}w(r)^2, \ldots, \sum_{r\in R}w(r)^n\right).$$

Proof. By Theorem 7.5.4, the sum of the weights of the different colourings is

$$\frac{1}{|G|}\sum_{g\in G}\overline{w}(g)$$

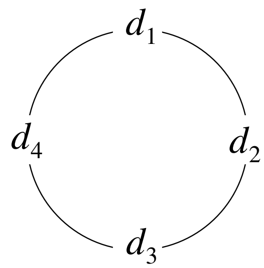
where $\overline{w}(g)$ is the sum of the weights of the elements fixed by g. Let the cycles of g partition D into $D_1, D_2, \ldots, D_k$. Then, the sum of the weights of the colourings fixed by G is

$$\left[w(1)^{|D_1|} + w(2)^{|D_1|} + \cdots + w(m)^{|D_1|}\right]\cdots\left[w(1)^{|D_k|} + w(2)^{|D_k|} + \cdots + w(m)^{|D_k|}\right].$$

Each of these terms is of the form $w_s = w(1)^s + w(2)^s + \cdots + w(m)^s$. The number of factors w_s is exactly the number of cycles of length s. Hence $\frac{1}{|G|}\sum_{g\in G}\overline{w}(g)$ is the same sum given by the RHS. ■

We conclude this section with three examples.

How many necklaces can be made from four beads – two blue, one red, and one yellow – up to equivalence by flips and rotations?



Let $D = \{d_1, d_2, d_3, d_4\}$ be the set of positions of the beads. Define the weights $w(\text{blue}) = b$, $w(\text{red}) = r$, and $w(\text{yellow}) = y$. The group of symmetries of the necklace is the dihedral group D_4. We have $|D_4| = 8$. We need to represent these 8 symmetries as a collection of permutations of the elements of D.

$$\begin{array}{lll}
(d_1)(d_2)(d_3)(d_4) & (d_1d_2d_3d_4) & (d_1)(d_2d_4)(d_3) \\
(d_1d_3)(d_2d_4) & (d_1d_4d_3d_2) & (d_1d_3)(d_2)(d_4) \\
(d_1d_2)(d_3d_4) & (d_1d_4)(d_2d_3) &
\end{array}$$

The cycle index polynomial is therefore

$$\begin{aligned}
\mathbb{Z}_{D_4}(x_1, x_2, x_3, x_4) &= \frac{1}{8} \sum_{g \in D_4} p(g) \\
&= \frac{1}{8}(x_1^4 + x_4^1 + x_1^2 x_2 + x_2^2 + x_4^1 + x_1^2 x_2 + x_2^2 + x_2^2) \\
&= \frac{1}{8}(x_1^4 + 2x_1^2 x_2 + 3x_2^2 + 2x_4^1),
\end{aligned}$$

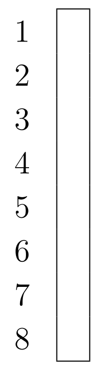
and so, by Polya's Theorem,

$$\begin{aligned}
inv(S) &= \mathbb{Z}_{D_4}(b + r + y, b^2 + r^2 + y^2, b^3 + r^3 + y^3, b^4 + r^4 + y^4) \\
&= \frac{1}{8}((b + r + y)^4 + 2(b + r + y)^2(b^2 + r^2 + y^2) + 3(b^2 + r^2 + y^2)^2 + 2(b^4 + r^4 + y^4)) \\
&= b^4 + r^4 + y^4 + b^3 r + b^3 y + br^3 + r^3 y + by^3 + ry^3 \\
&\quad + 2b^2 r^2 + 2b^2 y^2 + 2r^2 y^2 + 2b^2 ry + 2br^2 y + 2bry^2
\end{aligned}$$

We want the coefficient of $b^2 ry$, which is 2. Note that this generating function solves all similar counting problems. The number of necklaces with exactly two red beads is the sum of the coefficients of all terms involving r^2, which is 6.

Usually we are interested in just one coefficient rather than the whole inventory. The Multinomial Theorem can be helpful in obtaining individual coefficients.

A stick eight feet long is to be painted. Each foot may be painted a different colour. There are m colours to choose from. The only symmetry is rotation through $180°$*.*



Here $D = \{1, 2, \ldots, 8\}$ and $G = \{(1)(2)(3)(4)(5)(6)(7)(8), (18)(27)(36)(45)\}$. The cycle index polynomial is

$$\mathbb{Z}_G(x_1, x_2, \ldots, x_8) = \frac{1}{2}(x_1^8 + x_2^4).$$

Therefore, by Polya's Theorem, the number of ways to paint the stick is $\mathbb{Z}_G(m, m, \ldots, m) = \frac{1}{2}(m^8 + m^4)$.

The number of ways with three feet of blue, three feet of red, and two feet of green is the coefficient of $b^3r^3g^2$ in

$$\mathbb{Z}_G(b + r + g, b^2 + r^2 + g^2, \ldots, b^8 + r^8 + g^8)$$

where b, r, g are the weights of blue, red, and green, respectively. This is

$$\frac{1}{2}\left((b + r + g)^8 + (b^2 + r^2 + g^2)^4\right).$$

The coefficient of $b^3r^3g^2$ is half of the coefficient in $(b + r + g)^8$, since there's no contribution from $(b^2 + r^2 + g^2)^4$ because all of the exponents are even. This equals

$$\frac{1}{2}\binom{8}{332} = \frac{1}{2} \cdot \frac{8!}{3!3!2!} = 280.$$

We mostly use Polya's Theorem and its friends to count equivalence classes of colourings of some physical object, where two colourings are equivalent if there is a symmetry of the object that transforms one into the other. Really, though, the theorem is about equivalence classes of functions under permutations of the domain. Our final example illustrates that.

Find the number of equivalence classes of functions $f : \mathbb{Z}_2^3 \to \mathbb{Z}_2$, where functions g and h are deemed equivalent if $g(x, y, z) = h(y, x, z)$.

It seems clear that the group consists of the identity and the swap of the first two arguments to a function in the collection. There is an important point to notice. When we form the cycle index polynomial, the group has to be presented as a group of permutations of the entire domain $D = \{000, 001, 010, 011, 100, 101, 110, 111\}$. Let's consider the elements of D as the binary representations of the numbers 0 through 7. Then our group is $G = \{(0)(1)\cdots(7), (0)(1)(24)(35)(6)(7)\}$ (which is isomorphic to $\mathbb{Z}_2$), so that the cycle index polynomial is

$$\mathbb{Z}_G(x_1, x_2, \ldots, x_8) = \frac{1}{2}(x_1^8 + x_1^4x_2^2).$$

Since the size of the range is $m = 2$, the Polya-Redfield Theorem tells us that the number of equivalence classes is

$$\mathbb{Z}_G(2, 2, \ldots, 2) = \frac{1}{2}(2^8 + 2^4 2^2) = 170.$$

By contrast, since a function is determined by its values on all elements of the domain, there are $2^8 = 256$ different functions (each of the eight domain elements has two possible images).

If, instead, we want the number of inequivalent functions that send three elements of the domain to 0 and five to 1, then set $wt(0) = a$, $wt(1) = b$ and look for the coefficient of a^3b^5 in

$$\mathbb{Z}_G(a + b, a^2 + b^2, \ldots, a^8 + b^8) = \frac{1}{2}((a + b)^8 + (a + b)^4(a^2 + b^2)^2).$$

The coefficient is

$$\frac{1}{2}\left(\binom{8}{3\ 5} + \binom{4}{1\ 3}\binom{2}{1\ 1} + \binom{4}{3\ 1}\binom{2}{0\ 2}\right) = \frac{1}{2}(56 + 8 + 4) = 34.$$

7.6 Exercises

1. Prove Proposition 7.3.2.

2. Prove Proposition 7.3.3.

3. To *shuffle* a standard 52 card deck means to put the top 26 cards $1, 2, \ldots, 26$ into one pile, the bottom 26 cards $27, 28, \ldots, 52$ into a second pile, and then combine the piles so that the cards are ordered as $1, 27, 2, 28, \ldots, 26, 52$. How many shuffles are required to return a deck to its original order? (Hint: this is a question about permutations.)

4. Prove that
$$\begin{bmatrix} n \\ k \end{bmatrix} = \sum_{\lambda_1+\lambda_2+\cdots+\lambda_n=n} \frac{n!}{\lambda_1!\lambda_2!\cdots\lambda_n! 1^{\lambda_1} 2^{\lambda_2} \cdots n^{\lambda_n}}$$
where $\begin{bmatrix} n \\ k \end{bmatrix}$ is the Stirling number of the first kind.

5. What does Burnside's Lemma say when G consists of only the identity element?

6. Use Burnside's Lemma to show that the number of circular permutations of n different objects is $(n - 1)!$.

7. Use Burnside's Lemma to count the number of different dominoes with two squares, each of which is either blank of has d dots, where $1 \leq d \leq 6$.

8. Give two proofs that the number of ways that m distinct numbers from the set $\{1, 2, \ldots, n\}$ can be arranged in a circle is $\frac{n!}{m\,(n-m)!}$ where arrangements which differ only by rotation are considered the same.

9. How many different ways can we colour a roulette wheel of 15 locations with the three colours red, blue and green? (Such a wheel can be rotated, but not flipped into its mirror image.)

10. How many different ways can the 7 identical horses on a merry-go-round be painted with n colours to choose from? A merry-go-round can be rotated but not flipped. All horses face in the same direction. What is "special" about 7? Generalize your solution as far as you can.

11. A circular merry-go-round with 12 places is to be constructed using horses of three different types: Arabians, Belgians, and Clydesdales. The merry-go-round rotates clockwise, but the horses may be installed facing either direction.

    (a) How many different merry-go-rounds can be constructed?

    (b) How many of these use four Arabians, two Belgians, and six Clydesdales?

12. What does the formula in the Polya-Redfield theorem become when $G = S_n$?

13. Find the number of 2-colourings of the squares of an 8×8 chessboard under equivalence up to rotation.

14. (a) The faces of a cube are labelled Front = 1, Back = 2, Left = 3, Right = 5, Top = 6, Bottom = 4. Some permutations of the integers 1 through 6 correspond to physical rotations of the cube (e.g. (1)(2)(3456)) and others do not (e.g. ((135)(246)). Let G be the subgroup of S_6 which corresponds to the physical rotations of the cube. Show that G has 24 elements.

    (b) How many ways can the six faces of a cube be painted with 6 different colours, each face receiving a different colour, where two painted cubes are considered the same if one can be rotated to be the other?

15. Find the number of equivalence classes of functions $f : \mathbb{Z}_2 \to \mathbb{Z}_2$ if two functions g and h are deemed equivalent if $g(x, y) = h(1 - x, 1 - y)$.

16. Let $n = 4t$. Each unit length of each side of a $n \times n$ square is to be painted with one of m colours. Two painted squares are deemed equivalent if one can be flipped and / or rotated into the other.

    (a) Find the cycle index polynomial $Z_{D_4}(x_1, x_2, x_3, x_4)$.

    (b) Determine the number of equivalence classes of painted squares if two painted squares.

    (c) Suppose $m = 4$. How many of the painted squares in (b) have exactly n unit lengths of each colour? (Remember that n is a multiple of 4.)

17. Two n-digit sequences whose elements are chosen from $\{0, 1, 6, 8, 9\}$ are *rotationally equivalent* if reading one upside down produces the other. For example 689 and 986 are rotationally equivalent. Find the number of rotationally equivalent n-digit sequences.

18. Find the number of different ways to paint the vertices of a cube with the two colours red and blue, with equivalence being with respect to the group of symmetries of the cube.

19. (a) In how many ways can the faces of a tetrahedron be painted from a set of m different colours when two paintings are deemed equivalent if one can be transformed into the other by a symmetry of the tetrahedron?

(b) In how many of the ways in (a) can the faces be painted such that two are green, one is yellow and one is red?

20. Repeat Exercise 19 (a) for the other four platonic solids.

21. You are given a collection of g identical garden gnomes and t identical garden trolls. Determine an expression for the number of circular arrangements of these $g+t$ ornaments.