SUBQUOTIENTS OF HECKE C^*-ALGEBRAS

NATHAN BROWNLOWE, NADIA S. LARSEN, IAN F. PUTNAM, AND IAIN RAEBURN

In [2], Bost and Connes studied a particular Hecke C^*-algebra $\mathcal{C}_\mathbb{Q}$ arising in number theory. The algebra $\mathcal{C}_\mathbb{Q}$ can be realised as a semigroup crossed product $C^*(\mathbb{Q}/\mathbb{Z})\rtimes_\alpha\mathbb{N}^*$ by an endomorphic action α of the multiplicative semigroup $\mathbb{N}^*$ on the group C^*-algebra $C^*(\mathbb{Q}/\mathbb{Z})$ [7], and this realisation has provided useful insight into the analysis of $\mathcal{C}_\mathbb{Q}$ [5, 13]. Since individual elements of $\mathbb{Q}/\mathbb{Z}$ and $\mathbb{N}^*$ involve only finitely many primes, $C^*(\mathbb{Q}/\mathbb{Z})\rtimes_\alpha\mathbb{N}^*$ is the direct limit of subalgebras $C^*(G_F)\rtimes_\alpha\mathbb{N}^F$, where F is a finite set of primes, G_F is the subgroup of $\mathbb{Q}/\mathbb{Z}$ in which the denominators have all prime factors in F, and $\mathbb{N}^F$ acts through the embedding $(n_p)\mapsto\prod_{p\in F}p^{n_p}$ of $\mathbb{N}^F$ in $\mathbb{N}^*$ (see Section 1). One can therefore hope to understand the Hecke algebra $\mathcal{C}_\mathbb{Q}$ in terms of the finite-prime analogues $C^*(G_F)\rtimes_\alpha\mathbb{N}^F$.

Our goal is to analyse the structure of these finite-prime analogues of the Bost-Connes algebra. We started this analysis in [10], where we described a composition series for the two-prime analogue and identified the subquotients in familiar terms: there is a large type I ideal, a commutative quotient isomorphic to $C(\mathbb{T}^2)$, and the intermediate subquotient is isomorphic to a direct sum of Bunce-Deddens algebras. Here we describe a composition series for $C^*(G_F)\rtimes_\alpha\mathbb{N}^F$. Again there are a large type I ideal and a commutative quotient, and the intermediate subquotients are direct sums of simple C^*-algebras. We can describe the simple summands as ordinary crossed products by actions of $\mathbb{Z}^S$ for $S\subset F$. When $|S|=1$, these actions are odometers and the crossed products are Bunce-Deddens algebras; when $|S|>1$, the actions are an apparently new class of higher-rank odometer actions, and the crossed products are an apparently new class of classifiable AT-algebras.

We begin with a short section in which we describe the algebras we intend to study. In §2, we describe our composition series for the semigroup crossed product $C^*(G_F)\rtimes_\alpha\mathbb{N}^F$. It has $|F|+1$ subquotients, and all but two of them are direct sums of algebras stably isomorphic to ordinary crossed products of the form $C(\mathcal{U}(\mathbb{Z}_{F\setminus S}))\rtimes\mathbb{Z}^S$, where $S\subset F$ and $\mathcal{U}(\mathbb{Z}_{F\setminus S})$ is the group of units in the ring $\prod_{p\in F\setminus S}\mathbb{Z}_p$. Our main tools are the analysis of invariant ideals in semigroup crossed products from [9] and some technical lemmas on sums and intersections of ideals in C^*-algebras. We also use the general results of [16] to see that the simple summands are classifiable.

In §3, we show that when $S=\{q\}$ is a singleton, $C(\mathcal{U}(\mathbb{Z}_{F\setminus S}))\rtimes\mathbb{Z}^S$ is a direct sum of finitely many Bunce-Deddens algebras; as in [10], the number of summands depends on the orders of q in the finite groups $\prod_{p\neq q}\mathcal{U}(\mathbb{Z}/p^l\mathbb{Z})$ for large $l\in\mathbb{N}$. We then consider the case where $S=\{q,r\}$. By computing the K-theory of $C(\mathcal{U}(\mathbb{Z}_{F\setminus S}))\rtimes\mathbb{Z}^S$,

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we can see that they are not Bunce-Deddens algebras, for example. We expect these summands to be even harder to recognise when $|S| > 2$.

In §4, we use techniques like those of §2 to identify subquotients of the Bost-Connes algebra $C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_\alpha \mathbb{N}^*$. These include algebras stably isomorphic to $C(\mathcal{U}(\mathbb{Z}_{\mathcal{P}\setminus S})) \rtimes \mathbb{Z}^S$ when S is a cofinite subset of the set $\mathcal{P}$ of all primes; in this case, though, these crossed products are themselves simple, and even though the general theory of [16] no longer applies, we can see using results from [1] that they are classifiable AT-algebras. We finish with a purely number theoretic Appendix in which we identify the orders of an odd integer in the groups $\mathcal{U}(\mathbb{Z}/p^l\mathbb{Z})$ and their products. As in [10, Theorem 3.1], these are needed when we want to identify the number of simple summands in the various subquotients.

We thank N. Christopher Phillips for drawing our attention to [17], and for pointing out that our simple direct summands are AT-algebras rather than general AH-algebras.

1. Preliminaries

We denote by $\mathbb{N}^*$ the semigroup of positive integers under multiplication, and by $\mathbb{N}$ the semigroup of nonnegative integers under addition. It was shown in [7, Proposition 2.1] that there is an action α of $\mathbb{N}^*$ by endomorphisms of $C^*(\mathbb{Q}/\mathbb{Z})$ such that

$$\alpha_n(\delta_r) = \frac{1}{n}\sum_{ns=r} \delta_s \quad \text{for } r \in \mathbb{Q}/\mathbb{Z} \text{ and } n \in \mathbb{N}^*.$$

The corresponding semigroup crossed product $C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_\alpha \mathbb{N}^*$ is isomorphic to the Hecke C^*-algebra $\mathcal{C}_\mathbb{Q}$ of Bost and Connes [7, Corollary 2.10]. We denote by $(i_A, i_{\mathbb{N}^*})$ the canonical covariant representation of $(C^*(\mathbb{Q}/\mathbb{Z}), \mathbb{N}^*, \alpha)$ in $C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_\alpha \mathbb{N}^*$.

Let F be a set of prime numbers. The rational numbers of the form $k\big(\prod_{p\in F} p^{m_p}\big)^{-1}$ form a subgroup of $\mathbb{Q}$, whose image in $\mathbb{Q}/\mathbb{Z}$ we denote by G_F. The integrated form of the map $r \mapsto \delta_r : G_F \to UC^*(\mathbb{Q}/\mathbb{Z})$ is a homomorphism i_F of $C^*(G_F)$ into $C^*(\mathbb{Q}/\mathbb{Z})$; a standard duality argument shows that i_F is injective, so that we can identify $C^*(G_F)$ with the subalgebra $i_F(C^*(G_F))$ of $C^*(\mathbb{Q}/\mathbb{Z})$. When n has all its prime factors in F, α_n leaves this subalgebra invariant, and hence composing α with the map $(m_p)_{p\in F} \mapsto \prod_{p\in F} p^{m_p}$ gives an action of $\mathbb{N}^F$ on $C^*(G_F)$, which we also denote by α. The pair $(i_F, i_{\mathbb{N}^*}|_{\mathbb{N}^F})$ is a covariant representation of $(C^*(G_F), \mathbb{N}^F, \alpha)$ in $C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_\alpha \mathbb{N}^*$. Since i_F is injective, we can deduce from the main theorem of [11] (or by minor modifications to the argument in §3 of [7]) that the corresponding homomorphism

$$i_F \times i_{\mathbb{N}^*}|_{\mathbb{N}^F} : C^*(G_F) \rtimes_\alpha \mathbb{N}^F \to C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_\alpha \mathbb{N}^*$$

is also an injection. We use this injection to identify $C^*(G_F) \rtimes_\alpha \mathbb{N}^F$ with a subalgebra of $C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_\alpha \mathbb{N}^*$.

The crossed product $C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_\alpha \mathbb{N}^*$ is spanned by the elements of the form $i_A(\delta_r)i_{\mathbb{N}^*}(m)i_{\mathbb{N}^*}(n)^*$ [7, Lemma 3.2]. If F contains all the prime factors of m, n and the denominator of r, then $i_A(\delta_r)i_{\mathbb{N}^*}(m)i_{\mathbb{N}^*}(n)^*$ lies in $C^*(G_F) \rtimes_\alpha \mathbb{N}^F$. Thus $C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_\alpha \mathbb{N}^*$ is the direct limit $\overline{\bigcup_F C^*(G_F) \rtimes_\alpha \mathbb{N}^F}$ over increasing finite subsets F of the set $\mathcal{P}$ of prime numbers.

In the next section, we shall describe a composition series for $C^*(G_F)\rtimes_\alpha \mathbb{N}^F$ when F is a finite subset of $\mathcal{P}$, and identify the subquotients in terms of ordinary crossed products $C(X_S)\rtimes\mathbb{Z}^S$ associated to subsets S of F. The underlying space X_S is the group of units $\mathcal{U}(\mathbb{Z}_{F\setminus S})$ in the ring $\mathbb{Z}_{F\setminus S}:=\prod_{p\in F\setminus S}\mathbb{Z}_p$; as an additive group, $\mathbb{Z}_{F\setminus S}$ is the dual group of $G_{F\setminus S}$. The action of a prime $q\in S$ on

$$C(\mathcal{U}(\mathbb{Z}_{F\setminus S}))\subset C(\mathbb{Z}_{F\setminus S})\cong C^*(G_{F\setminus S})\subset C^*(\mathbb{Q}/\mathbb{Z})$$

induced by α_q is multiplication by q on $\mathcal{U}(\mathbb{Z}_{F\setminus S})$ (see [10, Lemma 1.1]), which because q is a unit in $\mathbb{Z}_{F\setminus S}$ is an automorphism. Thus the action of $\mathbb{N}^S$ on $C(\mathcal{U}(\mathbb{Z}_{F\setminus S}))$ extends to an action σ of $\mathbb{Z}^S$ such that

$$\sigma_{(m_p)}(f)(x)=f\big(\big(\textstyle\prod_{p\in S}p^{m_p}\big)^{-1}x\big)\text{ for }(m_p)\in\mathbb{N}^S.$$

As a matter of notation, we shall view a crossed product $A\rtimes_\beta G$ by an action of a group as the universal C^*-algebra generated by a copy of A and a unitary representation $i_G: G\to U(A\rtimes_\beta G)$ satisfying the covariance relation $\beta_s(a)=i_G(s)ai_G(s)^*$.

2. Finitely many primes

The object of this section is to prove the following theorem. For the definitions of AT-algebra, real rank zero and stable rank one, see [20] and the references given there.

Theorem 2.1. *Let F be a finite set of primes. Then there is a composition series $\{I_k\mid 1\le k\le |F|\}$ of ideals in $C^*(G_F)\rtimes_\alpha\mathbb{N}^F$ such that*

(a) $I_1\cong C(\mathcal{U}(\mathbb{Z}_F),\mathcal{K}(l^2(\mathbb{N}^F)))$;
(b) $I_{k+1}/I_k\cong\bigoplus_{S\subset F,\,|S|=k}\big(C(\mathcal{U}(\mathbb{Z}_{F\setminus S}))\rtimes_\sigma\mathbb{Z}^S\big)\otimes\mathcal{K}(l^2(\mathbb{N}^{F\setminus S}))$;
(c) $(C^*(G_F)\rtimes\mathbb{N}^F)/I_{|F|}\cong C(\mathbb{T}^F)$.

Each $C(\mathcal{U}(\mathbb{Z}_{F\setminus S}))\rtimes_\sigma\mathbb{Z}^S$ is a finite direct sum of simple AT-algebras with real rank zero and a unique tracial state.

The proof of the theorem will occupy the rest of the section. We will need some notation and a number of preliminary results.

Under the Fourier transform $C^*(G_F)\cong C(\mathbb{Z}_F)$ the action α becomes

$$\alpha_{(n_p)}(f)(x)=\begin{cases} f\big(\big(\prod_{p\in F}p^{n_p}\big)^{-1}x\big) & \text{if } x\in\big(\prod_{p\in F}p^{n_p}\big)\mathbb{Z}_F\\ 0 & \text{otherwise}\end{cases}$$

(see [10, Lemma 1.1]). For $S\subset F$, we set $\mathcal{Z}_S:=\{a\in\mathbb{Z}_F\mid a_p=0\text{ for }p\in S\}$, and we write $\mathcal{Z}_p$ for $\mathcal{Z}_{\{p\}}$. The next lemma identifies $C_0(\mathbb{Z}_F\setminus\mathcal{Z}_S)$ as the kind of ideal for which taking crossed products behaves well (see [9]).

Lemma 2.2. *For $S\subset F$, $C_0(\mathbb{Z}_F\setminus\mathcal{Z}_S)$ is an extendibly invariant ideal in $C(\mathbb{Z}_F)$.*

Proof. It suffices by [9, Theorem 4.3] to show that for each $n\in\mathbb{N}^F$, the endomorphism $x\mapsto\big(\prod_{p\in F}p^{n_p}\big)x$ of $\mathbb{Z}_F$ leaves both $\mathcal{Z}_S$ and $\mathbb{Z}_F\setminus\mathcal{Z}_S$ invariant. Certainly $\big(\prod_{p\in F}p^{n_p}\big)\mathcal{Z}_S$ is contained in $\mathcal{Z}_S$. If $x\notin\mathcal{Z}_S$, then $x_r\neq 0$ for some $r\in S$, $\prod_{p\in F}p^{n_p}x_r\neq 0$ for this r, and $\big(\prod_{p\in F}p^{n_p}\big)x\notin\mathcal{Z}_S$. □

Theorem 1.7 of [9] now allows us to identify $C_0(\mathbb{Z}_F \setminus \mathcal{Z}_S) \rtimes \mathbb{N}^F$ with an ideal J_S in $C(\mathbb{Z}_F) \rtimes_\alpha \mathbb{N}^F$ such that $(C(\mathbb{Z}_F) \rtimes_\alpha \mathbb{N}^F)/J_S = C(\mathcal{Z}_S) \rtimes \mathbb{N}^F$; we write J_p for $J_{\{p\}}$.

Lemma 2.3. $J_S = \sum_{p\in S} J_p$.

Proof. Since $\mathcal{Z}_S = \bigcap_{p\in S} \mathcal{Z}_p$, we have $\mathbb{Z}_F \setminus \mathcal{Z}_S = \bigcup_{p\in S} \mathbb{Z}_F \setminus \mathcal{Z}_p$, and $C_0(\mathbb{Z}_F \setminus \mathcal{Z}_S) = \sum_{p\in S} C_0(\mathbb{Z}_F \setminus \mathcal{Z}_p)$. It follows from Lemma 1.3 of [9] that if I, J and $I + J$ are extendibly invariant ideals in (A, P), then $(I + J) \rtimes P = (I \rtimes P) + (J \rtimes P)$. Thus the result follows from Lemma 2.2. □

For $1 \leq k \leq |F|$, we define

$$I_k := \prod_{S\subset F,\, |S|=k} J_S = \bigcap_{S\subset F,\, |S|=k} J_S. \tag{2.1}$$

It follows from [9, Lemma 1.3] that if I and J are extendibly invariant ideals in (A, P), then

$$(I \rtimes P)(J \rtimes P) = (IJ) \rtimes P,$$

and hence $I_k = C_0\big(\bigcap_{S\subset F,\, |S|=k}(\mathbb{Z}_F \setminus \mathcal{Z}_S)\big) \rtimes \mathbb{N}^F$. Therefore

$$I_1 = C_0\big(\bigcap_{p\in F}(\mathbb{Z}_F \setminus \mathcal{Z}_p)\big) \rtimes \mathbb{N}^F = C_0\big(\prod_{p\in F}(\mathbb{Z}_p \setminus \{0\})\big) \rtimes \mathbb{N}^F;$$

since $\mathbb{Z}_p \setminus \{0\}$ is homeomorphic to $\mathcal{U}(\mathbb{Z}_p) \times \mathbb{N}$ by [10, Lemma 2.3], part (a) of Theorem 2.1 follows from an argument similar to the one in the last paragraph of [10, page 176]. Similarly, we can prove part (c) by following the proof of (2.4) of [10], because $\big(C^*(G_F) \rtimes_\alpha \mathbb{N}^F\big)/I_{|F|} = \mathbb{C} \rtimes \mathbb{N}^F$.

To prove part (b) of Theorem 2.1, we need some lemmas. The first contains some general facts about families of ideals in C^*-algebras.

Lemma 2.4. *Suppose that $I_1, \dots, I_n$ are ideals in a C^*-algebra B.*

(a) *With $F_n = \{1, \dots, n\}$, we have*

$$\prod_{S\subset F_n,\, |S|=k} \big(\textstyle\sum_{i\in S} I_i\big) = \sum_{R\subset F_n,\, |R|=n-k+1} \big(\textstyle\prod_{j\in R} I_j\big) \text{ for } 1 \leq k \leq n. \tag{2.2}$$

(b) *Suppose that K is an ideal such that $I_i I_j \subset K$ for all i, j. Then $(\sum_{i=1}^n I_i)/K$ is naturally isomorphic to $\bigoplus_{i=1}^n (I_i/I_i \cap K)$.*

Proof. We prove (a) by induction on n. The statement is trivial for $n = 1, 2$. Suppose it holds for $n - 1$. When $k = 1$, both sides of (2.2) are $\prod_{i=1}^n I_i$, so we assume $k \geq 2$. Writing the left-hand side LHS of (2.2) as $(\prod_{n\in S})(\prod_{n\notin S})$ and applying the inductive hypothesis to F_{n-1} shows that

$$\text{LHS} = \Big(\prod_{|S|=k, n\in S} \big(I_n + \textstyle\sum_{i\in S\setminus\{n\}} I_i\big)\Big)\Big(\sum_{R\subset F_{n-1},\, |R|=n-k} \big(\textstyle\prod_{j\in R} I_j\big)\Big). \tag{2.3}$$

Because I_n is an ideal and $I_n^2 = I_n$, the first term of (2.3) simplifies to give

$$\text{LHS} = \Big(I_n + \prod_{S'\subset F_{n-1},\, |S'|=k-1} \big(\textstyle\sum_{i\in S'} I_i\big)\Big)\Big(\sum_{R\subset F_{n-1},\, |R|=n-k} \big(\textstyle\prod_{j\in R} I_j\big)\Big).$$

We can use the inductive hypothesis on F_{n-1} with k replaced by $k-1$ to get

$$\text{LHS} = \Big(I_n + \sum_{R'\subset F_{n-1},\,|R'|=n-k+1} \big(\textstyle\prod_{j\in R'} I_j\big)\Big)\Big(\sum_{R\subset F_{n-1},\,|R|=n-k} \big(\textstyle\prod_{j\in R} I_j\big)\Big), \tag{2.4}$$

which is contained in

$$\sum_{R\subset F_{n-1},\,|R|=n-k} \big(\textstyle\prod_{j\in R\cup\{n\}} I_j\big) + \sum_{R'\subset F_{n-1},\,|R'|=n-k+1} \big(\textstyle\prod_{j\in R'} I_j\big). \tag{2.5}$$

Since (2.5) is the same as the right-hand side RHS of (2.2), LHS $\subset$ RHS. On the other hand, every element of every $\prod_{j\in R'} I_j$ arises in (2.4) because we can pick $R\subset R'$, so RHS $\subset$ LHS.

To prove (b), note that the map $\phi_i : a+I_i\cap K \mapsto a+K$ is an injection of $I_i/(I_i\cap K)$ into $\big(\sum_{i=1}^n I_i\big)/K$, and

$$\phi_i(a+I_i\cap K)\phi_j(b+I_j\cap K) = ab+K = 0 \text{ for } i\neq j$$

because $ab\in I_iI_j\subset K$. So the ϕ_j combine to give an injection ϕ of $\bigoplus(I_i/I_i\cap K)$ into $(\sum_{i=1}^n I_i)/K$, which is clearly surjective. □

Lemma 2.5. *The ideals I_k of $C^*(G_F)\rtimes_\alpha \mathbb{N}^F$ defined in* (2.1) *satisfy*

$$I_{k+1}/I_k = \bigoplus_{S\subset F,\,|S|=k} \big(\textstyle\bigcap_{p\notin S} J_{S\cup\{p\}}\big)/J_S.$$

Proof. Lemma 2.4 (a) gives $I_{k+1} = \sum_{R\subset F,\,|R|=n-k}(\prod_{p\in R} J_p)$. The product of any two ideals $\prod_{p\in R} J_p$ with $|R|=n-k$ has at least $n-k+1$ factors J_p, and hence is contained in $I_k = \sum_{R\subset F,\,|R|=n-k+1}(\prod_{p\in R} J_p)$. Thus part (b) of Lemma 2.4 gives

$$I_{k+1}/I_k = \bigoplus_{R\subset F,\,|R|=n-k} \frac{\prod_{p\in R} J_p}{I_k\cap\big(\prod_{p\in R} J_p\big)}. \tag{2.6}$$

Now

$$I_k\cap\big(\textstyle\prod_{p\in R} J_p\big) = \sum_{|T|=n-k+1}\big(\prod_{q\in T} J_q\big)\big(\prod_{p\in R} J_p\big);$$

each of these summands has at least one factor J_q for $q\notin R$, and is then contained in $J_q(\prod_{p\in R} J_p)$. Using $I\cap J = IJ$ again gives

$$I_k\cap\big(\textstyle\prod_{p\in R} J_p\big) = \sum_{q\notin R} J_q\big(\prod_{p\in R} J_p\big) = \big(\sum_{q\notin R} J_q\big)\big(\prod_{p\in R} J_p\big),$$

and using the isomorphism $(I+J)/I = J/(I\cap J)$ and Lemma 2.3 gives

$$\frac{\prod_{p\in R} J_p}{I_k\cap(\prod_{p\in R} J_p)} = \frac{J_{F\setminus R} + \big(\prod_{p\in R} J_p\big)}{J_{F\setminus R}}.$$

Finally we observe that

$$J_{F\setminus R} + \big(\textstyle\prod_{p\in R} J_p\big) = \prod_{p\in R}\big(J_{F\setminus R}+J_p\big) = \prod_{p\in R} J_{(F\setminus R)\cup\{p\}}$$

and write $S = F\setminus R$ to deduce the result. □

Lemma 2.6. *The ideals J_S in $C^*(G_F)\rtimes_\alpha \mathbb{N}^F$ satisfy*

$$\big(\textstyle\bigcap_{p\in F\setminus S} J_{S\cup\{p\}}\big)/J_S \cong \big(C(\mathcal{U}(\mathbb{Z}_{F\setminus S}))\rtimes_\sigma \mathbb{Z}^S\big)\otimes\mathcal{K}(l^2(\mathbb{N}^{F\setminus S})).$$

Proof. We first realise $\big(\bigcap_{p\in F\setminus S} J_{S\cup\{p\}}\big)/J_S$ as a semigroup crossed product:

$$\begin{aligned}\textstyle\bigcap_{p\in F\setminus S} J_{S\cup\{p\}} &= C_0\big(\textstyle\bigcap_{p\in F\setminus S}(\mathbb{Z}_F\setminus \mathcal{Z}_{S\cup\{p\}})\big)\rtimes\mathbb{N}^F\\ &= C_0\big(\mathbb{Z}_F\setminus\big(\textstyle\bigcup_{p\in F\setminus S}\mathcal{Z}_{S\cup\{p\}}\big)\big)\rtimes\mathbb{N}^F.\end{aligned}$$

Thus

$$\begin{aligned}\big(\textstyle\bigcap_{p\in F\setminus S} J_{S\cup\{p\}}\big)/J_S &= C_0\big(\mathcal{Z}_S\setminus\big(\textstyle\bigcup_{p\in F\setminus S}\mathcal{Z}_{S\cup\{p\}}\big)\big)\rtimes\mathbb{N}^F\\ &= C_0\big(\big(\textstyle\prod_{p\in F\setminus S}\mathbb{Z}_p\setminus\{0\}\big)\times\big(\textstyle\prod_{p\in S}\{0\}\big)\big)\rtimes\mathbb{N}^F.\end{aligned}$$

The arguments of Corollary 2.4 and Lemma 2.5 of [10] show that this last crossed product is isomorphic to $\big(C(\mathcal{U}(\mathbb{Z}_{F\setminus S}))\rtimes_\sigma\mathbb{Z}^S\big)\otimes\mathcal{K}(l^2(\mathbb{N}^{F\setminus S}))$. □

Part (b) of Theorem 2.1 follows immediately from Lemmas 2.5 and 2.6.

To finish the proof of Theorem 2.1, it remains to prove the statements about the structure of $C(\mathcal{U}(\mathbb{Z}_{F\setminus S}))\rtimes_\sigma\mathbb{Z}^S$. Corollary A.6 implies that $H:=\overline{\mathbb{Z}^S}$ has finite index in $\mathcal{U}(\mathbb{Z}_{F\setminus S})$. The argument at the end of the proof of [10, Theorem 3.1] shows that $C(\mathcal{U}(\mathbb{Z}_{F\setminus S}))\rtimes_\sigma\mathbb{Z}^S$ is a finite direct sum of algebras isomorphic to $C(H)\rtimes_\sigma\mathbb{Z}^S$, which is simple because $\mathbb{Z}^S$ acts minimally and freely on H. Since H is an open and closed subset of $\mathcal{U}(\mathbb{Z}_{F\setminus S})$, it is totally disconnected, and it follows from [16, Theorem 6.11] that $C(H)\rtimes_\sigma\mathbb{Z}^S$ has real rank zero and stable rank one.

The space $\mathcal{U}(\mathbb{Z}_{F\setminus S})$ is the inverse limit of the finite groups $\mathcal{U}\big(\mathbb{Z}/\big(\prod_{p\in F\setminus S}p^{l_p}\big)\mathbb{Z}\big)$ over $l=(l_p)\in\mathbb{N}^{F\setminus S}$. The diagonally embedded copy of $\mathbb{N}$ is cofinal in $\mathbb{N}^{F\setminus S}$, and hence

$$\mathcal{U}(\mathbb{Z}_{F\setminus S})=\varprojlim\mathcal{U}\big(\mathbb{Z}/\big(\textstyle\prod_{p\in F\setminus S}p^n\big)\mathbb{Z}\big). \tag{2.7}$$

Let π_n denote the canonical surjection of $\mathcal{U}(\mathbb{Z}_{F\setminus S})$ onto $\mathcal{U}\big(\mathbb{Z}/\big(\prod_{p\in F\setminus S}p^n\big)\mathbb{Z}\big)$.

Lemma 2.7. *Let $H_n:=\pi_n(H)\subset\mathcal{U}\big(\mathbb{Z}/\big(\prod_{p\in F\setminus S}p^n\big)\mathbb{Z}\big)$ and let $\mathbb{Z}^S$ act on H_n via the embedding $(n_q)\mapsto\prod_{q\in S}q^{n_q}$ of $\mathbb{Z}^S$ in $\mathbb{Z}$. Then there are C^*-subalgebras A_n of $C(H)\rtimes_\sigma\mathbb{Z}^S$ such that $A_n\cong C(H_n)\rtimes\mathbb{Z}^S$ and $C(H)\rtimes_\sigma\mathbb{Z}^S=\overline{\bigcup A_n}$.*

Proof. The homomorphism π_n induces an injection π_n^* of $C(H_n)$ into $C(H)$, and then $C(H)=\overline{\bigcup_{n\in\mathbb{N}}\pi_n^*(C(H_n))}$. On $\mathbb{Z}^S\subset H$, π_n is reduction modulo $\prod_{p\in F\setminus S}p^n$, so π_n^* converts the action σ into the canonical action of $\mathbb{Z}^S$ by multiplication on H_n. Thus π_n^* induces a homomorphism $\pi_n^*\rtimes\mathrm{id}$ of $C(H_n)\rtimes\mathbb{Z}^S$ into $C(H)\rtimes_\sigma\mathbb{Z}^S$. The homomorphism π_n^* is faithful on $C(H_n)$ and intertwines the dual actions, and hence a standard argument shows that $\pi_n^*\rtimes\mathrm{id}$ is faithful on $C(H_n)\rtimes\mathbb{Z}^S$ (see, for example, [10, Lemma 4.2]). Since $\bigcup_n\pi_n^*(C(H_n))$ is dense in $C(H)$, we therefore have

$$C(H)\rtimes_\sigma\mathbb{Z}^S=\overline{\textstyle\bigcup_{n\in\mathbb{N}}\pi_n^*\rtimes\mathrm{id}(C(H_n)\rtimes\mathbb{Z}^S)},$$

as claimed. □

We can identify the subalgebras A_n explicitly.

Proposition 2.8. *Let F be a finite quotient of $\mathbb{Z}^k$. Then $C(F)\rtimes\mathbb{Z}^k$ is isomorphic to $C(\mathbb{T}^k, M_{|F|}(\mathbb{C}))$.*

Proof. Let H be the subgroup of $\mathbb{Z}^k$ with $F = \mathbb{Z}^k/H$. Then H is itself a free abelian group of rank k, and hence has the form $A\mathbb{Z}^k$ for some $A \in M_k(\mathbb{Z}) \cap GL_k(\mathbb{Q})$. The matrix A has a Smith normal form: there are matrices $P, Q \in GL_k(\mathbb{Z})$ such that $B := P^{-1}AQ^{-1}$ is diagonal [12, §3.22]. Then $H = A\mathbb{Z}^k = PBQ\mathbb{Z}^k = PB\mathbb{Z}^k \cong B\mathbb{Z}^k = b_{11}\mathbb{Z} \oplus \cdots \oplus b_{kk}\mathbb{Z}$. In other words, multiplying by P^{-1} gives an automorphism of $\mathbb{Z}^k$ which carries H into $\bigoplus b_{ii}\mathbb{Z}$. Thus

$$C(F) \rtimes \mathbb{Z}^k \cong C\big(\textstyle\prod_{i=1}^k (\mathbb{Z}/b_{ii}\mathbb{Z})\big) \rtimes \mathbb{Z}^k \cong \bigotimes_{i=1}^k \big(C(\mathbb{Z}/b_{ii}\mathbb{Z}) \rtimes_\tau \mathbb{Z}\big),$$

where τ is the action of $\mathbb{Z}$ by translation.

By [14, Corollary 2.5], $C(\mathbb{Z}/b\mathbb{Z}) \rtimes_\tau \mathbb{Z}$ is isomorphic to the induced algebra

$$\operatorname{Ind}_{(b\mathbb{Z})^\perp}^{\mathbb{T}} (C(\mathbb{Z}/b\mathbb{Z}) \rtimes_\tau (\mathbb{Z}/b\mathbb{Z}),\ \widehat{\tau}),$$

which is described in terms of a generator β of the dual action $\widehat{\tau}$ as the mapping torus

$$MT(\beta) = \{f : [0,1] \to C(\mathbb{Z}/b\mathbb{Z}) \rtimes_\tau (\mathbb{Z}/b\mathbb{Z}) \mid f(1) = \beta(f(0))\}. \tag{2.8}$$

Since $C(\mathbb{Z}/b\mathbb{Z}) \rtimes_\tau (\mathbb{Z}/b\mathbb{Z}) \cong B(l^2(\mathbb{Z}/b\mathbb{Z})) = M_{|b|}(\mathbb{C})$, the automorphism β is inner, and there is a continuous path β_t in $\operatorname{Aut} M_{|b|}(\mathbb{C})$ such that $\beta_0 = \mathrm{id}$ and $\beta_1 = \beta$. Now $\phi(f)(t) = \beta_t^{-1}(f(t))$ defines an isomorphism ϕ of (2.8) onto $C(\mathbb{T}, M_{|b|}(\mathbb{C}))$. We therefore have

$$C(F) \rtimes \mathbb{Z}^k \cong \bigotimes_{i=1}^k C(\mathbb{T}, M_{|b_{ii}|}(\mathbb{C})) \cong C(\mathbb{T}^k, M_{\prod_i |b_{ii}|}(\mathbb{C})),$$

and the result follows on observing that $\prod_i |b_{ii}| = |\det B| = |\det A| = |F|$. □

It follows from Proposition 2.8 and the decomposition $C(H) \rtimes_\sigma \mathbb{Z}^S = \overline{\bigcup A_n}$ that $C(H) \rtimes_\sigma \mathbb{Z}^S$ is an AH-algebra[1]. The K-theory of $C(H_n) \rtimes \mathbb{Z}^S$ is torsion free and this property is preserved under inductive limits, so $C(H) \rtimes_\sigma \mathbb{Z}^S$ has torsion free K-theory. Thus, since $C(H) \rtimes_\sigma \mathbb{Z}^S$ is a simple AH-algebra with real rank zero and no dimension growth, it is an AT-algebra by [17, Lemma 7.5].

(The long story here, basically rewritten from [17, Lemma 7.5], is as follows: Denote $A = C(H) \rtimes_\sigma \mathbb{Z}^S$. Then $K_*(A)$ is a weakly unperforated Riesz group. We need a modified version of Elliott's "range of the invariant" theorem, Theorem 8.3 from *On the classification of C^*-algebras of real rank zero*, J. reine angew. Math. **443** (1993). This modified theorem says that there is a simple AT-algebra B with real rank zero and with the same ordered K-theory as that of A. The modification in the proof of Elliott's Theorem 8.3 consists of noticing that since $K_*(A)$ is torsion free, all the algebras in the direct system constructed by Elliott must have primitive ideal space the circle or a point; in other words, no dimension-drop algebras occur, and the limit B will be an AT-algebra. Since all simple AH-algebras with real rank zero and no dimension growth have tracial rank zero by Proposition 2.6 of H. Lin, *Tracially AF C^*-algebras*, Trans. Amer. Math. Soc. **353** (2001), 693-722, it follows that A has tracial rank zero. In particular, B has tracial rank zero and hence $A \cong B$ by the

[1]To see that an inductive limit $\overline{\bigcup A_n}$ is an AH-algebra, it suffices to show that each A_n is a corner in a matrix algebra $M_N(C(X))$, or, equivalently, that A_n is a homogeneous algebra with vanishing Dixmier-Douady class. Since the Dixmier-Douady class $\delta(A)$ of an m-homogeneous algebra satisfies $m\delta(A) = 0$, and $H^3(\mathbb{T}^k, \mathbb{Z})$ has no torsion, it suffices to prove that each A_n is a homogeneous algebra with spectrum $\mathbb{T}^k$. In our situation we could prove this in several ways. However, Proposition 2.8 makes the stronger statement that A_n is isomorphic to $M_m(C(\mathbb{T}^k))$.

classification theorem for C^*-algebras with tracial rank zero: this is Theorem 5.2 of H. Lin, *Classification of simple C^*-algebras of tracial topological rank zero*, preprint.)

We also use the decomposition $C(H) \rtimes_\sigma \mathbb{Z}^S = \overline{\bigcup A_n}$ to prove that $C(H) \rtimes_\sigma \mathbb{Z}^S$ has a unique tracial state. Let μ denote the Haar measure on $H \subset \mathcal{U}(\mathbb{Z}_{F\setminus S})$. The action σ permutes the cylinder sets $\{\pi_n^{-1}(m) \mid m \in H_n\}$, so every invariant probability measure agrees with μ on cylinder sets. Since the characteristic functions of such sets span a dense subspace of $C(H)$, it follows that μ is the only invariant probability measure, and $C(H) \rtimes \mathbb{Z}^S$ has a unique tracial state by [3, Corollary VIII.3.8].

This completes the proof of Theorem 2.1.

3. The structure of $C(\mathcal{U}(\mathbb{Z}_{F\setminus S})) \rtimes_\sigma \mathbb{Z}^S$

3.1. **When S contains just one prime.** We consider $C(\mathcal{U}(\mathbb{Z}_{F\setminus S})) \rtimes_\sigma \mathbb{Z}^S$ when $S = \{q\}$. To simplify the notation, we relabel $F \setminus \{q\}$ as F. The following result generalises [10, Theorem 3.1] in two directions: to sets F with $|F| > 1$ and to sets F containing the even prime 2. If $\mathbf{l} = (l_p) \in \mathbb{N}^F$ is a multi-index, we write $o_{\mathbf{l}}(q)$ for the order of q in $\prod_{p\in F} \mathcal{U}(\mathbb{Z}/p^{l_p}\mathbb{Z})$.

Theorem 3.1. *Suppose F is a finite set of primes and q is a prime which does not belong to F. Then there are a multi-index $\mathbf{K} = (K_p) \in \mathbb{N}^F$ and $d \in \mathbb{N}$ such that*

$$o_{\mathbf{K}+\mathbf{l}}(q) = d\big(\textstyle\prod_{p\in F} p^{l_p}\big) \text{ for every } \mathbf{l} \in \mathbb{N}^F, \tag{3.1}$$

and $C(\mathcal{U}(\mathbb{Z}_F)) \rtimes_\sigma \mathbb{Z}$ is the direct sum of $\big(\prod_{p\in F}(p-1)p^{K_p-1}\big)/d$ copies of a Bunce-Deddens algebra with supernatural number $d\big(\prod_{p\in F} p^\infty\big)$.

The existence of $\mathbf{K}$ and d satisfying (3.1) is established in Proposition A.5. We saw in §2 that $C(\mathcal{U}(\mathbb{Z}_F)) \rtimes_\sigma \mathbb{Z}$ is the direct sum of copies of the simple algebra $C(H) \rtimes_\sigma \mathbb{Z}$, where H is the closure of $q^{\mathbb{Z}}$ in $\mathcal{U}(\mathbb{Z}_F)$. It remains to prove that $C(H) \rtimes_\sigma \mathbb{Z}$ is a Bunce-Deddens algebra and to calculate the index $|\mathcal{U}(\mathbb{Z}_F) : H|$, which is the number of simple direct summands.

Let $\{n_k\}$ be integers with $n_k \geq 2$, and let $X_k = \{0, 1, \dots, n_k - 1\}$. Addition by 1 with carry over is a homeomorphism of the totally disconnected space $X := \prod_{k\geq 0} X_k$ called an odometer action, and the resulting crossed product $C(X) \rtimes_\tau \mathbb{Z}$ is a *Bunce-Deddens algebra with supernatural number* $\mathbf{n} := \prod_{k\geq 0} n_k$ (see [3, Chapter VIII.4]).

Our claim that $C(H) \rtimes_\sigma \mathbb{Z}$ is a Bunce-Deddens algebra will follow from the next proposition, which generalises [10, Proposition 3.6].

Proposition 3.2. *Suppose $\{G_l \mid \in \mathbb{N}\}$ are finite groups and $G = \varprojlim(G_l, \pi_l)$. Fix $g \in G$ and let L denote the closed subgroup of G generated by g. Consider the action $\sigma : \mathbb{Z} \to \operatorname{Aut} C(G)$ such that $\sigma_n(f)(x) = f(g^{-n}x)$. Let $o_l(g)$ denote the order of $\pi_l(g)$ in G_l, and let*

$$d_l := \begin{cases} o_0(g) & \text{if } l = 0 \\ o_l(g)/o_{l-1}(g) & \text{if } l \geq 1. \end{cases} \tag{3.2}$$

Then $C(L) \rtimes_\sigma \mathbb{Z}$ is a Bunce-Deddens algebra with supernatural number $\prod_{l\geq 0} d_l$.

Proof. Let $X := \prod_{l\geq 0}\{0,1,\dots,d_l-1\}$. The argument in the proof of [10, Proposition 3.6] shows that the continuous maps $h_l : X \to G_l$ given by

$$h_l(\{a_n\}) = \pi_l(g^{a_0+a_1d_0+\cdots+a_ld_0d_1\dots d_{l-1}}) \tag{3.3}$$

combine to give an equivariant homeomorphism $h : X \to L$ which induces the required isomorphism. □

Our subgroup H of $\mathcal{U}(\mathbb{Z}_F)$ is the inverse limit $\varprojlim \pi_l(H)$, where $\pi_l : \mathcal{U}(\mathbb{Z}_F) \to \mathcal{U}(\mathbb{Z}/\big(\prod_{p\in F} p^{K_p+l}\big)\mathbb{Z})$ is the canonical surjection. Then Proposition 3.2 and (3.1) imply that $C(H)\rtimes_\sigma\mathbb{Z}$ is a Bunce-Deddens algebra with supernatural number $d(\prod_{p\in F} p)^\infty$ for $d = o_{\mathbf{K}}(q)$. By Corollary A.6, we have that

$$|\mathcal{U}(\mathbb{Z}_F) : H| = \big(\textstyle\prod_{p\in F}(p-1)p^{K_p-1}\big)/d, \tag{3.4}$$

which finishes the proof of Theorem 3.1.

3.2. **When S consists of two primes.** We now analyse $C(\mathcal{U}(\mathbb{Z}_{F\setminus S}))\rtimes\mathbb{Z}^S$ when $S=\{q,r\}$. For simplicity, we consider only the case $F=\{p,q,r\}$, so that we are interested in the crossed product $C(\mathcal{U}(\mathbb{Z}_p))\rtimes_\sigma\mathbb{Z}^2$, where

$$\sigma_{m,n}(f)(x) = f(q^{-m}r^{-n}x).$$

Theorem 3.3. *The C^*-algebra $C(\mathcal{U}(\mathbb{Z}_p))\rtimes_\sigma\mathbb{Z}^2$ is a finite direct sum of copies of a simple AT-algebra A which has real rank zero, a unique tracial state and K-theory satisfying two short exact sequences:*

$$\begin{aligned} &0 \longrightarrow \mathbb{Z}[p^{-1}] \longrightarrow K_0(A)\longrightarrow\mathbb{Z}\longrightarrow 0\\ &0\longrightarrow\mathbb{Z}\longrightarrow K_1(A)\longrightarrow\mathbb{Z}[p^{-1}]\longrightarrow 0.\end{aligned} \tag{3.5}$$

Everything except the assertion about K-theory was proved in Theorem 2.1; the simple C^*-algebra A is $C(H)\rtimes_\sigma\mathbb{Z}^2$, where H is the closure of $q^{\mathbb{Z}}r^{\mathbb{Z}}$ in $\mathcal{U}(\mathbb{Z}_p)$. We aim to analyse $C(H)\rtimes_\sigma\mathbb{Z}^2$ by writing it as an iterated crossed product $(C(H)\rtimes_{\sigma^q}\mathbb{Z})\rtimes\mathbb{Z}$. The inside crossed product is not simple unless $q^{\mathbb{Z}}$ is dense in H, and it is helpful to reduce to this case using the following lemma.

Lemma 3.4. *Let H_q denote the closure of $q^{\mathbb{Z}}$ in $\mathcal{U}(\mathbb{Z}_p)$. Then H_q has finite index $I(q)$ in H, and hence is an open and closed subset of H. The inclusion of $C(H_q)$ in $C(H)$ induces an isomorphism of $C(H_q)\rtimes_\sigma(\mathbb{Z}\times I(q)\mathbb{Z})$ onto the corner $\chi_{H_q}(C(H)\rtimes_\sigma\mathbb{Z}^2)\chi_{H_q}$.*

Proof. Corollary A.6 implies that H_q has finite index in $\mathcal{U}(\mathbb{Z}_p)$, so it certainly has finite index in H. The inclusion of $C(H_q)$ in $C(H)$ and the map

$$(m, I(q)n) \mapsto \chi_{H_q} i_{\mathbb{Z}^2}(m, I(q)n)\chi_{H_q}$$

form a covariant representation of $(C(H_q),\mathbb{Z}\times I(q)\mathbb{Z},\sigma)$ in $\chi_{H_q}(C(H)\rtimes_\sigma\mathbb{Z}^2)\chi_{H_q}$, and hence give a homomorphism

$$\phi : C(H_q)\rtimes_\sigma(\mathbb{Z}\times I(q)\mathbb{Z}) \to \chi_{H_q}(C(H)\rtimes_\sigma\mathbb{Z}^2)\chi_{H_q}.$$

We can identify $(\mathbb{Z}\times I(q)\mathbb{Z})^\wedge$ with $\mathbb{T}^2/(\mathbb{Z}\times I(q)\mathbb{Z})^\perp = \mathbb{T}^2/(1\times C_{I(q)})$, where C_n denotes the group of nth roots of unity, and then ϕ carries the dual action $\hat\sigma_{[w,z]}$ into $\hat\sigma_{w,z}$;

now a standard argument implies that ϕ is injective (or we could apply [19, Corollary 4.3], for example). We have

$$\chi_{H_q}(fi_{\mathbb{Z}^2}(m,n))\chi_{H_q} = (f\chi_{H_q})i_{\mathbb{Z}^2}(m,n)\chi_{H_q} = i_{\mathbb{Z}^2}(m,n)\sigma_{m,n}^{-1}(f\chi_{H_q})\chi_{H_q}.$$

Since the support of $\sigma_{m,n}^{-1}(f\chi_{H_q})$ is contained in $q^{-m}r^{-n}H_q = r^{-n}H_q$, we have

$$\sigma_{m,n}^{-1}(f\chi_{H_q})\chi_{H_q} = \begin{cases} \sigma_{m,n}^{-1}(f\chi_{H_q}) & \text{if } I(q) \text{ divides } n \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\begin{aligned} \chi_{H_q}(fi_{\mathbb{Z}^2}(m,n))\chi_{H_q} &= \begin{cases} i_{\mathbb{Z}^2}(m,n)\sigma_{m,n}^{-1}(f\chi_{H_q}) & \text{if } I(q) \text{ divides } n \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \phi((f\chi_{H_q})i_{\mathbb{Z}^2}(m,n)) & \text{if } I(q) \text{ divides } n \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Thus every $\chi_{H_q}(fi_{\mathbb{Z}^2}(m,n))\chi_{H_q}$ is in the range of ϕ, and ϕ is surjective. □

Corollary 3.5. *Define* $\gamma : \mathbb{Z} \to \operatorname{Aut}(C(H_q) \rtimes_{\sigma^q} \mathbb{Z})$ *by*

$$\gamma_m(fi_{\mathbb{Z}}(n)) = \sigma^r_{I(q)m}(f)i_{\mathbb{Z}}(n). \tag{3.6}$$

Then $(C(H_q) \rtimes_{\sigma^q} \mathbb{Z}) \rtimes_\gamma \mathbb{Z}$ *is isomorphic to a full corner in* $C(H) \rtimes_\sigma \mathbb{Z}^2$*.*

Proof. Theorem 4.1 of [15] gives $C(H_q) \rtimes_\sigma (\mathbb{Z} \times I(q)\mathbb{Z}) \cong (C(H_q) \rtimes_{\sigma^q} \mathbb{Z}) \rtimes I(q)\mathbb{Z}$, so the result follows from Lemma 3.4 on replacing $I(q)\mathbb{Z}$ by the isomorphic group $\mathbb{Z}$. □

The analysis in §3.1 shows that $C(H_q) \rtimes_{\sigma^q} \mathbb{Z}$ is a Bunce-Deddens algebra. The K-theory of Bunce-Deddens algebras is well-known. To state the version we need, recall that if $\mathbf{n} = (n_k)_{k\geq 0}$ is a sequence with $n_k \geq 2$, then $\mathbb{Z}[\mathbf{n}^{-1}]$ denotes the set of rational numbers with denominator $N_k = \prod_{i=0}^k n_i$ for some $k \geq 0$.

Proposition 3.6. *Suppose* $\mathbf{n} = (n_k)_{k\geq 0}$, $X_k = \{0, \dots, n_k - 1\}$, $X = \prod X_k$ *and* $\tau : \mathbb{Z} \to \operatorname{Aut} C(X)$ *is the associated odometer. Then there are isomorphisms* $\phi_0 : K_0(C(X) \rtimes_\tau \mathbb{Z}) \to \mathbb{Z}[\mathbf{n}^{-1}]$ *such that* $\phi_0([\chi_{J(a_0,\dots,a_k)}]) = N_k^{-1}$ *for each cylinder set* $J(a_0, \dots, a_k)$*, and* $\phi_1 : K_1(C(X) \rtimes_\tau \mathbb{Z}) \to \mathbb{Z}$ *such that* $\phi_1(i_{\mathbb{Z}}(1)) = 1$*.*

Proof. Because $K_1(C(X)) = 0$, the Pimsner-Voiculescu sequence for the system $(C(X), \mathbb{Z}, \tau)$ reduces to

$$0 \longrightarrow K_1(C(X) \rtimes_\tau \mathbb{Z}) \xrightarrow{\delta} K_0(C(X)) \xrightarrow{\operatorname{id} - \tau_*} K_0(C(X)) \xrightarrow{\operatorname{id}_*} K_0(C(X) \rtimes_\tau \mathbb{Z}) \longrightarrow 0.$$

Now let $C_k = \{J(a_0, \dots, a_k)\}$ be the set of cylinder sets of length $k+1$, and note that $C(X) = \overline{\bigcup_{k=1} A_k}$, where $A_k = \operatorname{span}\{\chi_J \mid J \in C_k\}$. Each χ_J for $J \in C_k$ is the sum of n_{k+1} basis elements of A_{k+1}, so the maps $[\chi_{J(a_0,\dots,a_k)}] \mapsto {N_k}^{-1}$ of A_k into $\mathbb{R}$ combine to give a homomorphism ψ_0 of $K_0(C(X)) = \varinjlim K_0(A_k)$ into $\mathbb{R}$ with range $\mathbb{Z}[\mathbf{n}^{-1}]$. Since the generating automorphism $\tau = \tau_1$ permutes C_k, the kernel of ψ_0 is the range of $\operatorname{id} - \tau_*$, and hence ψ_0 induces the required isomorphism ϕ_0 of $K_0(C(X) \rtimes_\phi \mathbb{Z})$ onto $\mathbb{Z}[\mathbf{n}^{-1}]$. To verify the statement about K_1, recall that δ is the coboundary map for the Toeplitz extension of $C(X) \rtimes_\tau \mathbb{Z}$ (see [18, §2]), and compute the index of $[i_{\mathbb{Z}}(1)]$ in $K_0(C(X) \otimes \mathcal{K}) \cong K_0(C(X))$. □

Proof of Theorem 3.3. We saw in the proof of Proposition 3.2 and in the paragraph following it that the homeomorphism h of $\prod_{k\geq 0} X_k$ onto the subgroup H_q of $\mathcal{U}(\mathbb{Z}_p)$ satisfies

$$\pi_k(h(\{a_n\})) = \pi_k(q^{a_0+a_1 o_p(q)+\cdots+a_k o_p(q)p^{k-1}}) \text{ for } k \geq 0,$$

and hence carries $J(a_0,\ldots,a_k)$ onto $Z(q^{a_0+a_1 o_p(q)+\cdots+a_k o_p(q)p^{k-1}})$, where

$$Z_k(n) = \{x \in \mathcal{U}(\mathbb{Z}_p) \mid \pi_k(x) = \pi_k(n)\}.$$

So we deduce from Proposition 3.6 that there is an isomorphism ϕ_0 of $K_0(C(H_q)\rtimes_\sigma\mathbb{Z})$ onto $\frac{1}{o_p(q)}\mathbb{Z}[p^{-1}]$ such that

$$\phi_0([\chi_{Z_k(m)}]) = \frac{1}{o_p(q)}\frac{1}{p^k}$$

for every integer m which lies in H_q.

Multiplying by the unit $r^{-I(q)l}$ carries $Z_k(m)$ into $Z_k(r^{-I(q)l}m)$, and hence $\phi_0 \circ (\gamma_l)_* = \phi_0$. Thus $(\gamma_l)_*$ is the identity on $K_0(C(H_q) \rtimes_\sigma \mathbb{Z})$. It is also the identity on $K_1(C(H_q)\rtimes_\sigma\mathbb{Z})$, and hence the Pimsner-Voiculescu sequence for $((C(H_q)\rtimes_\sigma\mathbb{Z}),\mathbb{Z},\gamma)$ collapses to the two short exact sequences

$$0 \longrightarrow \frac{1}{o_p(q)}\mathbb{Z}[p^{-1}] \longrightarrow K_0(C(H_q) \rtimes \mathbb{Z}^2)\longrightarrow\mathbb{Z} \longrightarrow 0$$

$$0 \longrightarrow \mathbb{Z} \longrightarrow K_1(C(H_q) \rtimes \mathbb{Z}^2)\longrightarrow\frac{1}{o_p(q)}\mathbb{Z}[p^{-1}] \longrightarrow 0.$$

From this and Corollary 3.5 we can deduce (3.5); since the isomorphism induced by Corollary 3.5 scales the class of [1], we have removed the factor $o_p(q)^{-1}$ by a further scaling to ensure that the final statement does not depend on the order of factors in our decomposition. □

Remark 3.7. The number of simple summands in Theorem 3.3 is $|\mathcal{U}(\mathbb{Z}_p) : H|$, and we can compute this using [10, Lemma 3.7]. For example, if p is odd and l is large, we have from (A.1) that

$$\begin{aligned}|\pi_l(H)| &= [o_{p^l}(q), o_{p^l}(r)] = [p^{l-L_p(q)}o_p(q), q^{l-L_p(r)}o_p(r)]\\ &= p^{l-\min(L_p(q),L_p(r))}[o_p(q), o_p(r)];\end{aligned}$$

thus we deduce

$$|\mathcal{U}(\mathbb{Z}_p) : H| = |\mathcal{U}(\mathbb{Z}/p^l\mathbb{Z}) : \pi_l(H)| = \frac{(p-1)p^{\min(L_p(q),L_p(r))-1}}{[o_p(q), o_p(r)]}.$$

We could carry out a similar analysis when $|F| > 1$, though it would not be so easy to work out some of the indices explicitly.

Remark 3.8. Theorem 2.1 implies in particular that $C(H)\rtimes_\sigma\mathbb{Z}^2$ satisfies the hypotheses of the classification theorem of Elliott for AT-algebras [20, Theorem 3.2.6]. We can tell from the computation of K-theory in Theorem 3.3 that $C(H) \rtimes_\sigma \mathbb{Z}^2$ is not a Bunce-Deddens algebra, but it is still closely related to an odometer. The homeomorphism of $\prod_{k\geq 0} X_k$ onto H_q identifies the action of the first copy of $\mathbb{Z}$ (multiplication by q on H_q) with an odometer (addition of 1 with carry over). The action of the second copy of $\mathbb{Z}$ (multiplication by r on H_q) also acts as a permutation on each X_k:

it moves X_0 around in a different order, and this action carries over into X_1 when the marker in X_0 returns to the starting point. So we can think of the action of $\mathbb{Z}^2$ as two independent odometers on the same set. We can normalise the scale so that either copy of $\mathbb{Z}$ acts by addition of 1 with carry over, but not so that both act this way at once.

4. The Bost-Connes algebra

The Hecke C^*-algebra $\mathcal{C}_{\mathbb{Q}}$ of Bost and Connes [2] is isomorphic to the semigroup crossed product $C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_\alpha \mathbb{N}^*$. The Fourier transform takes $C^*(\mathbb{Q}/\mathbb{Z})$ onto the algebra of continuous functions on the compact group $\mathcal{Z} := \prod_{p\in\mathcal{P}} \mathbb{Z}_p$ and carries α into the action given by (see [6, §3.1])

$$\alpha_n(f)(x) = \begin{cases} f(x/n) & \text{if } n \text{ divides } x \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2.2 is valid with F replaced by $\mathcal{P}$ and $\mathbb{Z}_F$ by $\mathcal{Z}$. Thus for $S \subset \mathcal{P}$, an application of [9, Theorem 1.7] gives that $J_S := C_0(\mathcal{Z} \setminus \mathcal{Z}_S) \rtimes_\alpha \mathbb{N}^*$ is an ideal of $\mathcal{C}_{\mathbb{Q}} = C(\mathcal{Z}) \rtimes \mathbb{N}^*$, with quotient isomorphic to $C(\mathcal{Z}_S) \rtimes \mathbb{N}^*$. Choose $a \in \mathcal{Z}$ such that $a_p = 0 \iff p \in S$. Then $\{\mathbb{Q}_+^* a \cap \mathcal{Z}\}$ has closure $\mathcal{Z}_S$, so $C_0(\mathcal{Z} \setminus \mathcal{Z}_S)$ is the kernel of the representation π_a considered in [8, page 440], and it follows from [10, Lemma 4.2] that J_S is the kernel of the representation $\pi_a \times V$ described in [8, page 440]. We can now deduce that $S \mapsto J_S$, as S runs through the proper subsets of $\mathcal{P}$, is the parametrisation of $(\operatorname{Prim} C_{\mathbb{Q}}) \setminus \widehat{\mathbb{Q}_+^*}$ given in [8, Theorem 2.8].

Theorem 4.1. *Suppose that S is a proper subset of $\mathcal{P}$.*

(a) *If $\mathcal{P} \setminus S$ is infinite, then $J_S = \bigcap_{p\notin S} J_{S\cup\{p\}}$.*

(b) *If $0 < |\mathcal{P} \setminus S| < \infty$, then*

$$\big(\textstyle\bigcap_{p\notin S} J_{S\cup\{p\}}\big)/J_S \cong \big(C(\mathcal{U}(\mathbb{Z}_{\mathcal{P}\setminus S})) \rtimes_\sigma \mathbb{Z}^S\big) \otimes \mathcal{K}(l^2(\mathbb{N}^{\mathcal{P}\setminus S})).$$

(c) *$\mathcal{C}_{\mathbb{Q}}/J_{\mathcal{P}}$ is isomorphic to $C^*(\mathbb{Q}_+^*) = C(\widehat{\mathbb{Q}_+^*})$.*

Moreover, $C(\mathcal{U}(\mathbb{Z}_{\mathcal{P}\setminus S})) \rtimes_\sigma \mathbb{Z}^S$ is a simple AT-algebra with real rank zero and a unique tracial state.

It follows from [8, §2] that every basic open neighbourhood of J_S has the form

$$U_G = \{J_T \mid T \subset \mathcal{P}, T \cap G = \emptyset\}$$

for some finite subset G of $\mathcal{P}\setminus S$. When $\mathcal{P}\setminus S$ is infinite, there are always lots of $J_{S\cup\{p\}}$ in U_G, and thus $J_S \in \overline{\{J_{S\cup\{p\}} \mid p \notin S\}}$; this says precisely that $\bigcap_{p\notin S} J_{S\cup\{p\}} \subset J_S$. The other inclusion is trivial, and (a) follows. Part (c) is true because $\mathcal{Z}_{\mathcal{P}} = \{0\}$. To prove (b) we just need to replace F by $\mathcal{P}$ and $\mathbb{Z}_F$ by $\mathcal{Z}$ in the proof of Lemma 2.6.

It remains to prove the statements about $C(\mathcal{U}(\mathbb{Z}_{\mathcal{P}\setminus S})) \rtimes_\sigma \mathbb{Z}^S$. The Chinese Remainder Theorem implies that $\mathbb{Z}$ is dense in $\mathbb{Z}_{\mathcal{P}\setminus S}$, and hence $\mathbb{Z}^S = \mathbb{Z} \cap \mathcal{U}(\mathbb{Z}_{\mathcal{P}\setminus S})$ is dense in $\mathcal{U}(\mathbb{Z}_{\mathcal{P}\setminus S})$. Thus $\mathbb{Z}^S$ acts minimally and freely on $\mathcal{U}(\mathbb{Z}_{\mathcal{P}\setminus S})$, and $C(\mathcal{U}(\mathbb{Z}_{\mathcal{P}\setminus S})) \rtimes_\sigma \mathbb{Z}^S$ is simple. However, since $|S| = \infty$, we cannot apply the results of [16] to conclude that $C(\mathcal{U}(\mathbb{Z}_{\mathcal{P}\setminus S})) \rtimes_\sigma \mathbb{Z}^S$ has real rank zero and stable rank one, as we did in Section 2 for $C(H) \rtimes_\sigma \mathbb{Z}^S$. Instead we aim to use Theorems 1 and 2 of [1], and to do this we need to show that $C(\mathcal{U}(\mathbb{Z}_{\mathcal{P}\setminus S})) \rtimes_\sigma \mathbb{Z}^S$ is an AH-algebra with the extra property of slow dimension growth.

Since $\mathcal{P} \setminus S$ is finite, we have as in (2.7) that $\mathcal{U}(\mathbb{Z}_{\mathcal{P}\setminus S})$ is the inverse limit of the finite groups $\mathcal{U}\big(\mathbb{Z}/\big(\prod_{p\in\mathcal{P}\setminus S} p^{l_p}\big)\mathbb{Z}\big)$ over $l = (l_p) \in \mathbb{N}^{\mathcal{P}\setminus S}$. Hence $\mathcal{U}(\mathbb{Z}_{\mathcal{P}\setminus S}) = \varprojlim F_n$, where

$$F_n := \mathcal{U}\big(\mathbb{Z}/\big(\textstyle\prod_{p\in\mathcal{P}\setminus S} p^n\big)\mathbb{Z}\big). \tag{4.1}$$

We denote the canonical surjection of $\mathcal{U}(\mathbb{Z}_{\mathcal{P}\setminus S})$ onto F_n by π_n. The analogue of Lemma 2.7 for F_n and the canonical action of $\mathbb{Z}^S$ by multiplication on F_n implies that $C(\mathcal{U}(\mathbb{Z}_{\mathcal{P}\setminus S})) \rtimes_\sigma \mathbb{Z}^S$ is the closed union of C^*-subalgebras isomorphic to $C(F_n) \rtimes \mathbb{Z}^S$.

Towards applying Proposition 2.8, we note that the infinite direct sum $\mathbb{Z}^S$ is the union of the subgroups $\mathbb{Z}^E$ associated to finite subsets E of S. Thus by using an argument similar to that in Lemma 2.7 we have that $C(F_n) \rtimes \mathbb{Z}^S$ is the closed union of subalgebras isomorphic to $C(F_n) \rtimes \mathbb{Z}^E$. However, by choosing a particular sequence E_n of finite subsets of S, we can show that $C(\mathcal{U}(\mathbb{Z}_{\mathcal{P}\setminus S})) \rtimes_\sigma \mathbb{Z}^S$ has slow dimension growth.

Indeed, since $\mathbb{Z}/\big(\prod_p p^n\big)\mathbb{Z} \cong \prod_p \mathbb{Z}/p^n\mathbb{Z}$, we have $F_n \cong \prod_{p\in\mathcal{P}\setminus S} \mathcal{U}(\mathbb{Z}/p^n\mathbb{Z})$. Thus F_n is a product of at most $|\mathcal{P}\setminus S|+1$ cyclic groups (the $+1$ allows for the possibility that $2 \in \mathcal{P} \setminus S$), and hence has a generating set $\{x_{n,i}\}$ with at most $|\mathcal{P} \setminus S| + 1$ elements. By Dirichlet's Theorem, there are primes $q_{n,i}$ such that

$$q_{n,i} \equiv x_{n,i} \pmod{\textstyle\prod_{p\in\mathcal{P}\setminus S} p^n},$$

and each $q_{n,i}$ belongs to S because it is a unit modulo $\prod_{p\in\mathcal{P}\setminus S} p^n$. Now let $E'_n := \{q_{n,i}\}$, list the primes in S as $\{r_n \mid n \in \mathbb{N}\}$, and take

$$E_n := \big(\textstyle\bigcup_{m\le n} E'_m\big) \cup \{r_1, \dots, r_n\}.$$

We then have $\pi_n(\mathbb{Z}^{E_n}) = F_n$, $E_m \subset E_n$ for $m \le n$, and $\bigcup E_n = S$.

We have now realised $C(\mathcal{U}(\mathbb{Z}_{\mathcal{P}\setminus S})) \rtimes_\sigma \mathbb{Z}^S$ as the closure of an increasing union $\bigcup_{n\in\mathbb{N}} B_n$ in which B_n is isomorphic to the crossed product $C(F_n) \rtimes \mathbb{Z}^{E_n}$ by a transitive action of $\mathbb{Z}^{E_n}$. By an argument identical to the one at the end of Section 2 we conclude that $C(\mathcal{U}(\mathbb{Z}_{\mathcal{P}\setminus S})) \rtimes \mathbb{Z}^S$ has a unique tracial state.

We prove next that $C(\mathcal{U}(\mathbb{Z}_{\mathcal{P}\setminus S})) \rtimes \mathbb{Z}^S$ is an AH-algebra with real rank zero. Proposition 2.8 implies that $B_n \cong C(F_n) \rtimes \mathbb{Z}^{E_n} \cong C(\mathbb{T}^{|E_n|}, M_{|F_n|}(\mathbb{C}))$. But

$$\frac{|E_n|}{|F_n|} \le \frac{n(|\mathcal{P} \setminus S| + 2)}{\prod_p (p-1)p^{n-1}} \to 0 \text{ as } n \to \infty,$$

and thus the sequence $B_n \cong C(F_n) \rtimes \mathbb{Z}^{E_n}$ of subalgebras of $C(\mathcal{U}(\mathbb{Z}_{\mathcal{P}\setminus S})) \rtimes \mathbb{Z}^S$ has slow dimension growth. It now follows from [1, Theorem 1] that $C(\mathcal{U}(\mathbb{Z}_{\mathcal{P}\setminus S})) \rtimes \mathbb{Z}^S$ has topological stable rank one. Since the projections in $C(\mathcal{U}(\mathbb{Z}_{\mathcal{P}\setminus S})) \rtimes \mathbb{Z}^S$ trivially separate the unique tracial state, [1, Theorem 2] implies that $C(\mathcal{U}(\mathbb{Z}_{\mathcal{P}\setminus S})) \rtimes \mathbb{Z}^S$ has real rank zero. The K-groups of $C(\mathcal{U}(\mathbb{Z}_{\mathcal{P}\setminus S})) \rtimes \mathbb{Z}^S$ are inductive limits of torsion free groups, and hence are themselves torsion free, so it follows as in Section 2 that $C(\mathcal{U}(\mathbb{Z}_{\mathcal{P}\setminus S})) \rtimes \mathbb{Z}^S$ is an AT-algebra.

This completes the proof of Theorem 4.1.

Appendix A. The orders of a prime in groups of units

For p prime and $m \in \mathbb{N}$ such that $(m,p)=1$, we denote by $o_{p^l}(m)$ the order of m in $\mathcal{U}(\mathbb{Z}/p^l\mathbb{Z})$. It was shown in [10, Theorem 3.1] that if p is odd, there is a positive integer $L_p(m)$ such that

$$o_{p^l}(m) = \begin{cases} o_p(m) & \text{if } 1 \le l \le L_p(m) \\ p^{l-L_p(m)} o_p(m) & \text{if } l > L_p(m); \end{cases} \tag{A.1}$$

the proof uses that the groups $\mathcal{U}(\mathbb{Z}/p^l\mathbb{Z})$ are cyclic. We will now show how to modify the arguments of [10, §3] to obtain an analogue of (A.1) for $p=2$, in which case $\mathcal{U}(\mathbb{Z}/2^l\mathbb{Z})$ are no longer cyclic.

Proposition A.1. *If m is an odd integer and $m \equiv 1 \pmod 4$, then there exists a positive integer $K = L_2(m)$ such that*

$$o_{2^l}(m) = \begin{cases} 1 & \text{if } 1 \le l \le K \\ 2^{l-K} & \text{if } l > K; \end{cases} \tag{A.2}$$

if $m \equiv 3 \pmod 4$, then there exists a positive integer $L = L_2(m)$ such that

$$o_{2^l}(m) = \begin{cases} 1 & \text{if } l = 1 \\ 2 & \text{if } 1 < l \le L \\ 2^{l-(L-1)} & \text{if } l > L. \end{cases} \tag{A.3}$$

To prove Proposition A.1 we use general properties of cyclic groups as in [10, §3]. We begin with a lemma.

Lemma A.2. *Suppose $l \ge 3$. Then the group $\{n \in \mathcal{U}(\mathbb{Z}/2^l\mathbb{Z}) \mid n \equiv 1 \pmod 4\}$ is the cyclic subgroup $\langle 5 \rangle_l$ of $\mathcal{U}(\mathbb{Z}/2^l\mathbb{Z})$ generated by 5.*

Proof. Theorem $2'$ of [4, Chapter 4.1] says that $|\langle 5 \rangle_l| = 2^{l-2}$. For $k \ge 0$ we have

$$5^k = (4+1)^k = \sum_{n=0}^{k} \binom{k}{n} 4^n = 4 \sum_{n=1}^{k} \binom{k}{n} 4^{n-1} + 1,$$

so $5^k \equiv 1 \pmod 4$. Hence, if $n \equiv 5^k \pmod{2^l}$ for some $0 \le k < 2^{l-2}$, then $n \equiv 1 \pmod 4$. Since the order of $\{n \in \mathcal{U}(\mathbb{Z}/2^l\mathbb{Z}) \mid n \equiv 1 \pmod 4\}$ is also 2^{l-2}, the result follows. □

Corollary A.3. *An element of $\mathcal{U}(\mathbb{Z}/2^l\mathbb{Z})$ is congruent to $3 \pmod 4$ if and only if it is congruent to $-5^k \pmod{2^l}$ for some k satisfying $0 \le k < 2^{l-2}$.*

Corollary A.4. *Suppose $m \in \mathbb{Z}$ satisfies $m \equiv 1 \pmod 4$. Then for every $l > 0$ we have*

$$o_{2^l}(m) = \begin{cases} o_{2^{l+1}}(m) & \text{if } 2 \text{ does not divide } o_{2^{l+1}}(m) \\ o_{2^{l+1}}(m)/2 & \text{if } 2 \text{ divides } o_{2^{l+1}}(m). \end{cases} \tag{A.4}$$

Proof. Since a number is coprime to 2^l if and only if it is coprime to 2^{l+1}, the reduction map $\pi : \mathcal{U}(\mathbb{Z}/2^{l+1}\mathbb{Z}) \to \mathcal{U}(\mathbb{Z}/2^l\mathbb{Z})$ is a surjective homomorphism. Lemma A.2 implies that $m \equiv 5^r \pmod{2^{l+1}}$, where $r = o_{2^{l+1}}(5)/o_{2^{l+1}}(m) = 2^{l-1}/o_{2^{l+1}}(m)$. Hence, by applying [10, Lemma 3.2] to the restriction of π to a homomorphism of $\langle 5\rangle_{l+1}$ onto $\langle 5\rangle_l$, we have

$$\begin{aligned} o_{2^l}(m) &= o(\pi(5^r)) \\ &= \begin{cases} 2^{l-1}/(2^{l-1}/o_{2^{l+1}}(m), 2^{l-1}) & \text{if } 2^{l-1} \text{ divides } 2^{l-1}/o_{2^{l+1}}(m) \\ 2^{l-1}/\big(2(2^{l-1}/o_{2^{l+1}}(m), 2^{l-1})\big) & \text{if } 2^{l-1} \text{ does not divide } 2^{l-1}/o_{2^{l+1}}(m), \end{cases} \end{aligned}$$

which simplifies to (A.4). □

Proof of Proposition A.1. Suppose first that $m \equiv 1 \pmod 4$. For fixed N, there exists an $l \in \mathbb{N}$ satisfying $m^N < 2^l$. Then $o_{2^l}(m) > N$ and hence the sequence $\{o_{2^l}(m) \mid l \in \mathbb{N}\}$ must be unbounded. In particular, $\{o_{2^l}(m)\}$ is not a constant sequence. Let K be the first integer such that $o_{2^K}(m) < o_{2^{K+1}}(m)$. Then $o_{2^l}(m) = o_2(m) = 1$ for $1 \le l \le K$, and by Corollary A.4 we have $o_{2^{K+1}}(m) = 2o_2(m) = 2$. Since $o_{2^{K+1}}(m)$ divides $o_{2^l}(m)$ for all $l > K$, it follows that 2 divides $o_{2^l}(m)$ for all $l > K$. We now apply Corollary A.4 $l-K$ times to deduce that $o_{2^l}(m) = 2^{l-K}o_{2^K}(m) = 2^{l-K}$.

Now suppose that $m \equiv 3 \pmod 4$. Certainly $o_2(m) = 1$. For $l > 1$, Corollary A.3 tells us that $m \equiv -5^k$ for some $0 \le k < 2^{l-2}$. Thus $m^2 \equiv 5^{2k} \pmod{2^l}$, and therefore $m^2 \in \langle 5\rangle_l$. Let L be the first integer such that $o_{2^L}(m^2) < o_{2^{L+1}}(m^2)$. Applying Corollary A.4 to m^2 and repeating the argument of the preceding paragraph gives (A.3) because $o_{2^l}(m) = 2o_{2^l}(m^2)$. □

We now need to extend these results to cover actions on $\mathcal{U}(\mathbb{Z}_F)$ for an arbitrary finite set F of primes. We write $F = \{p_1, \dots, p_n\}$ and fix a prime q which is not in F. We denote by $o_{(l_1,\dots,l_n)}(q)$ the order of $(q, \dots, q)$ in $\prod_{i=1}^n \mathcal{U}(\mathbb{Z}/{p_i}^{l_i}\mathbb{Z})$.

Proposition A.5. *There exist positive integers $K_1, \dots, K_n$ and d such that*

$$o_{(K_1+l_1,\dots,K_n+l_n)}(q) = d{p_1}^{l_1}\dots{p_n}^{l_n} \tag{A.5}$$

for every $(l_1, \dots, l_n) \in \mathbb{N}^F$.

Proof. Suppose first that $p_1, \dots, p_n$ are distinct odd primes, and let $L_{p_i}(q)$ be as in (A.1). Let

$$z_i := \max\{z \mid {p_i}^z \text{ divides } o_{p_j}(q) \text{ for some } j \in \{1, \dots, n\}\},$$

and define $K_i := L_{p_i}(q) + z_i$ and $d := [o_{p_1}(q), \dots, o_{p_n}(q)]$, where $[r_1, \dots, r_n]$ is the least common multiple of the integers r_i. In general, if g_i are elements of order r_i in finite groups G_i, then the order of $(g_1, \dots, g_n)$ in $G_1 \times \dots \times G_n$ is $[r_1, \dots, r_n]$. Thus from the properties of $L_{p_i}(q)$ we obtain

$$\begin{aligned} o_{(K_1+l_1,\dots,K_n+l_n)}(q) &= [p_1^{(K_1+l_1)-L_{p_1}(q)}o_{p_1}(q), \dots, p_n^{(K_n+l_n)-L_{p_n}(q)}o_{p_n}(q)] \\ &= [p_1^{z_1+l_1}o_{p_1}(q), \dots, p_n^{z_n+l_n}o_{p_n}(q)] \\ &= {p_1}^{l_1}\dots{p_n}^{l_n}[o_{p_1}(q), \dots, o_{p_n}(q)], \end{aligned} \tag{A.6}$$

which is (A.5).

Now suppose that $2 \in F$, say $p_1 = 2$. If $q \equiv 1 \pmod 4$, we let

$$z_i := \max\{z \mid {p_i}^z \text{ divides } o_{p_j}(q) \text{ for some } j \in \{2, \dots, n\}\},$$

and define $K_1 := L_2(q) + z_1$, $K_i := L_{p_i}(q) + z_i$ for $i > 1$, and $d := [o_{p_2}(q), \ldots, o_{p_n}(q)]$. Reasoning as in (A.6) gives (A.5).

If $q \equiv 3 \pmod 4$, we let

$$z_1 = \max(1, \max\{z \mid 2^z \text{ divides } o_{p_j}(q) \text{ for some } j \in \{2, \ldots, n\}\}),$$
$$z_i = \max\{z \mid {p_i}^z \text{ divides } o_{p_j}(q) \text{ for some } j \in \{2, \ldots, n\}\}$$

for $i > 1$, and define $K_1 := L_2(q) + z_1 - 1$, $K_i := L_{p_i}(q) + z_i$ for $i > 1$ and $d := [2, o_{p_2}(q), \ldots, o_{p_n}(q)]$. Again, reasoning as in (A.6) gives (A.5). □

Corollary A.6. *The closure H of $q^{\mathbb{Z}}$ in $\mathcal{U}(\mathbb{Z}_F)$ is a subgroup of finite index*

$$|\mathcal{U}(\mathbb{Z}_F) : H| = \big(\textstyle\prod_{i=1}^n (p_i - 1) p_i^{K_i - 1}\big)/d.$$

Proof. Apply Proposition A.5 to $\mathbf{l} = (l, l, \ldots, l)$ to see that $|\pi_l(H)| = d\big(\prod_{i=1}^n p_i^l\big)$ for large l, and the result follows from [10, Lemma 3.7]. □

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School of Mathematical and Physical Sciences, University of Newcastle, NSW 2308, Australia

E-mail address: Nathan.Brownlowe@studentmail.newcastle.edu.au

Department of Mathematics, Institute for Mathematical Sciences, University of Copenhagen, Universitetsparken 5, 2100 Copenhagen, Denmark

E-mail address: nadia@math.ku.dk

Department of Mathematics and Statistics, University of Victoria, British Columbia V8W 3P4, Canada

E-mail address: putnam@math.uvic.ca

School of Mathematical and Physical Sciences, University of Newcastle, NSW 2308, Australia

E-mail address: iain@frey.newcastle.edu.au