Lifting factor maps to resolving maps

Ian F. Putnam [*],
Department of Mathematics and Statistics,
University of Victoria,
Victoria, B.C., Canada

Abstract

We consider Smale spaces; that is, homeomorphisms of a compact metric spaces possessing canonical coordinates of contracting (stable) and expanding (unstable) directions. Examples of such dynamical systems include the basic sets for Smale's Axiom A systems. We also assume that each point of the space is non-wandering and that there is a dense orbit. We show that any almost one-to-one factor map between two such systems may be lifted in a certain sense to a factor map which is injective on the local stable sets (i.e. s-resolving). We derive several corollaries. One is a refinement of Bowen's result that every irreducible Smale space is a factor of an irreducible shift of finite type by an almost one-to-one factor map. We are able to show that there exists such a factor which is the composition of an s-resolving map and a u-resolving map.

1 Introduction

In this paper, we consider the dynamical systems called Smale spaces. Smale introduced the notion of an Axiom A system. This is a smooth dynamical system where the restriction of the action to the non-wandering set has a hyperbolic structure. Of critical importance was Smale's observation that even though the system was smooth, the basic sets need not be manifolds. Motivated by this, Ruelle introduced the notion of a Smale space as an attempt to axiomatize the dynamics on a basic set of an Axiom A system.

Roughly speaking, a Smale space (X, d, f) is a compact metric space (X, d) together with a homeomorphism, f, which has canonical coordinates of contracting and expanding directions. Let us explain the central features. There is an absolute positive constant ϵ_X and a map

$$[,] : \{(x, y) \mid x, y \in X, d(x, y) \leq \epsilon_X\} \to X$$

[*]Supported in part by a grant from NSERC, Canada

satisfying various conditions (see [Rue]). There are sets

$$\begin{aligned} V^s(x,\epsilon) &= \{x' \in X \mid [x',x] = x\} \\ V^u(x,\epsilon) &= \{x' \in X \mid [x,x'] = x\} \end{aligned}$$

for every x in X and $0 < \epsilon \le \epsilon_X$, called the local stable and unstable sets of x. For $\epsilon \le \epsilon_X/2$, the map

$$[,] : V^u(x,\epsilon) \times V^s(x,\epsilon) \to X$$

is a homeomorphism to a neighbourhood of x. These are the 'canonical coordinates' in X.

Moreover, there is a constant $0 < \lambda < 1$ such that

$$d(f(y), f(z)) \le \lambda d(y,z), \text{ for all } y,z \in V^s(x,\epsilon),$$

and also

$$d(f^{-1}(y), f^{-1}(z)) \le \lambda d(y,z), \text{ for all } y,z \in V^u(x,\epsilon),$$

which reflects the hyperbolicity of the map.

Every Smale space is expansive, meaning that there is a constant ϵ such that, for all pairs x, x', if $d(f^n(x), f^n(x')) \le \epsilon$ for all integers n implies that $x = x'$. We assume that our constant ϵ_X is chosen sufficiently small to satisfy the expansiveness condition.

An important class of examples is the shifts of finite type [Rue]. In fact, these are exactly the zero-dimensional Smale spaces (see 18.7.8 of [KH]).

We say that a Smale space is irreducible if every point is non-wandering and there is a dense orbit.

We are concerned with maps between such systems. Given two systems (X, f) and (Y, g), a factor map between them is a continuous function

$$\pi : X \to Y$$

such that

$$\pi \circ f = g \circ \pi.$$

We write this as

$$\pi : (X, f) \to (Y, g).$$

Such a map is *finite-to-one* if there is a constant M such that $\#\pi^{-1}\{y\} \le M$, for every y in Y. Such a map is *almost one-to-one* if there is a point y in Y such that $\#\pi^{-1}\{y\} = 1$. Here, $\#A$ denotes the cardinality of the set A.

When the systems are Smale spaces, there are two special classes of factor maps. A map π is s-resolving if $\pi \mid V^s(x,\epsilon)$ is injective for every x in X and some $\epsilon > 0$. Similarly, it is u-resolving if $\pi \mid V^u(x,\epsilon)$ is injective. It is shown in [Put2] that such an s-resolving map is actually a homeomorphism on the local stable sets.

Such maps are finite-to-one. If we compose an s-resolving map with a u-resolving one, the result is finite-to-one, but not resolving in general. It is an interesting and subtle question to ask whether any finite-to-one factor can be decomposed as a composition of (two or more) resolving maps. In [KMT] (or see [LM]), Kitchens gives an example of a finite-to-one factor between two shifts of finite type which has no such decomposition. (Another example of a non-resolving map which has no non-trivial decompositions at all is given in [Boy1].)

In another direction, Trow [T] and Boyle [Boy1] have considered the problem of how many decompositions a given factor map may have.

We pursue a slightly different direction. We show that a finite-to-one map between two irreducible Smale spaces may be lifted to an s-resolving map between two others which factor onto the originals by u-resolving maps. The precise statement is given below.

Before stating this, let us take a moment to describe two sources of motivation. This first is purely dynamical. The well-known result of Bowen [Bow] states that every Smale space is the image of a shift of finite type under an almost one-to-one factor map. Can this be done by resolving maps? Of course, any image of a shift of finite type under an s-resolving map will have local stable sets which are totally disconnected. So the correct formulation is to ask for a composition of (two or more) resolving maps. We show that this is indeed the case as a corollary of our lifting result.

The second motivation comes the study of C^*-algebras. It is possible to construct C^*-algebras from an irreducible Smale space (X, f) [Put1]. There are two principal ones, denoted $S(X, f)$ and $U(X, f)$. Unfortunately, the issue of functoriality of this construction is subtle. In [Put2], it is shown that for s-resolving maps, the construction of S is contra-variant while that of U is co-variant. For an irreducible matrix A which defines a shift of finite type (X_A, f_A) (see [LM]) the algebras S and U are both AF-algebras. Their K_0 groups are the dimension groups associated with the matrices A^T (denoting the transpose of A) and A, respectively. This functoriality at the level of dimension groups has been observed already [BMT].

Let us now state our main result.

Theorem 1.1. Let (X, d_X, f) and (Y, d_Y, g) be irreducible Smale spaces and suppose that

$$\pi : (X, f) \to (Y, g)$$

is an almost one-to-one factor map. Then there exist irreducible Smale spaces, $(\tilde{X}, \tilde{f})$ and $(\tilde{Y}, \tilde{g})$ and factor maps $\alpha, \beta, \tilde{\pi}$ as shown.

$$\begin{array}{ccc} (\tilde{X}, \tilde{f}) & \xrightarrow{\tilde{\pi}} & (\tilde{Y}, \tilde{g}) \\ \alpha \downarrow & & \downarrow \beta \\ (X, f) & \xrightarrow{\pi} & (Y, g) \end{array}$$

Moreover, the diagram is commutative, α and β are u-resolving and $\tilde{\pi}$ is s-resolving.

The proof is very long and we relegate it to the next section. Let us make some simple remarks. If the map π happens to be s-resolving, then we can take $\tilde{\pi} = \pi$, $\alpha = id$ and $\beta = id$. On the other hand, if π is u-resolving, we take $\beta = \pi$, $\alpha = id$ and $\tilde{\pi} = id$. We also note that to have a solution to this lifting problem under the condition $\alpha = id$, is just to factor π as an s-resolving map followed by a u-resolving one.

However, if we accept the Theorem for the moment, we can then obtain our desired corollaries. First, we note the following topological result. Its proof is elementary and we leave it as an exercise for the interested reader.

Lemma 1.2. Let

$$\pi : X \to Y$$

be a continuous, finite-to-one surjection between compact metric spaces. If Y is totally disconnected, then so is X.

Corollary 1.3. Let (Y, g) be an irreducible Smale space such that $V^u(y, \epsilon)$ is totally disconnected for every y in Y and $0 < \epsilon < \epsilon_Y$. Then there is an irreducible shift of finite type (Σ, σ) and an u-resolving factor map

$$\beta : (\Sigma, \sigma) \to (Y, g).$$

Proof. We apply Bowen's result to find an irreducible shift of finite type, (X, f), and an almost one-to-one factor map

$$\pi : (X, f) \to (Y, g).$$

We apply our main result to find $(\tilde{X}, \tilde{f}), \alpha, \beta$ and $\tilde{\pi}$. From Lemma 1.2, $(\tilde{X}, \tilde{f})$ is totally disconnected and hence both local stable and unstable sets in $(\tilde{X}, \tilde{f})$ are totally disconnected. As $\tilde{\pi}$ is s-resolving, it is a homeomorphism on local stable sets. Hence the local stable sets in $(\tilde{Y}, \tilde{g})$ are also totally disconnected.

As the map β is u-resolving, it is a local homeomorphism from the local stable sets in $(\tilde{Y}, \tilde{g})$ to those of (Y, g), which are totally disconnected by hypothesis. We conclude that $(\tilde{Y}, \tilde{g})$ is totally disconnected and hence is a shift of finite type. Letting $(\Sigma, \sigma) = (\tilde{Y}, \tilde{g})$ completes the proof. □

Corollary 1.4. Let (Y, g) be an irreducible Smale space. Then there is another irreducible Smale space, (Ω, ω), an irreducible shift of finite type, (Σ, σ), and two factor maps

$$\pi_1 : (\Sigma, f) \to (\Omega, f)$$

and

$$\pi_2 : (\Omega, f) \to (X, f)$$

such that π_1 is s-resolving while π_2 is u-resolving.

Proof. Again we begin with Bowen's result and let (X, f) be a shift of finite type and

$$\pi : (X, f) \to (Y, g)$$

be an almost one-to-one factor map. We apply our main theorem to obtain $(\tilde{X}, \tilde{f}), \alpha, \beta$ and $\tilde{\pi}$. Again applying 1.2, we see that $(\tilde{X}, \tilde{f})$ is totally disconnected and hence a shift of finite type. If we then let $(\Sigma, \sigma) = (\tilde{X}, \tilde{f})$, $(\Omega, \omega) = (\tilde{Y}, \tilde{g})$, $\pi_1 = \tilde{\pi}$ and $\pi_2 = \beta$, this completes the proof. □

Since a preliminary version of this paper appeared, Mike Boyle [Boy2] has extended these results in the zero-dimensional case, where Smale spaces are shifts of finite type. He also relates this construction with earlier work of Nasu [Na].

We mention two final remarks. First, it seems natural to ask whether, for a given π, there exists a minimal lifting, in an obvious sense. If so, is it unique? Secondly, it would be desirable to extend the result to the case where Y, g is merely expansive. In this case, Y, g is a finitely presented system [Fr]. Boyle has now given affirmative answers to both of these questions in the zero-dimensional case [Boy2].

Acknowledgements: It is a pleasure to thank Mike Boyle and Paul Trow for several helpful discussions. It is also a pleasure to thank the referee for a diligent reading of the paper and many helpful suggestions.

2 Proof of the main result

We begin this section with two basic results on factor maps between Smale spaces. Both are probably known (see [Boy1] for the first), but we can find no proof in the literature set in the generality of Smale spaces.

Lemma 2.1. Suppose that (X, d_X, f) and (Y, d_Y, g) are irreducible Smale spaces and suppose that

$$\pi : (X, f) \to (Y, g)$$

is a finite-to-one factor map. If x, x' are any points in X which are both stably and unstably equivalent and so that $\pi(x) = \pi(x')$, then $x = x'$.

Proof. We provide only a rough sketch of the proof. It relies on the notion of shadowing. We refer the reader to [KH] for details.

We suppose that x and x' are as above, but unequal. We choose ϵ so that $d(x, x') > \epsilon > 0$ and sufficiently small so that $d(z, z') < \epsilon$, for any z, z' in X, implies $d(\pi(z), \pi(z')) < \epsilon_Y/2$. Then we choose $\delta > 0$ such that every δ-pseudo-orbit is ϵ-shadowed by an orbit. Next we use the fact that x and x' are both stably and unstably equivalent to find a positive integer n sufficiently large so that

$$d(f^n(x), f^n(x')), d(f^{-n}(x), f^{-n}(x')) < \delta/2$$

Next, we use irreducibility to choose an point y and positive integer m such that

$$d(y, f^{n+1}(x)), d(f^{m+1}(y), f^{-n}(x)) < \epsilon.$$

Let p and p' denote the respective finite sequences of points in X,

$$\begin{aligned} p &= f^{-n}(x), f^{-n+1}(x), \ldots, f^n(x), y, f(y), \ldots, f^m(y) \\ p' &= f^{-n}(x'), f^{-n+1}(x'), \ldots, f^n(x'), y, f(y), \ldots, f^m(y) \end{aligned}$$

For each positive integer k, we may form a δ-pseudo-orbit:

$$p^\infty.(p')^k p^\infty$$

(here the decimal indicates the space between coordinates -1 and 0). By the shadowing property, this is ϵ-shadowed by the orbit of a point x_k in X. From the choice of ϵ it is easy to see that the x_k are all distinct. It is also easy to see that for any k, k' and integer n

$$d(g^n(\pi(x_k)), g^n(\pi(x_{k'}))) < \epsilon_Y$$

and so $\pi(x_k) = \pi(x_{k'})$, for all k, k' and this contradicts the hypothesis that π is finite-to-one. □

Lemma 2.2. Let

$$\pi : (X, f) \to (Y, g)$$

be a factor map. There exists a positive $\epsilon \leq \epsilon_X$ such that, for all x, x' in X with $d(x, x') \leq \epsilon$, we have $d(\pi(x), \pi(x')) < \epsilon_Y$ and

$$[\pi(x), \pi(x')] = \pi([x, x']).$$

Proof. From the uniform continuity of π and the bracket in Y, we may find $\epsilon' \leq \epsilon_X/2$ such that

$$\begin{aligned} d(\pi(x), \pi(x')), d(\pi(x), [\pi(x), \pi(x')]) &< \epsilon_Y/2, \\ d(\pi(x'), [\pi(x), \pi(x')]) &< \epsilon_Y/2, \end{aligned}$$

whenever $d(x, x') < \epsilon'$. Next, from the uniform continuity of the bracket in X, we choose $0 < \epsilon < \epsilon'$ such that

$$d(x, [x, x']), d(x', [x, x']) < \epsilon',$$

whenever $d(x, x') < \epsilon$. We must verify the conclusion of the Lemma for such x, x'. One checks by induction that for any $n \geq 0$, we have $f^n([x, x'])$ is in $V^s(f^n(x), \epsilon)$ and therefore, in particular,

$$d(f^n(x), f^n([x, x'])) \leq \epsilon',$$

and hence

$$d(g^n(\pi(x)), g^n(\pi([x, x']))) = d(\pi(f^n(x)), \pi(f^n([x, x']))) < \epsilon_Y/2,$$

for all $n \geq 0$. In a similar way, we check that

$$d(g^n(\pi(x')), g^n(\pi([x, x']))) = d(\pi(f^n(x')), \pi(f^n([x, x']))) < \epsilon_Y/2,$$

for all $n \leq 0$.

On the other hand, the same inequalities hold replacing $\pi([x, x'])$ by $[\pi(x), \pi(x')]$. Therefore we see that

$$d(g^n(\pi([x, x'])), g^n([\pi(x), \pi(x')])) \leq \epsilon_Y,$$

for all integers n. By expansiveness, this means that

$$\pi([x, x']) = [\pi(x), \pi(x')]$$

as desired.

□

Henceforth, we replace the constant ϵ_X with the smaller ϵ given in the last Lemma. We may assume that the conclusion of the Lemma holds for $d(x, x') \leq \epsilon_X$. Let us also assume that ϵ_X is chosen sufficiently small so that, for any x, x' in X, $d_X(x, x') < \epsilon_X/2$ implies that $d_Y(\pi(x), \pi(x')) < \epsilon_Y/2$. As each of X, f and Y, g is a Smale space, there are constants $0 < \lambda_X, \lambda_Y < 1$, which control the rate of contraction in each system. We let λ denote the larger of the two. This single value works for both systems.

We now turn to the proof of the main result in section 1. It will be done in a long series of lemmas. We begin by sketching a broad outline of our method of proof. We begin with irreducible Smale spaces (X, d_X, f) and (Y, d_Y, g) and a finite-to-one factor map

$$\pi : (X, f) \to (Y, g).$$

We begin by finding a periodic point in Y with a unique pre-image in X. We define W to be the weak unstable set of this point; that is, all points unstably equivalent to a point in its orbit.

Based on a parameter, ρ, we will define new metrics, δ_X on W and δ_Y on $\pi(W)$. These will involve the map π. The basic idea of these new metrics is to measure distances in the unstable direction exactly as the originals do. But for two stably equivalent points to be close in δ, they must be close in d and be compatible with respect to π in a sense we describe below. Our new spaces will be the completion of W and $\pi(W)$ in these new metrics.

Proposition 2.3. Let $\pi : (X, f) \to (Y, g)$ be an almost one-to-one factor map between irreducible Smale spaces. Then there is a periodic point y_0 in Y such that $\#\pi^{-1}\{y_0\} = 1$.

Proof. Let y be in Y with $\#\pi^{-1}\{y_0\} = 1$. Suppose that $\pi(x) = y$. Let y_n be a sequence of periodic points in Y converging to y. Suppose that for each n, there are $x_n \neq x'_n$ with $\pi(x_n) = \pi(x'_n) = y_n$. By passing to subsequences we may assume that x_n and x'_n are convergent. Their limit points must map to y and hence both sequences converge to x. We choose $\epsilon > 0$ such that every ϵ-pseudo-orbit in X is $\epsilon_X/2$-shadowed by an orbit. Next, select n sufficiently large that $d(x_n, x'_n) < \epsilon$. As y_n is periodic and π is finite-to-one, the points

x_n and x'_n are also periodic. Choose $N \geq 1$ such that $f^N(x_n) = x_n$ and $f^N(x'_n) = x'_n$. Let p and p' denote the finite sequences $f^i(x_n), i = 0, 1, \ldots, N-1$ and $f^i(x'_n), i = 0, 1, \ldots, N-1$, respectively. By making any sequence using p and p', we obtain an ϵ-pseudo-orbit in X. Each is shadowed by an orbit. Moreover, since f is expansive and $x_n \neq x'_n$, $d(f^i(x_n), f^i(x'_n)) \geq \epsilon_X$, for some $0 \leq i < N$. This means that the orbits we obtain in this way are distinct for different sequences of p and p'. On the other hand, an easy argument using $\pi(x_n) = \pi(x'_n)$ and the expansiveness of g shows that they all equal under π. This contradicts π being finite-to-one. Therefore, for some n, $\pi^{-1}\{y_n\}$ contains one point. □

We select a periodic point y_0 in Y such that $\pi^{-1}\{y_0\} = \{x_0\}$. As noted in the last proof, x_0 is also periodic. We then define

$$W = \cup_{n \in \mathbb{Z}} V^u(f^n(x_0))$$

to be the union of the unstable sets of the points in the orbit of x_0. We summarize the properties of W in the following Lemma.

Lemma 2.4.

1. $\pi(W) = \cup_{n \in \mathbb{Z}} V^u(g^n(y_0))$.
2. $\pi^{-1}(\pi(W)) = W$.
3. $f(W) = W$.
4. $g(\pi(W)) = \pi(W)$.
5. If x and x' in W are stably equivalent and $\pi(x) = \pi(x')$, then $x = x'$.

Proof. Let N be the period of x_0; so $g^N(y_0) = y_0$ also. For the first part, the containment $\subset$ is obvious. For the reverse, we suppose that x is in X with $\pi(x)$ in $V^u(g^n(\pi(y_0)))$, for some n in $\mathbb{Z}$. Consider the sequence $f^{-iN}(x), i = 1, 2, \ldots$ in X. Let x' in X be any limit point of this sequence. Select a subsequence indexed by i_k which converges to x'. Applying π, we obtain a sequence which converges to $g^n(y_0)$. It follows that $\pi(f^{-n}(x')) = y_0$ and this implies that $f^{-n}(x') = x_0$. We may then choose an i_k sufficiently large so that $d_X(f^n(x_0), f^{-i_k N}(x)) \leq \epsilon_X$ and $\pi(f^{-i_k N}(x))) \in V^u(g^n(y_0), \epsilon_Y)$. Then both $\pi([f^n(x_0), f^{-i_k N}(x)])$ and $[g^n(y_0), \pi(f^{-i_k N}(x))]$ are defined and hence equal, by Lemma 2.4. But the later is just $g^n(y_0)$. This implies that $[f^n(x_0), f^{-i_k N}(x)]) = f^n(x_0)$. This completes the proof.

The second part follows immediately from the first. The third and fourth parts are clear. As for the last, we suppose that x is in $V^u(f^i(x_0)$ and x' is in $V^u(f^j(x_0)$, with $0 \leq i, j < N$. Consider the sequences $f^{-nN}(x)$ and $f^{-nN}(x')$, $n = 1, 2, \ldots$. These converge to $f^i(x_0)$ and $f^j(x_0)$ respectively. The map π maps these to the same sequence and so their limit points are identified by π. This implies that they are equal. Hence, x and x' are in the same set $V^u(f^i(x_0)$. In particular, they are unstably equivalent to each other. As they are stably equivalent as well and the map π is finite-to-one, they are equal. □

Definition 2.5. Let y_1, y_2 be in $\pi(W)$ with y_2 in $V^s(y_1, \epsilon_Y/2)$. A *compatibility map from y_1 to y_2* is a map

$$\nu : \pi^{-1}\{y_1\} \to \pi^{-1}\{y_2\}$$

such that $\nu(x) \in V^s(x, \epsilon_X/2)$, for all x in $\pi^{-1}\{y_1\}$. For such a map, we define

$$|\nu| = \sup\{d_X(x, \nu(x)) \mid x \in \pi^{-1}\{y_1\} \leq \epsilon_X/2.$$

Lemma 2.6. Let y_1, y_2 be in W with y_2 in $V^s(y_1, \epsilon_Y/2)$. A compatibility map from y_1 to y_2 is injective and unique.

Proof. Let ν be a compatibility map from y_1 to y_2. Suppose x, x' are in $\pi^{-1}\{y_1\}$ and hence in W. The $\pi(x) = \pi(x') = y_1$. On the other hand, x and $\nu(x)$ are stably equivalent as are x' and $\nu(x')$. If $\nu(x) = \nu(x')$, then by part 5 of Lemma 2.4, $x = x'$. As for uniqueness, suppose that ν and ν' are two compatibility maps. Then $\nu(x)$ and $\nu'(x)$ are both in $\pi^{-1}\{y_2\}$, hence in W and both are stably equivalent to x. Again by part 5 of Lemma 2.4, they are equal. □

The following result is an easy consequence of the properties of a Smale space and we omit the proof.

Lemma 2.7. Let y_1, y_2 be in $\pi(W)$ with y_2 in $V^s(y_1, \epsilon_Y/2)$ and let ν be a compatibility map from y_1 to y_2. Then, for any $k \geq 1$, $g^k(y_2)$ in $V^s(g^k(y_1), \lambda^k\epsilon_Y/2) \subset V^s(g^k(y_1), \epsilon_Y/2)$, $f^k \circ \nu \circ f^{-k}$ is a compatibility map from $g^k(y_1)$ to $g^k(y_2)$ and

$$|f^k \circ \nu \circ f^{-k}| \leq \lambda^k |\nu|.$$

Definition 2.8. We say that two points, y_1, y_2 in $\pi(W)$, are *compatible* if

1. $y_2 \in V^s(y_1, \epsilon_Y/2)$ and
2. there are compatibility maps from y_1 to y_2 and from y_2 to y_1.

An argument analogous to those in Lemma 2.6 shows that the two compatibility maps in the definition above must actually be inverses to one another and, in particular, they are bijections. It is worth noting that compatibility is a reflexive and symmetric relation, but may not be transitive.

The notion of compatibility we will use in our definition of the metric is a slightly stronger one. The compatibility should extend between points in the local unstable sets. Based on a parameter ρ, we call this ρ-compatibility and it is stated precisely as follows.

Definition 2.9. Let $0 < \rho < \epsilon_Y/2$. We say that y_1 and y_2 in $\pi(W)$ are ρ-compatible if y_2 is in $V^s(y_1, \epsilon_Y/2)$ and

1. for all y in $V^u(y_1, \rho)$, there is a compatibility map from y to $[y, y_2]$, and
2. for all y in $V^u(y_2, \rho)$, there is a compatibility map from y to $[y, y_1]$.

If it were the case that $[V^u(y_1,\rho),y_2] = V^u(y_2,\rho)$, then the conditions of the definition are simply that y and $[y,y_2]$ are compatible for all y in $V^u(y_1,\rho)$. The formalism above will be easier in practise. Again, ρ-compatibility is reflexive and symmetric, but not necessarily transitive.

Lemma 2.10. If y_1 and y_2 in Y are ρ-compatible then, for all $k \geq 1$, $g^k(y_1)$ and $g^k(y_2)$ are ρ-compatible.

Proof. It clearly suffices to do the case $k=1$. As y_1 and y_2 are ρ-compatible, the latter is in $V^s(y_1,\epsilon_Y/2)$. It follows that $g^k(y_2)$ is in $V^s(g(y_1),\epsilon_Y/2)$. Suppose that y is in $V^u(g(y_1),\rho)$. It follows that $g^{-1}(y)$ is in $V^u(y_1,\rho)$. As y_1 and y_2 are ρ-compatible, there is a compatibility map ν from $g^{-1}(y)$ to $g^{-1}(y_2)$. Then the map $f\circ\nu\circ f^{-1}$ is a compatibility map from y to $[y,g(y_2)]$. The other direction is done in an analogous way. □

The converse of this result is not true. Later, in the definition of the metric, it will be built in. The following result is an easy exercise in topology which we leave to the reader.

Lemma 2.11. Suppose that $\pi: X\to Y$ is a continuous, finite-to-one surjection between compact metric spaces. For y in Y and $\epsilon>0$, there is a $\delta>0$ such that, if x' is in X with $d_Y(\pi(x'),y)<\delta$, then there is x in X with $\pi(x)=y$ and $d_X(x,x')<\epsilon$.

We now want to fix a parameter ρ with certain properties. These are outlined in the following.

Proposition 2.12. There exist $0<\rho<\epsilon_Y/2$, a finite set A and a map $\gamma:\pi(W)\to A$ such that

1. if y_1,y_2 are in $\pi(W)$, y_2 is in $V^s(y_1,\epsilon_Y/2)$ and $\gamma(y_1)=\gamma(y_2)$, then y_1 and y_2 are ρ-compatible,
2. if y_n is a sequence in $\pi(W)\cap V^s(y_0,\epsilon_Y/2)$ converging to y_0, then there is $N\geq 1$ such that, for all $n\geq N$, y_n and y_0 are ρ-compatible.

Proof. First, we choose $0<\epsilon<\epsilon_X/2$ such that for all x,x' in X with $d_X(x,x')<2\epsilon$, $d_X(x,[x,x']),d_X(x,[x',x])<\epsilon_X/2$. By Lemma 2.11, for each y' in Y, we may find $0<\delta_{y'}<\epsilon_Y/2$ such that, if x is in X with $d_Y(\pi(x),y')\leq\epsilon$, there is x' in $\pi^{-1}\{y'\}$ and with $d_X(x,x')<\delta_{y'}$.

The collection $B(y',\delta_{y'}/2), y'\in Y$ is an open cover of Y. We select a finite subcover, $B(y',\delta_{y'}/2), y'\in F$, where $F\subset Y$ is finite. We define

$$A=\bigcup_{y'\in F}2^{\pi^{-1}\{y'\}},$$

where, for any set S, 2^S denotes the collection of all subsets of S. The map γ is defined as follows. Let y be in $\pi(W)$. Select any y' in F with $d_Y(y,y')<\delta_{y'}/2$. Then set

$$\gamma(y)=\{x'\in\pi^{-1}\{y'\}\mid \text{ there is } x\in X, d_X(x',x)<\epsilon, \pi(x)\in V^u(y,\epsilon_Y/2)\}.$$

Note that $\pi(\gamma(y)) = \{y'\}$. The function γ is neither necessarily unique nor continuous. We choose ρ such that $0 < \rho < \min\{\delta_{y_0}/2, \delta_y/2 \mid y \in F\}$.

To show the first conclusion holds, suppose that $\gamma(y_1) = \gamma(y_2)$ and y_2 is in $V^s(y_1, \epsilon_Y/2)$. Let y be in $V^u(y_1, \rho)$. Select y' in F such that $\pi(\gamma(y_1)) = \pi(\gamma(y_2)) = \{y'\}$. This implies that $d_Y(y_1, y'), d_Y(y_2, y') < \delta_{y'}/2$. We define a map $\nu : \pi^{-1}\{y\} \to \pi^{-1}\{[y, y_2]\}$ as follows. Suppose that $\pi(x) = y$. Then we have

$$\begin{aligned} d_Y(\pi(x), y') &\leq d_Y(\pi(x), y_1) + d_Y(y_1, y') \\ &\leq \rho + \delta_{y'}/2 \\ &\leq \delta_{y'}. \end{aligned}$$

From the definition of $\delta_{y'}$, there is x' in X with $\pi(x') = y'$ and $d_X(x, x') < \epsilon$. This means that x' is in $\gamma(y_1) = \gamma(y_2)$ and hence there is z in X such that $\pi(z)$ is in $V^u(y_2, \epsilon_Y/2)$ and $d(x', z) < \epsilon$. We estimate

$$d_X(x, z) \leq d_X(x, x') + d_X(x', z) < 2\epsilon < \epsilon_X.$$

We let $\nu(x) = [x, z]$. By the choice of ϵ, we have $\nu(x)$ is in $V^s(x, \epsilon_X/2)$ and finally, $\pi(\nu(x)) = \pi[x, z] = [\pi(x), \pi(z)] = y_2$, since $\pi(z)$ is in $V^u(y_2, \epsilon_Y/2)$. The case for y in $V^u(y_2, \rho)$ is done analogously.

For the second part, we first note that $\rho < \delta_{y_0}/2$. By the choice of δ_{y_0}, this means that if x' is in X with $d_Y(\pi(x'), y_0) < \rho$, there is x in X with $\pi(x) = y_0$ and $d_X(x, x') < \epsilon$. we may choose N so that, for all $n \geq N$. But x_0 is the unique pre-image of y_0 so we may restate the conclusion in this case as $d(x', x_0) < \epsilon$.

Now suppose that $d_Y(y_n, y_0) < \delta_{y_0}/2$. If y is in $V^u(y_0, \rho)$ and x is in $\pi^{-1}\{y\}$, then $d_X(x, x_0) < \epsilon$. Let x_n be any pre-image of y_n. Then $d_X(x_n, x_0) < \epsilon$. Then we have $d_X(x, x_n) < 2\epsilon$ and we define $\nu(x) = [x, x_n]$. It is easy to check that this is a compatibility map from y to $[y, y_n]$. On the other hand, if y is in $V^u(y_n, \rho)$. Then $d_Y(y, y_0) < \delta_{y_0}$. If x is any point in $\pi^{-1}\{y\}$, then $d_X(x, x_0) < \epsilon$ and we define $\nu(x) = [x, x_0]$. Again it is easy to see this is a compatibility map from y_n to y_0. This completes the proof. □

We make a choice of ρ, A and γ as in the above Proposition and fix them for the remainder of the paper. We are now ready to begin defining our new metric on $\pi(W)$.

Definition 2.13. For any points y_1 and y_2 in $\pi(W)$, a *rectangular path* (or simply a *path*) from y_1 to y_2 is a finite sequence $p = (p_0, \ldots, p_{2l})$, of points in $\pi(W)$ such that

1. $y_1 = p_0$ and $y_2 = p_{2l}$,
2. for all $0 \leq i < l$, $p_{2i+2} \in V^u(p_{2i+1}, \epsilon_Y/2)$, and
3. for all $0 \leq i < l$, $p_{2i+1} \in V^s(p_{2i}, \epsilon_Y/2)$ and p_{2i} and p_{2i+1} are ρ-compatible.

For such a path p, we define its length $l(p)$ to be

$$l(p) = \sum_{i=0}^{2l-1} d_Y(p_i, p_{i+1}).$$

Clearly we can think of a path as being a sequence of 'moves', alternating between moving in the s and u directions in Y, each of distance less than ρ. If one wanted to make a path from x to y by first moving in the u-direction to z and then in the u-direction *again* to get to y, then this could be realized by the path (x, x, z, z, y). That is, we can make a 'trivial' move in the s-direction. This does not alter the length of the path. So our definition which insists on alternating the directions really loses no generality in doing so.

Notice that by the triangle inequality, we have

$$l(p) \geq d_Y(y_1, y_2)$$

for any path from y_1 to y_2.

Let us choose now a positive constant D which is greater than the diameter of both Y and X. Its role is to simply put a bound on the length of paths in Y. At the same time, its use in the following definition takes care of the possibility that there may be no paths between a particular pair of points. Our first definition of δ_Y^0 which follows, is preliminary to our final definition of the new metric on $\pi(W)$.

Definition 2.14. Let y_1 and y_2 be in $\pi(W)$. We define

$$\delta_Y^0(y_1, y_2) = \inf\{D, l(p) \mid p \text{ is a rectangular path from } y_1 \text{ to } y_2\}.$$

Let us observe some elementary properties of δ^0.

Lemma 2.15. 1. $\delta_Y^0(y_1, y_2) \geq d_Y(y_1, y_2)$.

2. δ_Y^0 is a metric on Y.

3. If y_2 is in $V^u(y_1, \epsilon_Y/2)$, then $\delta_Y^0(y_1, y_2) = d(y_1, y_2)$.

4. If y_1 is in $V^s(y_2, \epsilon_Y/2)$ and y_1 and y_2 are ρ-compatible, then

$$\delta_Y^0(y_1, y_2) = d_Y(y_1, y_2).$$

Proof. 1. The first statement follows from the observation earlier that $l(p) \geq d(y_1, y_2)$, for any path p from y_1 to y_2 and the fact that D was chosen to be greater than the diameter of Y.

2. This is a routine type of argument. We omit the details.

3. This is clear from the definition, part 1 and the observation that (y_1, y_1, y_2) is a path from y_1 to y_2 with length equal to $d_Y(y_1, y_2)$.

4. This is clear from the definition, part 1 and the observation that (y_1, y_2, y_2) is a path from y_1 to y_2 with length equal to $d_Y(y_1, y_2)$.

□

The last statement provides a formula on δ_Y^0 for points which are ρ-compatible. We now want a kind of converse, namely that if points y_1 and y_2 in the same local stable set are close in δ_Y^0 then y_1 and y_2 are almost ρ-compatible, in a sense made precise below.

Lemma 2.16. There exists $0 < \epsilon_c < \epsilon_Y/2$ and a positive integer K such that, for any y_1, y_2 in $\pi(W)$ with y_2 in $V^s(y_1, \epsilon_Y/2)$, if

$$\delta_Y^0(y_1, y_2) < \epsilon_c,$$

then $g^K(y_1)$ and $g^K(y_2)$ are ρ-compatible.

Proof. Choose $K \geq 2$ sufficiently large so that $\lambda^K \cdot 2 \cdot \#A < 1$. Next, we claim that there is a constant $0 < \epsilon_c < (1-\lambda)\rho$ such that, for every y_1, y_2 with $d_Y(y_1, y_2) \leq \epsilon_c$, we have

$$[V^u(y_1, \lambda^2\rho), y_2] \subset V^u(y_2, \lambda\rho).$$

This basically follows from the continuity of the bracket and the fact that $\lambda^2\rho < \lambda\rho$. We omit the details. We also require that $\epsilon_c < \epsilon_X/2$ is such that $d_Y(y, y') < \epsilon_c$ implies that $d_Y(y, [y, y']), d_Y(y, [y', y]) < (1-\lambda)\rho$.

Now suppose y_1 and y_2 are as in the statement. We find with a rectangular path $p = (p_0, \ldots, p_{2N})$ in $\pi(W)$ from y_1 to y_2 of length $l(p) < \epsilon_c$. It follows from the definitions and the triangle inequality that $d_Y(p_i, p_j) \leq l(p) < \epsilon_c$, for all $0 \leq i, j \leq 2N$. We consider the sequence $q_n = [y_1, p_{2n}]$, for all $0 \leq n \leq N$. Note that $q_0 = p_0 = y_1$ and $q_N = p_{2N} = y_2$. Easy computations and the choice of ϵ_c show that q_n is in $V^s(y_1, \epsilon_Y/2)$, and in $V^s(q_m, \epsilon_Y/2)$, for all $1 \leq m, n \leq N$. We also estimate $d_Y(q_n, p_{2n}) = d_Y([y_1, p_{2n}], p_{2n}) < (1-\lambda)\rho$. Putting all these things together, we see that, for all $0 \leq n < N$,

$$V^u(q_n, \lambda\rho) \subset V^u(p_{2n}, \rho).$$

We will construct a sequence $0 = n_0 < n_1 < \cdots < n_I = N$ such that $I \leq 2\#A$ and, for all $0 \leq i < I$ and y in $V^u(y_1, \lambda^2\rho)$, a compatibility map, denoted ν_i, from $[y, q_{n_i}]$ to $[y, q_{n_{i+1}}]$. First note that from our choice of ϵ_c, $[y, q_n]$ is in $V^u(q_n, \lambda\rho)$, for all $0 \leq n < N$. We set $n_0 = 0$. The value of n_1 is found as follows. Consider $\gamma(q_0) = \gamma(y_1)$. If there exists $n > 0$ with $\gamma(q_n) = \gamma(q_0)$, then select n_1 to be the largest such n. That is, $\gamma(q_0)$ does not occur as $\gamma(q_{n'})$ for $n' > n_1$. In the case that there is no such n, we set $n_1 = 0 + 1 = 1$. We continue in this fashion. So that for each i, either $\gamma(q_{n_i}) = \gamma(q_{n_{i+1}})$ or else $n_{i+1} = n_i + 1$. Since the values of $\gamma(q_{n_i})$ do not occur for any $n' > n_{i+1}$ in either case, each element of A occurs as $\gamma(q_{n_i})$ for at most two (necessarily consecutive) values of i. From this we conclude that $I \leq 2\#A$. Next, we consider y as above and the maps ν_i. In the case that $n_{i+1} = n_i + 1$, we use the fact that $[y, q_n]$ is in

$V^u(q_n, \lambda\rho)$ and hence in $V^u(p_{2n}, \rho)$. As p_{2n} and p_{2n+1} are compatible, there is a compatibility map ν from $[y, q_n]$ to $[[y, q_n], p_{2n+1}] = [y, p_{2n+1}]$. Finally, we note that p_{2n+1}, p_{2n+2} and q_{n+1} are all in the same local unstable set so $[y, p_{2n+1}] = [y, q_{n+1}]$. In this case, we set $\nu_i = \nu$. In the case that $\gamma(q_{n_i}) = \gamma(q_{n_{i+1}})$, we know from Lemma 2.12 that q_{n_i} and $q_{n_{i+1}}$ are ρ-compatible. Hence there is a compatibility map which we choose for ν_i.

Then, $\nu = \nu_{I-1} \circ \cdots \circ \nu_0$ is a map from $\pi^{-1}\{y\} = \pi^{-1}\{[y, q_{n_0}]\}$ to $\pi^{-1}\{[y, q_{n_I}]\}$. We also estimate, for any x in $\pi^{-1}\{y\}$,

$$\begin{aligned} d_X(f^K(x), f^K(\nu(x))) &\leq \sum_{i=1}^{I-1} d_X(f^K(\nu_{i-1} \circ \cdots \circ \nu_0(x)), f^K(\nu_i \circ \cdots \circ \nu_0(x))) \\ &\leq \sum_{i=1}^{I-1} \lambda^K d_X(\nu_{i-1} \circ \cdots \circ \nu_0(x), \nu_i \circ \cdots \circ \nu_0(x)) \\ &\leq \lambda^K \cdot I \cdot \epsilon_X/2 \\ &\leq \lambda^K \cdot 2\#A\epsilon_X/2 \\ &\leq \epsilon_X/2. \end{aligned}$$

We are now ready to prove that $g^K(y_1)$ and $g^K(y_2)$ are ρ-compatible. Let y' be any point in $V^u(g^K(y_1), \rho)$. Then $y = g^{-K}(y')$ is in $V^u(y_1, \lambda^K\rho) \subset V^u(y_1, \lambda\rho)$. From the argument above the map $f^K \circ \nu_{I-1} \circ \cdots \circ \nu_0 \circ f^{-K}$ is a compatibility map from y' to $[y', g^K(y_2)]$. Reversing the rôles of y_1 and y_2 then shows the other direction. This completes the proof. □

Now we are ready to define our new metric on $\pi(W)$.

Definition 2.17. Fix $0 < r < 1 - \lambda$. For each positive integer k and y_1, y_2 in $\pi(W)$, we define

$$\delta_Y^k(y_1, y_2) = r^k \delta_Y^0(g^{-k}(y_1), g^{-k}(y_2)).$$

We also define

$$\delta_Y(y_1, y_2) = \sum_{k=0}^{\infty} \delta_Y^k(y_1, y_2).$$

Lemma 2.18. Let y_1, y_2 be in $\pi(W)$.

1. $\delta_Y(y_1, y_2) \geq d_Y(y_1, y_2)$.
2. For all $k \geq 0$, δ_Y^k is a metric on $\pi(W)$.
3. δ_Y is a metric on $\pi(W)$.
4. If y_2 is in $V^u(y_1, \epsilon_Y/2)$ then

$$\delta_Y(y_1, y_2) \leq (1 - r\lambda)^{-1} d_Y(y_1, y_2).$$

Proof. All parts, except for 4, follow easily from the definitions and earlier properties of δ_Y^0. As for part 4, we note that if y_2 is in $V^u(y_1, \rho_Y)$, then, for every $k \geq 0$,

$$d_Y(g^{-k}(y_1), g^{-k}(y_2)) \leq \lambda^k d_Y(y_1, y_2).$$

Moreover, $g^{-k}(y_2)$ is in $V^u(g^{-k}(y_1), \rho_Y)$, for every $k \geq 0$. We can apply part 3 of Lemma 2.15 to each term in the series for $\delta_Y(y_1, y_2)$ and the conclusion follows at once. ☐

Suppose that the sequence y_n converges to y. We can consider $[y_n, y]$ and $[y, y_n]$, which are defined for sufficiently large n. Similarly, if k is a fixed positive integer, the expression $[g^{-k}(y_n), g^{-k}(y)]$ is defined for sufficiently large n. We will speak of the sequence $[g^{-k}(y_n), g^{-k}(y)]$, suppressing this subtlety.

Lemma 2.19. Suppose that y_n is a sequence in $\pi(W)$ converging to y in Y, d_Y. Then the following conditions are equivalent.

1. y_n is Cauchy in δ_Y.
2. y_n is Cauchy in δ_Y^k, for all $k \geq 0$.
3. $g^{-k}(y_n)$ is Cauchy in δ_Y^0, for all $k \geq 0$.
4. $[y, y_n]$ is Cauchy in δ_Y.
5. $[y, y_n]$ is Cauchy in δ_Y^k, for all $k \geq 0$.
6. $[g^{-k}(y), g^{-k}(y_n)]$ is Cauchy in δ_Y^0, for all $k \geq 0$.
7. For all $k \geq 0$, there is $N \geq 1$ such that $[g^{-k}(y), g^{-k}(y_n)]$ and $[g^{-k}(y), g^{-k}(y_m)]$ are ρ-compatible for all $m, n \geq N$.

Moreover, if any of these conditions hold, then the sequence obtained by intertwining y_n and $[y, y_n]$ is also Cauchy in δ_Y.

Proof. The equivalence of the first three conditions follows immediately from the definitions. This is also true for the equivalence of the fourth, fifth and sixth conditions.

We will prove that condition (7) implies (3), (6) implies (7) and that (3) and (6) are equivalent. This will complete the proof.

Let us assume (7) and show that (3) holds. Fix $k \geq 0$. Let $N \geq 1$ be as in (7) and assume that $n, m \geq N$. From (7), it follows that

$$(g^{-k}(y_n), g^{-k}(y_n), [g^{-k}(y), g^{-k}(y_n)], [g^{-k}(y), g^{-k}(y_m)], g^{-k}(y_m))$$

is a rectangular path from $g^{-k}(y_n)$ to $g^{-k}(y_m)$. The length of this path is

$$\begin{aligned} & d_Y(g^{-k}(y_n), [g^{-k}(y), g^{-k}(y_n)]) \\ + \quad & d_Y([g^{-k}(y), g^{-k}(y_n)], [g^{-k}(y), g^{-k}(y_m)]) \\ + \quad & d_Y([g^{-k}(y), g^{-k}(y_m)], g^{-k}(y_m)) \end{aligned}$$

which clearly tends to zero for n, m large since y_n converges to y.

Now let us assume that (6) holds. Let $k \geq 0$ be fixed. Using K as given in Lemma 2.16 and applying hypothesis (6) for the sequence $[g^{-k-K}(y), g^{-k-K}(y_n)]$, we may choose N such that, for all $m, n \geq N$,

$$\delta_Y^0([g^{-k-K}(y), g^{-k-K}(y_m)], [g^{-k-K}(y), g^{-k-K}(y_n)]) < \epsilon_c.$$

It follows that

$$d_Y([g^{-k-K}(y), g^{-k-K}(y_m)], [g^{-k-K}(y), g^{-k_K}(y_n)]) < \epsilon_Y/2,$$

for all $m, n \geq N$. It follows from Lemma 2.16 that $g^K([g^{-k-K}(y), g^{-k-K}(y_m)]) = [g^{-k}(y), g^{-k}(y_m)]$ and $g^K([g^{-k-K}(y), g^{-k-K}(y_m)]) = [g^{-k}(y), g^{-k}(y_m)]$ are ρ-compatible. The equivalence of (3) and (6) follows from the triangle inequality and the fact that, for n sufficiently large,

$$\delta_Y^0([g^{-k}(y), g^{-k}(y_n)], g^{-k}(y_n)) = d_Y([g^{-k}(y), g^{-k}(y_n)], g^{-k}(y_n)),$$

by part 3 of Lemma 2.15, and hence tends to zero.

We now turn to the last statement. From part 4 of Lemma 2.18, we have

$$\delta_Y(y_n, [y, y_n]) \leq (1 - r\lambda)^{-1} d_Y(y_n, [y, y_n])$$

which tends to zero as n tends to infinity. This completes the proof. □

Our new space $\tilde{Y}$ will be defined as the completion of $\pi(W)$ in the metric δ_Y. This means that, formally, it consists of equivalence classes of Cauchy sequences. As noted earlier, each sequence is also Cauchy and hence convergent in Y, d_Y. A consequence of the last Lemma is that every equivalence class may be represented by one lying in the local stable set of the limit point.

Lemma 2.20. Let y be in Y and let y_n be sequence in $\pi(W) \cap V^s(y, \epsilon_Y/2)$ which converges to y in d_Y and is Cauchy in δ_Y. Let y' in $V^u(y, \epsilon_Y/2)$. Then the sequence $[y', y_n]$ is Cauchy in δ_Y.

Proof. Let $k \geq 0$. We will show that for sufficiently large m, n, $g^{-k}([y', y_m])$ and $g^{-k}([y', y_n])$ are ρ-compatible. The conclusion then follows from Lemma 2.19. Select a positive integer M such that $2\lambda^M \epsilon_Y < \rho < \epsilon_Y/2$. We may find N such that, for all $m, n \geq N$, $g^{-k-M}(y_m)$ and $g^{-k-M}(y_n)$ are ρ-compatible. By the continuity of the bracket, we may also choose N such that $[y', y_n]$ is in $V^u(y_n, \epsilon_Y)$, for all $n \geq N$. Finally, we require N to be sufficiently large that, for all $n \geq N$, $d_Y(g^{-k}(y), g^{-k}(y_n)) < \epsilon_Y/2$. It follows from the last requirement that, for such n,

$$\begin{aligned} d_Y(g^{-k}(y'), g^{-k}(y_n)) &\leq d_Y(g^{-k}(y'), g^{-k}(y)) + d_Y(g^{-k}(y), g^{-k}(y_n)) \\ &\leq d_Y(y', y) + \epsilon_Y/2 \\ &\leq \epsilon_Y \end{aligned}$$

From this it follows that $g^{-k}([y', y_m]) = [g^{-k}(y'), g^{-k}(y_n)]$.

Let $m, n \geq N$ and let z be in $V^u(g^{-k}([y', y_m]), \rho)$. Then we have

$$\begin{aligned} d_Y(g^{-M}(z), g^{-M-k}(y_m)) &\leq d_Y(g^{-M}(z), g^{-M}([g^{-k}(y'), g^{-k}(y_m)])) \\ &\quad + d_Y(g^{-M}([g^{-k}(y'), g^{-k}(y_m)]), g^{-M-k}(y_m)) \\ &\leq \lambda^M \rho + \lambda^M \epsilon_Y \\ &\leq \rho/2 + \rho/2 \\ &= \rho. \end{aligned}$$

There is a compatibility map ν from $g^{-M}(z)$ to $[g^{-M}(z), g^{-k-M}(y_n)]$. Then, by Lemma 2.7, $f^M \circ \nu \circ f^{-M}$ is a compatibility map from z to $[z, g^{-k}(y_n)]$. Reversing the rôles of y_n and y_m completes the proof. □

At this point, we are ready to begin to define the new metric on W. The starting point is extending the notion of rectangular path to the points W.

Definition 2.21. For any points x_1 and x_2 in W, a *rectangular path* (or simply a *path*) from x_1 to x_2 is a finite sequence $p = (p_0, \ldots, p_{2l})$, of points in W such that

1. $x_1 = p_0$ and $x_2 = p_{2l}$,
2. for all $0 \leq i < l$, $p_{2i+2} \in V^u(p_{2i+1}, \epsilon_X/2)$, and
3. for all $0 \leq i < l$, $p_{2i+1} \in V^s(p_{2i}, \epsilon_X/2)$ and $\pi(p_{2i})$ and $\pi(p_{2i+1})$ are ρ-compatible.

For such a path p, we define its length $l(p)$ to be

$$l(p) = \sum_{i=0}^{2l-1} d_X(p_i, p_{i+1}).$$

The following result is an immediate consequence of the definitions. We omit the proof.

Lemma 2.22. If $p = (p_0, \ldots, p_{2l})$ is a rectangular path in W from x_1 to x_2, then

$$\pi(p) = (\pi(p_0), \ldots, \pi(p_{2l}))$$

is a rectangular path in $\pi(W)$ from $\pi(x_1)$ to $\pi(x_2)$.

We also define δ_X^0 in analogous fashion as for δ_Y^0.

Definition 2.23. Let x_1 and x_2 be in W. We define

$$\delta_X^0(x_1, x_2) = \inf\{D, l(p) \mid p \text{ is a rectangular path from } x_1 \text{ to } x_2\}.$$

We observe some elementary properties of δ_X^0. The proof of the first is is exactly as before and is omitted.

Lemma 2.24. Let x_1, x_2 be in W.

1. $\delta_X^0(x_1, x_2) \geq d_X(x_1, x_2)$.
2. δ_X^0 is a metric on X.
3. If x_2 is in $V^u(x_1, \rho_X)$, then $\delta_X^0(x_1, x_2) = d_X(x_1, x_2)$.
4. If x_1 is in $V^s(x_2, \rho_X)$ and $\pi(x_1)$ and $\pi(x_2)$ are ρ-compatible, then

$$\delta_X^0(x_1, x_2) = d_X(x_1, x_2).$$

Lemma 2.25. There exists $0 < \epsilon_c' < \epsilon_X/2$ and $K' \geq 1$ such that if x_1, x_2 are in W with x_2 in $V^s(x_1, \epsilon_X/2)$ and

$$\delta_X^0(x_1, x_2) < \epsilon_c',$$

then $\pi(x_1)$ and $\pi(x_2)$ are ρ-compatible.

Proof. We choose ϵ_c' such that, for any x_1, x_2 in X, $d_X(x_1, x_2) < \epsilon_c'$ implies $d_y(\pi(x_1), \pi(x_2)) < \epsilon_c$. Now suppose that x_1, x_2 are as in the hypotheses. Let $p = (p_0, \ldots, p_{2N})$ be a rectangular path from x_1 to x_2 with $l(p) < \epsilon_c'$. Then for all $0 \leq i, j \leq 2N$, we have $d_X(p_i, p_j) \leq l(p) < \epsilon_c'$. Then $\pi(p) = (\pi(p_0), \ldots, \pi(p_{2N}))$ is a rectangular path from $\pi(x_1)$ to $\pi(x_2)$ and $d_Y(\pi(p_i), \pi(p_j)) < \epsilon_c$, for all $0 \leq i, j \leq 2N$. The remainder of the argument given in the proof of Lemma 2.16 may be used to show that $g^K(\pi(x_1)) = \pi(f^K(x_1))$ and $g^K(\pi(x_2)) = \pi(f^K(x_2))$ are ρ-compatible. □

Definition 2.26. For each positive integer k and x_1, x_2 in W, we define

$$\delta_X^k(x_1, x_2) = r^k \delta_X^0(f^{-k}(x_1), f^{-k}(x_2)).$$

We also define

$$\delta_X(x_1, x_2) = \sum_{k=0}^{\infty} \delta_X^k(x_1, x_2).$$

The following three results are analogous to earlier ones for the metric δ_Y. The proofs are either also analogous or follow easily from the earlier ones and we omit them.

Lemma 2.27. Let x_1, x_2 be in W.

1. $\delta_X(x_1, x_2) \geq d_X(x_1, x_2)$.
2. For all $k \geq 0$, δ_X^k is a metric on X.
3. δ_X is a metric on X.
4. If x_2 is in $V^u(x_1, \epsilon_x/2)$ then

$$\delta_X(x_1, x_2) \leq (1 - r\lambda)^{-1} d_X(x_1, x_2).$$

Lemma 2.28. Suppose that x_n is a sequence in W converging to x in d_X. Then the following conditions are equivalent.

1. x_n is Cauchy in δ_X.
2. x_n is Cauchy in δ_X^k, for all $k \geq 0$.
3. $f^{-k}(x_n)$ is Cauchy in δ_X^0, for all $k \geq 0$.
4. $[x, x_n]$ is Cauchy in δ_X.
5. $[x, x_n]$ is Cauchy in δ_X^k, for all $k \geq 0$.
6. $[f^{-k}(x), f^{-k}(x_n)]$ is Cauchy in δ_X^0, for all $k \geq 0$.
7. For all $k \geq 0$, there is $N \geq 1$ such that $\pi([f^{-k}(x), f^{-k}(x_n)])$ and $\pi([f^{-k}(x), f^{-k}(x_m)])$ are ρ-compatible for all $m, n \geq N$.

Moreover, if any of these conditions hold, then the sequence obtained by intertwining x_n and $[x, x_n]$ is also Cauchy in δ_X.

Lemma 2.29. Let x be in X and let x_n be sequence in $W \cap V^s(x, \epsilon_X/2)$ which converges to x in d_X and is Cauchy in δ_X. Let x' in $V^u(x, \epsilon_X/2)$. Then the sequence $[x', x_n]$ is Cauchy in δ_X.

Having constructed the metrics δ_Y and δ_X, we are ready to define our new systems, $(\tilde{Y}, \tilde{g})$ and $(\tilde{X}, \tilde{f})$.

Definition 2.30.

1. We define $\tilde{X}$ to be the completion of the metric space (W, δ_X). We use δ_X to denote the natural extension of the metric to $\tilde{X}$.
2. We define $\tilde{Y}$ to be the completion of the metric space $(\pi(W), \delta_Y)$. We use δ_Y to denote the natural extension of the metric to $\tilde{Y}$.

Theorem 2.31. The spaces $\tilde{X}$ and $\tilde{Y}$ are both compact.

Proof. We consider the case of $\tilde{Y}$; that of $\tilde{X}$ is analogous. If y'_n is any sequence in $\tilde{Y}$, we may find y_n in $\pi(W)$ such that $\delta_Y(y'_n, y_n) < \frac{1}{n}$. It then suffices to show that y_n has a convergent subsequence. As Y, d_Y is compact, we may replace y_n by a subsequence which is convergent to some y in Y, d_Y. Next, by replacing y_n by $[y, y_n]$, we may assume that y_n is in $V^s(y, \epsilon_Y/2)$, for all n. In view of Lemma 2.16, this does not alter the property of having a convergent subsequence. We consider the sequence $\gamma(y_n)$. As the range of γ is finite, we may find y_n^0, a subsequence of y_n such that $\gamma(y_n^0)$ is constant. We continue inductively in the same way, choosing y_n^k to be a subsequence of y_n^{k-1} such that $\gamma(g^{-k}(y_n^k))$ is constant, for all $k \geq 1$. Now the sequence y_n^n is a subsequence of y_n and satisfies $\gamma(g^{-k}(y_m^m)) = \gamma(g^{-k}(y_n^n))$, provided $m, n \geq k$, for all $k \geq 0$. For such values of k, m, n, $g^{-k}(y_m^m)$ and $g^{-k}(y_n^n)$ are ρ-compatible, by Lemma 2.12. By Lemma 2.19, this subsequence is Cauchy in δ_Y and hence convergent. □

Theorem 2.32. There are continuous surjections

$$\alpha : \tilde{X} \to X$$

and

$$\beta : \tilde{Y} \to Y$$

such that $\alpha(x) = x$, for all x in W and $\beta(y) = y$, for all y in $\pi(W)$. Moreover, if x_n is any Cauchy sequence in (W, δ_X) converging to $\tilde{x}$ in $\tilde{X}$, then it is also Cauchy and hence convergent in d_X and we have

$$\alpha(\tilde{x}) = \lim_{n\to\infty} x_n,$$

where the limit is taken in X. Also, if y_n is any Cauchy sequence in $(\pi(W), \delta_Y)$ converging to $\tilde{y}$ in $\tilde{Y}$, then it is also Cauchy and hence convergent in d_Y and we have

$$\beta(\tilde{y}) = \lim_{n\to\infty} y_n,$$

where the limit is taken in Y.

Proof. It follows from Lemmas 2.27 and 2.18 that any sequence in W (respectively, $\pi(W)$) which is Cauchy in δ_X (δ_Y, respectively) is also Cauchy in d_X (d_Y, respectively). The existence of the maps follows and their continuity is an easy exercise. Surjectivity follows from continuity, the fact that W maps onto $\pi(W)$ and that these sets are dense. □

Theorem 2.33. The map f extends to a homeomorphism of $\tilde{X}$, which we denote by $\tilde{f}$. The map g extends to a homeomorphism of $\tilde{Y}$, which we denote by $\tilde{g}$.

Proof. We will prove the second part only. The first is done in exactly the same fashion.

Let y_n be a sequence in $\pi(W)$. We will show that it is Cauchy in δ_Y if and only if $g(y_n)$ is. The result follows from this.

First suppose that it is Cauchy. Then by the discussion following Lemma 2.19, we may assume that y_n converges to y in Y and that y_n is in $V^s(y, \epsilon_Y/2)$, for all n. By Lemma 2.19 for every $k \geq 0$, we have N such that $g^{-k}(y_n)$ and $g^{-k}(y_m)$ are ρ-compatible. By Lemma 2.10,the same N which works in condition 7 for $k = 0$ will also work for $k = 1$. Another application of Lemma 2.19 shows that $g(y_n)$ is Cauchy in δ_Y.

On the other hand, we can calculate directly from the definition of δ that

$$\begin{aligned} \delta_Y(g(y_n), g(y_m)) &= \sum_{k\geq 0} r^k \delta_Y^0(g^{1-k}(y_n), g^{1-k}(y_m)) \\ &= \delta_Y^0(g(y_n), g(y_m)) + r\delta_Y(y_n, y_m) \end{aligned}$$

from which it follows that

$$\delta_Y(y_n, y_m) \leq r^{-1}\delta_Y(g(y_n), g(y_m)).$$

So if the sequence $g(y_n)$ is Cauchy in δ_Y, then so is y_n. □

Theorem 2.34. The systems $(\tilde{X}, \delta_X, \tilde{f})$ and $(\tilde{Y}, \delta_Y, \tilde{g})$ are Smale spaces.

Proof. Let us first consider the system $(\tilde{Y}, \tilde{g})$. We choose $\epsilon_{\tilde{Y}} > 0$ such that for all y, y' in Y, $d_Y(y, y') < \epsilon_{\tilde{Y}}$ implies that

$$d_Y(y, y'), d_Y(y, [y', y]), d_Y(y', [y', y]) < \epsilon_Y.$$

If $\tilde{y}$ and $\tilde{y}'$ are two points of $\tilde{Y}$ with $\delta_Y(\tilde{y}, \tilde{y}') < \epsilon_{\tilde{Y}}$, we may find sequences y_n and y'_n in $\pi(W)$ which are converging to $\tilde{y}$ and $\tilde{y}'$, respectively, in δ_Y. These sequences converge to points, say y and y' respectively, in d_Y and we may assume that y_n is in $V^s(y, \epsilon_Y/2)$ and $V^s(y', \epsilon_Y/2)$, respectively.We have

$$d_Y(y, y') = \lim_n d_Y(y_n, y'_n) \leq \lim_n \delta_Y(y_n, y'_n) = \delta_Y(\tilde{y}, \tilde{y}') < \epsilon_{\tilde{Y}}.$$

It follows from Lemma 2.20 (applied to y_n, y and $[y', y]$) that the sequence $[y', y_n] = [[y', y]y_n]$ is Cauchy in δ_Y and we define

$$[\tilde{y}', \tilde{y}] = \lim_n [y', y_n]$$

where the limit is taken in $(\tilde{Y}, \delta_Y)$. The proof that this is well-defined and satisfies all the appropriate conditions for a Smale space is rather routine. We leave the details to the reader.

The proof for the space $\tilde{X}$ is done in a similar way. We omit the details. □

Theorem 2.35. The Smale spaces $\tilde{X}, \tilde{f}$ and $\tilde{Y}, \tilde{g}$ are irreducible.

Proof. We make the following claim: for any point $\tilde{y}$ in $\tilde{Y}$ and $\epsilon > 0$, there are ϵ-pseudo-orbits from y_0 to $\tilde{y}$ and from $\tilde{y}$ to y_0. The conclusion for $\tilde{Y}$ follows from this [KH]. Assume that $\epsilon < \epsilon_Y/2$. We choose y in $\pi(W)$ such that $\delta_Y(y, \tilde{y}) < \epsilon$. From the definition of W, there is $n \leq 0$ such that $g^n(y)$ is in $V^u(y_0, (1 - r\lambda)\epsilon)$. Then by Lemma 2.18,

$$\delta_Y(g^n(y), y_0) \leq (1 - r\lambda)^{-1} d_Y(g^n(y), y_0) < \epsilon,$$

and $y_0, g^{n+1}(y), \ldots, g^{-1}(y), \tilde{y}$ is an ϵ-pseudo-orbit from y_0 to $\tilde{y}$.

For the other direction, we first choose y' in $\pi(W)$ with $\delta_Y(\tilde{g}(\tilde{y}), g(y'))\epsilon_Y/2$. Then, using the irreducibility of Y, g, we choose y'' in the stable set of y_0 sufficiently close to y' that $d_Y(y', [y'', y']) < (1 - r\lambda)\epsilon/2$. We let $y = [y'', y']$, which is in the stable set of y_0, in the unstable set of y' and hence in $\pi(W)$ and has

$$\delta_Y(\tilde{y}, y) \leq \delta(\tilde{y}, y') + \delta_Y(y', y) = \delta_Y(\tilde{y}, y') + (1 - r\lambda) d_Y(y', y) < \epsilon.$$

Now the sequence $g^n(y), n \geq 0$, eventually satisfies the hypothesis of part 2 of Lemma 2.12. Let p be the perioc of y. For every $k \geq 0$, the sequence $g^{kp}(y_n)$ also converges to $g^{kp}(y_0) = y_0$ and the same argument shows that, for n sufficiently large, $g^{kp}(y_n)$ is ρ-compatible to $g^{kp}(y_0$. The same statement is true for any $g^k(y_n)$ and $g^k(y_0)$ by Lemma 2.10. Hence y_n converges to y_0 in δ_Y. We select $n \geq 1$ such that $\delta_Y(g^n(y), y_0) < \epsilon$. Then the sequence $\tilde{y}, g(y), \ldots, g^{n-1}(y), y_0$ is an ϵ-pseudo-orbit from $\tilde{y}$ to y_0.

The proof for $\tilde{X}, \tilde{f}$ is done in an analogous way. We omit the details. □

Theorem 2.36. The maps α and β are u-resolving.

Proof. We prove the statement for α only. Suppose that $\tilde{x}$ and $\tilde{x}'$ are in $\tilde{X}$ and $\tilde{x}'$ is in $V^u(\tilde{x}, \epsilon_{\tilde{X}})$. Also suppose that $\alpha(\tilde{x}) = \alpha(\tilde{x}')$. We will show that $\tilde{x} = \tilde{x}'$.

We find a sequence x_n converging to $\tilde{x}$ in $(\tilde{X}, \delta_X)$. Now x_n must have a limit, say x in X. By definition, $\alpha(\tilde{x}) = x$. Using the fact that $\tilde{x}'$ is in $V^u(\tilde{x}, \epsilon_{\tilde{X}})$ and the definition of the bracket on $\tilde{X}$, we have

$$\begin{aligned} \tilde{x}' &= [\tilde{x}', \tilde{x}] \\ &= \lim_n [\alpha(\tilde{x}'), x_n] \\ &= \lim_n [\alpha(\tilde{x}), x_n] \\ &= \lim_n [x, x_n]. \end{aligned}$$

But for sufficiently large n, we have

$$\begin{aligned} \delta_X(\tilde{x}, [x, x_n]) &\leq \delta_X(\tilde{x}, x_n) \\ &\quad + \delta_X(x_n, [x, x_n]) \\ &\leq \delta_X(\tilde{x}, x_n) \\ &\quad (1 - r\lambda)^{-1} d_X(x_n, [x, x_n]) \end{aligned}$$

by part 4 of Lemma 2.27. Clearly the right hand side above tends to 0 as n tends to infinity. We conclude that

$$\tilde{x}' = \lim_n [x, x_n] = \tilde{x}$$

as desired. □

Theorem 2.37. The map

$$\pi : W \to \pi(W)$$

extends to a continuous map

$$\tilde{\pi} : \tilde{X} \to \tilde{Y}.$$

Moreover, $\tilde{\pi}$ is an s-resolving factor map and we have

$$\beta \circ \tilde{\pi} = \pi \circ \alpha.$$

Proof. We first note that it is an easy consequence of the equivalence of parts 1 and 7 in Lemmas 2.19 and 2.28 that π maps Cauchy sequences in (W, δ_X) to Cauchy sequences in $(\pi(W), \delta_Y)$. It therefore extends to a well-defined map

$$\tilde{\pi} : \tilde{X} \to \tilde{Y}$$

We want to show that $\tilde{\pi}$ is continuous. Suppose that $\tilde{x}_n$ is a sequence in $\tilde{X}$, converging to $\tilde{x}$. We must show that $\tilde{\pi}(\tilde{x}_n)$ converges to $\tilde{\pi}(\tilde{x})$. As Y is compact it suffices to show that the only accumulation point of the sequence $\tilde{\pi}(\tilde{x}_n)$ is $\tilde{\pi}(\tilde{x})$. Suppose that y is some accumulation point. Let us pass to a convergent subsequence, without changing our notation. For each n, we may find a sequence in W converging to $\tilde{x}_n$. The image of this sequence under π then converges to $\tilde{\pi}(\tilde{x}_n)$. So we may find a single point x_n in W such that

$$\delta_X(x_n, \tilde{x}_n), \delta_Y(\pi(x_n), \tilde{\pi}(\tilde{x}_n)) < \frac{1}{n}.$$

Now the sequence x_n is converging to $\tilde{x}$ in $\tilde{X}$. Hence $\pi(x_n)$ is converging in $\tilde{Y}$ to $\tilde{\pi}(\tilde{x})$. On the other hand, it is converging to the same thing as $\tilde{\pi}(\tilde{x}_n)$ which is y. Hence, we see that $\tilde{\pi}(\tilde{x}) = y$ as desired. This completes the proof that $\tilde{\pi}$ is continuous.

The map $\tilde{\pi}$ is onto since it maps W onto a dense subset $\pi(W)$ of $\tilde{Y}$ and since $\tilde{\pi}(\tilde{X})$ is compact and hence closed. It satisfies the equation

$$\tilde{g} \circ \tilde{\pi} = g \circ \pi = \pi \circ f = \tilde{\pi} \circ \tilde{f}$$

on a dense set W in $\tilde{X}$ and hence by continuity

$$\tilde{g} \circ \tilde{\pi} = \tilde{\pi} \circ \tilde{f}$$

We now want to see that it is s-resolving. Suppose that $\tilde{x}$ and $\tilde{x}'$ are in $\tilde{X}$ with $\tilde{x}'$ in $V^s(\tilde{x}, \epsilon_{\tilde{X}})$ and $\tilde{\pi}(\tilde{x}) = \tilde{\pi}(\tilde{x}')$. We choose sequences x_n and x'_n in W converging in δ_X to $\tilde{x}$ and $\tilde{x}'$ respectively with x_n in $V^s(x, \epsilon_{\tilde{X}})$ and x'_n in $V^s(x', \epsilon_{\tilde{X}})$.

By Lemma 2.28, these sequences converge in X. Let

$$\lim_n x_n = x, \lim_n x'_n = x'.$$

By definition $\alpha(\tilde{x}) = x$ and $\alpha(\tilde{x}') = x'$. Also, both sequences $\pi(x_n)$ and $\pi(x'_n)$ converge to $\tilde{\pi}(\tilde{x}) = \tilde{\pi}(\tilde{x}') = \tilde{y}$ in δ_Y. Let $y = \beta(\tilde{y})$ and so we have

$$y = \lim_n \pi(x_n) = \lim_n \pi(x'_n)$$

in d_Y. Then the sequence which intertwines $\pi(x_n)$ and $\pi(x'_n)$ is also Cauchy in δ_Y. Hence for any $k \geq 0$, $\pi(f^{-k}(x_n))$ and $\pi(f^{-k}(x'_n))$ are ρ-compatible.

We first claim that $x = x'$. As f is expansive, we may find k such that

$$d_X(f^{-k}(x'), f^{-k}(x)) > \epsilon_X.$$

As $\tilde{x}'$ is in $V^s(\tilde{x}, \epsilon_{\tilde{X}})$, we see that $x' = \alpha(\tilde{x}')$ is in $V^s(x, \epsilon_X)$ and this means that k cannot be negative. We choose $k \geq 0$ such that the estimate above holds. Next, we choose n such that

$$d_X(f^{-k}(x'_n), f^{-k}(x_n)) > \epsilon_X.$$

Since $\pi(f^{-k}(x_n))$ and $\pi(f^{-k}(x'_n))$ are ρ-compatible, we may find a compatibility map ν from $\pi(f^{-k}(x_n))$ to $\pi(f^{-k}(x'_n))$ with $|\nu| < \epsilon_X$. The points

$\nu(f^{-k}(x_n))$ and $f^{-k}(x'_n)$ are both stably equivalent to $f^{-k}(x_n)$. They cannot be equal because of the estimate above and the fact that

$$d_X(f^{-k}(x_n), \nu(f^{-k}(x'_n))) \leq |\nu| < \epsilon_X.$$

But this means we have two points in W, $\nu(f^{-k}(x_n))$ and $f^{-k}(x'_n)$, which are stably equivalent and have the same image under π. This is a contradiction. We conclude that $x = x'$.

Now it is an easy consequence of Lemma 2.28 that the sequence obtained by intertwining x_n and x'_n is Cauchy in δ_X. This implies that $\tilde{x} = \tilde{x}'$ and we are done. □

References

[AP] J.E. Anderson and I.F. Putnam, *Topological invariants for substitution tilings and their associated C^*-algebras*, Ergodic Theory and Dynamical Systems 18(1998), 509-537.

[BMT] M. Boyle, B. Marcus and P. Trow, *Resolving maps and the dimension group for shifts of finite type*, Memoirs A.M.S., 70, No. 377 (1987).

[Boy1] M. Boyle, *Factoring factor maps*, J. Lond. Math. Soc. (2), 57 (1998), No. 2, 491-502.

[Boy2] M. Boyle, *Putnam's resolving maps in dimension zero*, preprint.

[Bow] R. Bowen, *Markov partitions for Axiom A diffeomorphisms*, Amer. J. Math. 92 (1970), 725-747.

[Fr] D. Fried, *Finitely presented dynamical systems*, Ergod. Th. and Dynam. Sys. 7(1987), 489-507.

[KH] A. Katok and B. Hasselblatt, *Introduction to the modern theory of dynamical systems*, Encyclopedia of math. and its applications, Cambridge University Press, Cambridge, 1995.

[KMT] B. Kitchens, B. Marcus and P. Trow, *Eventual factor maps and compositions of closing maps*, Ergod. Th. and Dynam. Sys. 11 (1991), 85-113.

[LM] D. Lind and B. Marcus, *An introduction to symbolic dynamics and coding*, Cambridge University Press, Cambridge, 1995.

[Na] M. Nasu, *Topological conjugacy for sofic systems and extensions of automorphisms of finite subsystems of topological Markov shifts*, Procedings of Maryland Special Year in Dynamics, 1986-87, Springer Lecture Notes in Mathematics 1342 (1988), 564-607, Springer-Verlag.

[Put1] I.F. Putnam, *C^*-algebras from Smale spaces*, Canad. J. Math. 48 (1996), 175-195.

[Put2] I.F. Putnam, *Functoriality of the C^*-algebras associated with hyperbolic dynamical systems*, J. Lond. Math. Soc. (2) 62 (2000), 873-884.

[Rue] D. Ruelle, *Non-commutative algebras for hyperbolic diffeomorphisms*, Invent. Math. 93 (1988), 1-13.

[Sm] S. Smale, *Differentiable dynamical systems*, Bull. A. M. S. 73(1967), 747-817.

[T] P. Trow, *Decompositions of finite-to-one factor maps*, Israel J. Math. 91 (1995), 129-155.