Orbit equivalence of Cantor minimal systems: A survey and a new proof

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Abstract

We give a new proof of the classification, up to topological orbit equivalence, of minimal AF-equivalence relations and minimal actions of the group of integers on the Cantor set. This proof relies heavily on the structure of AF-equivalence relations and the theory of dimension groups; we give a short survey of these topics.

1 Introduction

The papers [8, 9, 10, 13, 20] present a study of orbit equivalence for certain minimal dynamical systems on a Cantor set; i.e. a compact, totally disconnected, metrizable space with no isolated points. This is a parallel program to that initiated by Dye in ergodic theory (for example, see [2]) and also that in Borel equivalence relations (for example, see [16]). By a dynamical system, we mean to include actions of countable groups but more generally étale equivalence relations. We explain the terminology in the next section. Group actions which are free are a special case, the underlying equivalence

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relation is the orbit relation: the equivalence classes are the orbits of the action. An orbit equivalence between two such systems is a homeomorphism between the underlying spaces which carries the equivalence classes of one system to those of the other. The terminology is motivated by the case of a group action. In this setting, an equivalence relation is said to be minimal if every equivalence class is dense.

Aside from group actions, the most important class of étale equivalence relations is the so-called AF-equivalence relations. The precise definition is given in the next section but means, roughly, that these can be approximated by subequivalence relations where the equivalence classes are finite; AF stands for "approximately finite". This turns out to be an extremely interesting class since it is rich enough to exhibit very intricate structures but the finite approximations make them quite tractable.

In [10], a complete classification was given for those minimal equivalence relations on a Cantor set which arise from actions of the group of integers, $\mathbb{Z}$, or are AF. Subsequently, this was extended to include actions of $\mathbb{Z}^2$ [8, 12] and $\mathbb{Z}^d, d \geq 3$ [9]. The result is to give a complete invariant of orbit equivalence.

Here we present a new proof of the classification for AF-equivalence relations and for $\mathbb{Z}$-actions. There are several reasons for doing so. The first is that in [10], the result is really shown, first, for $\mathbb{Z}$-actions; the extension to the AF case is derived from that. There is a certain sense in which this is backward: the subsequent extensions always rely on proving that a given group action is orbit equivalent to an AF-equivalence relation and using the classification result there. That is, the most natural progression would be: AF-equivalence relations, $\mathbb{Z}$-actions, $\mathbb{Z}^2$-actions, etc. So we give a direct proof of the result for AF-equivalence relations and also show how the result to include $\mathbb{Z}$-actions can be derived from it.

The second reason for a new proof is that the one given in [10] relies on a non-trivial result in homological algebra. It has not been clear if this is really needed. Our proof here avoids it, resolving that question. The third reason is the rôle of a result we refer to as the absorption theorem in the program. Very roughly, the absorption theorem asserts that a minimal AF-equivalence relation is orbit equivalent to a "small" extension of itself. (The result is very technical; so much so, that there are three different versions in the literature [11, 7, 19].) The results which extend the classification from AF-equivalence relations to actions of $\mathbb{Z}^d, d \geq 1$, are done by showing that the orbit relation for actions of these groups can be realized as such a small extension of some AF-subequivalence relation, so that the absorption theorem implies that the orbit classification of the group action is the same as for the AF-subequivalence relation. In fact, the key step in our proof here of the classification of AF-equivalence relations also relies on this same absorption theorem. This seems to stress the utility and importance of this rather technical result. That is, all our results on orbit equivalence for Cantor minimal systems rely on the absorption theorem.

The paper is organized as follows. The second section is a presentation of background material, definitions and results. The understanding of the structure of AF-equivalence relations relies on invariants which are ordered groups, called dimension groups. We present a short treatment which, hopefully, will be accessible to readers who are not familiar with the basics. Many of these facts will also be needed in the proof to follow. We do not present many proofs, but we try to give references and the key ideas behind the results when they are simple to understand and illustrate the main points. The third section is the statement of our main technical result, Theorem 3.1, and its consequences the classification of minimal AF-equivalence relations up to orbit equivalence, Corollary 3.2, and the extension to include minimal $\mathbb{Z}$-actions, Corollary 3.3. The proof of the technical result is given in the last section separately because of its length.

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2 Preliminary definitions and results

2.1 Étale equivalence relations

We begin with a discussion of topological equivalence relations, étale equivalence relations and, particularly, AF-equivalence relations.

Let X be a locally compact metrizable space. For convenience, we also assume that X is second countable and will often also assume X is compact. Let R be an equivalence relation on X. We assume that R is endowed with a topology, $\mathcal{T}$, which is second countable. We say that R or (X, R) is étale, or more accurately that $(R, \mathcal{T})$ is étale, if it satisfies the following conditions [22, 11]. First, the set

$$R^2 = \{((x, y), (y, z)) \in R \times R\}$$

is closed in $R \times R$ and the maps sending $((x, y), (y, z))$ in R^2 to (x, z) in R and (x, y) in R to (y, x) in R are continuous. Second, the diagonal, $\{(x, x) \mid$

$x \in X\}$ is open in R. Finally, the maps $r, s : R \to X$ defined by $r(x, y) = x$, $s(x, y) = y$ are open and are local homeomorphisms. We say $U \subset R$ is an R-set if it is open and $r|_U, s|_U$ are homeomorphisms. The collection of R-sets forms a neighbourhood base for the topology of R. We remark that the topology of R is rarely the same as the relative topology from $R \subset X \times X$, except in trivial cases. The equivalence relation (X, R) is *minimal* if every R-equivalence class is dense in X.

An action of a group G on a space X is a collection of homeomorphisms, $\varphi^s, s \in G$, satisfying

$$\begin{aligned} \varphi^{st} &= \varphi^s \circ \varphi^t, s, t \in G, \\ \varphi^e(x) &= x, x \in X. \end{aligned}$$

The action is free if, for s is in G and x in X, $\varphi^s(x) = x$ only if $s = e$. Given such an action,

$$R_\varphi = \{(x, \varphi^s(x)) \mid s \in G, x \in X\}$$

is an equivalence relation. Assume now that G is countable and we endow it with the discrete topology. There is an étale topology on R_φ which can be described in either of two ways [22]. First, notice the map sending (x, s) in $X \times G$ to $(x, \varphi^s(x))$ in R_φ is surjective by the definition of R_φ and injective since the action is free. We simply transfer the product topology from $X \times G$ to R_φ with this map. The second description is to consider

$$\{\varphi^s|U \mid s \in G, U \subset X \text{ open }\}.$$

It is easy to see these sets form a neighbourhood base of R_φ-sets for this topology.

This is a convenient place to discuss the notion of an invariant measure for an étale equivalence relation [22]. By measure, we shall always mean a Borel probability measure. Let (X, R) be an étale equivalence relation. A measure μ on X is said to be R-invariant if $\mu(r(E)) = \mu(s(E))$, for every R-set U and Borel set $E \subset U$. We let $M(X, R)$ denote the set of all R-invariant measures.

2.2 Isomorphism and orbit equivalence

We introduce the notions of isomorphism and orbit equivalence [11].

Definition 2.1. *Let* (X_1, R_1) *and* (X_2, R_2) *be étale equivalence relations. They are* isomorphic *and we write* $(X_1, R_1) \cong (X_2, R_2)$ *if there exists a homeomorphism* $h : X_1 \to X_2$ *such that* $h \times h(R_1) = R_2$*, equivalently,* h *maps each* R_1*-equivalence class to an* R_2*-equivalence class, and*

$$h \times h : R_1 \to R_2$$

is a homeomorphism.

Definition 2.2. *Let* (X_1, R_1) *and* (X_2, R_2) *be equivalence relations. They are* orbit equivalent *and we write* $(X_1, R_1) \sim (X_2, R_2)$ *if there exists a homeomorphism* $h : X_1 \to X_2$ *such that* $h \times h(R_1) = R_2$*, equivalently,* h *maps each* R_1*-equivalence class to an* R_2*-equivalence class.*

For further discussion and the notion of weak orbit equivalence, also see [13].

2.3 AF-equivalence relations

A rich and tractable class of étale equivalence relations is given by the so-called AF-equivalence relations [11, 22]. To construct such an example, we begin with a Bratteli diagram, (V, E). This is an infinite directed graph. The vertex set V is the union of a sequence of finite, non-empty, pairwise disjoint sets, $V_n, n \geq 0$. The set V_0 is assumed for convenience to consist of a single vertex, v_0. Similarly, the edge set is the union of a sequence of finite, non-empty, pairwise disjoint sets, $E_n, n \geq 1$. An edge e in E_n has initial vertex $i(e)$ in V_{n-1} and terminal vertex $t(e)$ in V_n. We assume always that our graph has no sources other than v_0 and no sinks: that is, $i^{-1}\{v\}$ and $t^{-1}\{v\}$ are non-empty for any v in V (other than $t^{-1}\{v_0\}$). An example of a Bratteli diagram is drawn in section 3.

For v in V_{n-1} and v' in V_n, we let $E_n(v, v')$ denote the set of all edges e with $i(e) = v, t(e) = v'$. A path in the diagram from V_m to V_n is a finite list of edges $p = (p_{m+1}, p_{m+2}, \ldots, p_n)$ such that $p_i \in E_i$ and $t(p_i) = i(p_{i+1})$, $m < i < n$. The initial and terminal vertices of the path are $i(p) = i(p_{m+1})$ and $t(p) = t(p_n)$, respectively.

The (infinite) path space of the diagram is the set

$$X = X(V, E) = \{(x_1, x_2, \ldots) \mid x_n \in E_n, t(x_n) = i(x_{n+1}), n \geq 1\}.$$

It is endowed with the relative topology of the infinite product space $\Pi_n E_n$, where each E_n is given the discrete topology. The space X is compact, metrizable and totally disconnected. For each path $p = (p_1, \ldots, p_n)$, we let $X(p) = \{x \in X \mid x_i = p_i, 1 \leq i \leq n\}$. It is easy to see that such a set is both closed and open and, allowing n and p to vary, these sets form a neighbourhood base for the topology on X.

The equivalence relation $R = R(V, E)$ is tail equivalence: two elements x and y in X are tail equivalent if, for some $N \geq 0$, $x_n = y_n$, for all $n > N$. We let $R_N = R_N(V, E)$ denote the set of those pairs (x, y) which satisfy this condition for a fixed N. That is, R is the union of the R_N and, clearly, $R_N \subset R_{N+1}$, for all $N \geq 0$. For fixed N, the equivalence relation R_N is finite in the sense that all equivalence classes are finite.

The topology on R can be described in two ways. The first is to endow each R_N with the relative topology of $X \times X$ in which it is an étale equivalence relation. Then R is endowed with the inductive limit topology; that is, a subset U is open if and only if $U \cap R_N$ is open in R_N, for all $N \geq 0$. In the second description, we consider a pair of paths p, q from V_0 to V_N such that $t(p) = t(q)$ and define

$$R_N(p, q) = \{(x, y) \in X \times X \mid x_i = p_i, y_i = q_i, 1 \leq i \leq N, x_n = y_n, n > N\}.$$

Varying N, p, q, such sets form a neighbourhood base for a topology in which R is étale. These two descriptions yield the same topology.

This gives us enough information to define an AF-equivalence relation.

Definition 2.3. *Let R be an étale equivalence on the compact, metrizable space X. We say that (X, R) (or just R) is an AF-equivalence relation if X is totally disconnected and R is the union of an increasing sequence of compact, open subequivalence relations.*

We remark that there is a more general definition allowing X to be locally compact [11, 22], but we will not need that here. Of course, our construction above using Bratteli diagrams gives examples of such equivalence relations. In fact, it provides all of them (up to isomorphism) - see [11].

Theorem 2.4. *Let (X, R) be an AF-equivalence relation. There exists a Bratteli diagram (V, E) such that (X, R) is isomorphic to $(X(V, E), R(V, E))$.*

Suppose that (V, E) is a Bratteli diagram and we choose an increasing sequence $m_0 = 0 < m_1 < m_2 < \ldots$. The *telescope* of (V, E) to this sequence is the Bratteli diagram (V', E') with $V'_k = V_{m_k}$ and E'_k is the collection of all paths from $V_{m_{k-1}}$ to V_{m_k}. The initial and terminal maps have already been defined. There is an obvious map from $X(V,E)$ to $X(V',E')$ which simply takes an infinite path in (V,E) and groups the entries so that it becomes an infinite path in (V',E'). It is easy to verify this map is a homeomorphism and defines an isomorphism between $(X(V,E),R(V,E))$ and $(X(V',E'),R(X,E'))$. We define equivalence of Bratteli diagrams to be the smallest equivalence relation such that a diagram is equivalent to any telescope of itself. Equivalence of diagrams implies isomorphism of the associated AF-equivalence relations.

We note the following description of minimality for AF-equivalence relations. We say that the diagram (V,E) is *simple* if it satisfies the following condition: for every $m \geq 0$, there exists an $n > m$ such that for every pair of vertices, v in V_m and v' in V_n, there exists a path from v to v'. As an easy consequence of the definitions, we have:

Theorem 2.5. *A Bratteli diagram,* (V,E)*, is simple if and only if its associated AF-equivalence relation,* $(X(V,E),R(V,E))$*, is minimal.*

It is fairly easy to see that any simple Bratteli diagram, provided $X(V,E)$ is infinite, may be telescoped to one where the numbers $\min\{\#E_n(v,v') \mid v \in V_{n-1}, v' \in V_n\}$ grow at some prescribed rate. Such a condition can also be achieved for the cardinality of the vertex sets as follows. Consider the diagram obtained by inserting a vertex in the midpoint of each edge. More precisely, the vertex set at level $n \geq 0$ is $V_{n/2}$, for n even, and $E_{(n+1)/2}$, for n odd. The edge set at level n is $E_{n/2}$, for n even and $E_{(n+1)/2}$, for n odd. The initial and terminal maps are either the same as for the original graph, or the identity. Telescoping this diagram to its even levels yields the original diagram, while telescoping to its odd levels gives diagram whose vertex sets are the E_n's and hence grow in cardinality. Further telescoping will retain such growth in the size of the vertex sets, while causing the size of the edge sets to grow again also.

2.4 Invariants

We now discuss invariants for étale equivalence relations. These invariants will be partially ordered abelian groups. For brevity, we will use the term ordered abelian group instead. An ordered abelian group [14] consists of an abelian group G together with a subset, G^+, called the positive elements, which satisfy:

1. G^+ is a subsemigroup; that is, $G^+ + G^+ \subset G^+$,
2. $G^+ \cap (-G^+) = \{0\}$ and
3. G^+ generates the group; that is, $G^+ - G^+ = G$.

We obtain an order on the group by defining $a \geq b$ if and only if $a - b$ is in G^+. This relation is a partial order in the usual sense. We say that an element is *strictly positive* if it is in $G^+ - \{0\}$. Our groups will also have a distinguished positive element u which is an *order unit*: if a is any element of G, then there is a positive integer n such that $nu \geq a$. The simplest example of an ordered abelian group is the integers, $\mathbb{Z}$, with $\mathbb{Z}^+ = \{0, 1, 2, \ldots\}$.

Let (X, R) be an étale equivalence relation. We let $C(X, \mathbb{Z})$ be the set of continuous integer-valued functions on X, which is an abelian group under pointwise addition. For any clopen set $E \subset X$, its characteristic function, denoted χ_E, is in $C(X, \mathbb{Z})$.

Definition 2.6. *Let (X, R) be a minimal AF-equivalence relation or the minimal étale equivalence relation generated by the action of the group of integers, $\mathbb{Z}$. For any compact, open R-set U, we define $\partial\chi_U = \chi_{r(U)} - \chi_{s(U)}$ and let $B(X, R)$ be the subgroup generated by all such functions. We define*

$$K^0(X, R) = C(X, \mathbb{Z})/B(X, R),$$

For f in $C(X, \mathbb{Z})$, we let $[f]$ denote its coset in the quotient group. We also define an order structure on this groups by setting the positive cone to be

$$K^0(X, R)^+ = \{[f] \mid f \geq 0\}$$

and we let 1 denote the constant function 1, so that $[1]$ is on order unit.

Some comments are in order on two points. The first is the notation and the second is that this definition can obviously be given verbatim for étale equivalence relations (although it probably does not make too much sense without the hypothesis that X is totally disconnected). As described in [22], it is possible to construct a C^*-algebra from an étale equivalence relation and one may then consider the (ordered) K-zero group of that C^*-algebra. In the cases of minimal AF-equivalence relations and minimal $\mathbb{Z}$-actions this coincides with the definition given above. This explains our choice of notation. Going further, if one were to consider a free, minimal action of $\mathbb{Z}^2$, that statement would be false. (Roughly speaking, the fact that the C^*-algebra contains $C^*(\mathbb{Z}^2)$ as a subalgebra contributes a Bott element to its K-theory, which is not accounted for in the group in our definition. A much more complete explanation of this case can be found in [6].)

We now make the following definition in the more general setting of étale equivalence relations.

Definition 2.7. *Let (X, R) be an étale equivalence relation. We define a subgroup of $C(X, \mathbb{Z})$ by*

$$B_m(X, R) = \{f \in C(X, \mathbb{Z}) \mid \int_X f d\mu = 0, \text{ for all } \mu \in M(X, R)\}.$$

and

$$D_m(X, R) = C(X, \mathbb{Z})/B_m(X, R)$$

with the positive cone

$$D_m(X, R)^+ = \{[f] \mid f \geq 0\},$$

and order unit $[1]$.

It is clear from the definitions that $B(X, R) \subset B_m(X, R) \subset C(X, \mathbb{Z})$, for minimal AF-equivalence relations and minimal $\mathbb{Z}$-actions and it follows that $D_m(X, R)$ is a quotient of $K^0(X, R)$.

Theorem 2.8. *Let (X_1, R_1) and (X_2, R_2) be étale equivalence relations. If (X_1, R_1) and (X_2, R_2) are orbit equivalent then*

$$D_m(X_1, R_1) \cong D_m(X_2, R_2),$$

as ordered abelian groups with order units, meaning that there is a group isomorphism $\alpha : D_m(X_1, R_1) \to D_m(X_2, R_2)$ such that $\alpha(D_m(X_1, R_1)^+) = D_m(X_2, R_2)^+$ and $\alpha[1_{X_1}] = [1_{X_2}]$.

Proof. Suppose that $h : X_1 \to X_2$ is an orbit equivalence. It is clear that the map sending f in $C(X_1, \mathbb{Z})$ to $f \circ h^{-1}$ in $C(X_2, \mathbb{Z})$ is an isomorphism. We will prove that, if μ is any R_2-invariant measure, then its pullback $h^*(\mu)$ is R_1-invariant. Once this is done, it is an easy matter to see that the map $\alpha[f] = [f \circ h^{-1}]$ is the desired order isomorphism.

Fix $i = 1, 2$. Notice that the map $r \times s : R_i \to X_i \times X_i$ is really just the usual inclusion, but written in this fashion, we regard the domain as having the topology in which it is étale while the range has the usual product topology. Hence, this map is not usually a homeomorphism to its image. However, it follows from the definition of étale that it is continuous. Using the hypothesis that R_i is second countable, it can be shown that it is an isomorphism between the associated Borel spaces. We will not quite get that far, but we provide the main ideas. Notice that from the continuity of the map $r \times s$, every Borel subset of R_i with the relative topology is also Borel in the étale topology. In addition, if K is a compact subset of R_i, then the restriction of $r \times s$ to it is a homeomorphism to its image which is also compact in $X_i \times X_i$. So a subset of K is Borel in the étale topology if and only if it is Borel in the product topology.

We know that R_i is second countable. Take a countable neighbourhood base $\mathcal{N}_i$ for its topology and consider those elements whose closure is compact and contained in an R_i-set. (All R_i-sets are pre-compact, but we will not prove this.) We claim this subcollection is also a neighbourhood base. Let (x, y) be in R_i and let U be an open set containing (x, y). Since R_i-sets form a neighbourhood base, find V which is an R_i-set with $(x, y) \in V \subset U$. Now choose an open neighbourhood O of x such that its closure is contained in the interior of $r(V)$. It is easy to check that the closure of $V' = (r \mid V)^{-1}(O)$ is a compact subset of V and contains (x, y). Next, find a W in $\mathcal{N}_i$ such that $x \in W \subset V'$. It is clear that W is in our subcollection. This establishes the claim. We list the elements of this subcollection, $U_i^n, n \in \mathbb{N}$.

Now let U be an R_1-set and $E \subset U$ be a Borel set. Define, inductively, $E^1 = E \cap U_1^1$ and $E^m = (E \cap U_1^m) - E^{m-1}$, for $m \geq 2$, so that we have

1. E^m is a Borel subset of U_1^m,

2. E is the disjoint union of all $E^m, m \geq 1$.

Notice that the range map on each E^m is a bijection and the images are pairwise disjoint. The same holds for the map s. As a consequence of the first condition above, $h \times h(E^m)$ is a Borel subset of R_2. Holding m fixed for the moment, we do the same thing in R_2, letting $F^1 = h \times h(E^m) \cap U_2^1$ and $F^n = h \times h(E^m) \cap U_2^n - F^{n-1}$, for $n \geq 2$, so that

1. F^n is a Borel subset of U_2^n,

2. $h \times h(E^m)$ is the disjoint union of all $F^n, n \geq 1$.

Then we may compute

$$\begin{aligned} h^*(\mu)(r(E^m)) &= \mu(h(r(E^m))) \\ &= \mu(r(h \times h(E^m))) \\ &= \sum_{n=1}^{\infty} \mu(r(F^n)) \\ &= \sum_{n=1}^{\infty} \mu(s(F^n)) \\ &= \mu(s(h \times h(E^m))) \\ &= \mu(h(s(E^m))) \\ &= h^*(\mu)(s(E^m)). \end{aligned}$$

Finally, we have

$$h^*(\mu)(r(E)) = \sum_{m=1}^{\infty} h^*(\mu)(r(E^m)) = \sum_{m=1}^{\infty} h^*(\mu)(s(E^m)) = h^*(\mu)(s(E))$$

and it follows that $h^*(\mu)$ is R_1-invariant. □

2.5 Invariants for AF-equivalence relations

If the AF-relation (X, R) is tail equivalence on the Bratteli diagram (V, E), then the invariant $K^0(X, R)$ may be computed directly from the diagram. The aim of this subsection is to describe this computation. For any finite set A, we let $\mathbb{Z}A$ denote the free abelian group on A. That is, a typical element is a formal integral combination of the elements of A. Of course, it is isomorphic as a group to $\mathbb{Z}^n$, where n is the number of elements of A, but our notation allows us to consider A as a subset of the group. We denote by $\mathbb{Z}^+A$ the subsemigroup with identity generated by the elements of A; that is, it consists of non-negative integral combinations of A.

Suppose that V and V' are two finite sets of vertices and E is a set of edges between them, meaning that there are initial and terminal maps $i : E \to V$ and $t : E \to V'$. We may define a group homomorphism, $\varepsilon : \mathbb{Z}V \to \mathbb{Z}V'$, by setting

$$\varepsilon(v) = \sum_{i(e)=v} t(e), v \in V.$$

This defines ε on the generators of $\mathbb{Z}V$ and has a unique extension which is a group homomorphism. Equivalently, if we let $E(v, v')$ denote the set of edges e with $i(e) = v, t(e) = v'$, and $\varepsilon(v, v') = \#E(v, v')$, for any $v \in V, v' \in V'$, then

$$\varepsilon(v) = \sum_{v' \in V'} \varepsilon(v, v')v', v \in V.$$

It is clear this homomorphism is positive in the sense that it maps the positive cone in its domain into the positive cone in the range.

It is worth noting that the converse is also true: if $h : \mathbb{Z}V \to \mathbb{Z}V'$ is a group homomorphism, we define, for $v \in V, v' \in V'$, $h(v, v')$ to be the unique integer such that

$$h(v) = \sum_{v' \in V'} h(v, v')v',$$

If, in addition, h is a positive group homomorphism, then we have $h(v, v') \geq 0$, for all v, v', and we may choose an edge set E such that $\#E(v, v') = h(v, v')$, for all v, v'.

We note that if V, V' and V'' are three vertex sets, E are edges from V to V' and E' are edges from V' to V'', then for all v in V and v'' in V'', we have

$$\varepsilon' \circ \varepsilon(v, v'') = \sum_{v' \in V'} \varepsilon(v, v')\varepsilon'(v', v'').$$

From the Bratteli diagram, (V, E), we may construct a sequence of abelian groups and homomorphisms:

$$\mathbb{Z}V_0 \stackrel{\varepsilon_1}{\to} \mathbb{Z}V_1 \stackrel{\varepsilon_2}{\to} \mathbb{Z}V_2 \cdots$$

where ε_n is the group homomorphism obtained as above from the edge set E_n, for $n \geq 1$. Notice each group is given the standard order and each homomorphism is positive. For convenience, for any $m < n$, we let

$$\varepsilon_{m,n} = \varepsilon_n \circ \cdots \circ \varepsilon_{m+1} : \mathbb{Z}V_m \to \mathbb{Z}V_n.$$

The inductive limit of such a system, which we denote by $K^0(V, E)$, is defined as follows. Consider the disjoint union of the groups, which we denote $\bigsqcup_n \mathbb{Z}V_n$. We define an equivalence relation: if a is in $\mathbb{Z}V_m$ and a' is in $\mathbb{Z}V_{m'}$, $a \sim a'$ if there exists $n > m, m'$ such that $\varepsilon_{m,n}(a) = \varepsilon_{m',n}(a')$. Alternately, $\sim$ is the equivalence relation generated by $a \sim \varepsilon_{n+1}(a)$, for $n \geq 0$ and a in $\mathbb{Z}V_n$. Let $K^0(V, E)$ denote the quotient of $\bigsqcup_n \mathbb{Z}V_n$ by this equivalence relation. If

a is in $\mathbb{Z}V_n$, we let $[a, n]$ denote its class in $K^0(V, E)$. Although $\bigsqcup_n \mathbb{Z}V_n$ is not itself a group, it is easy to see that $K^0(V, E)$ has a group structure by defining $[a, m] + [a', m'] = [\varepsilon_{m,n}(a) + \varepsilon_{m',n}(a'), n]$, where $n > m, m'$. The group has a positive cone, $K^0(V, E)^+ = \{[a, n] \mid n \geq 1, a \in \mathbb{Z}^+V_n\}$. A word of warning is in order. It is entirely possible that a is in $\mathbb{Z}V_m$ and is not positive there, yet $\varepsilon_{m,n}(a)$ *is* in $\mathbb{Z}^+V_n$, for some $n > m$. In this case, $[a, m]$ is in $K^0(V, E)^+$. We also note that this group has a distinguished positive element, $[v_0, 0]$. We also observe that since the diagram has no sinks, if v is in V_n, then $[v, n] \neq 0$; i.e. it is strictly positive.

The next result gives our combinatorial description of the invariant $K^0(X, R)$, when (X, R) is an AF-equivalence relation. From the statement given, the proof is just a matter of checking the claimed map is well defined and does indeed define an isomorphism.

Theorem 2.9. *Let (V, E) be a Bratteli diagram. Then $K^0(X(V, E), R(V, E))$ is isomorphic to $K^0(V, E)$ as ordered abelian groups with distinguished order units. Moreover, for a path p from v_0 to $t(p)$ in V_n, the isomorphism carries $[\chi_{X(p)}]$ in $K^0(X(V, E), R(V, E))$ to $[t(p), n]$ in $K^0(V, E)$.*

The following seminal result shows the power of the invariant and also the naturality of the class of AF-relations.

Theorem 2.10 (Elliott-Krieger)**.** *Let (V, E) and (V', E') be two Bratteli diagrams. The following are equivalent:*

1. *The diagrams (V, E) and (V', E') are equivalent.*
2. *The AF-equivalence relations $(X(V, E), R(V, E))$ and $(X(V', E'), R(V', E'))$ are isomorphic.*
3. *$K^0(V, E)$ and $K^0(V', E')$ are isomorphic as ordered abelian groups with distinguished order units.*

Let us say a word or two about the history of this result. Originally, Bratteli defined the diagrams which now carry his name as a combinatorial description for inductive systems of finite dimensional semisimple algebras. He also showed that the equivalence class of the diagram was a complete invariant for the limit of the algebras (taken either in the category of algebras or in the category of C^*-algebras). It follows from the basic properties of

K-theory that the group we call $K^0(V,E)$ is the K-theory of the limit C^*-algebra. The equivalence of conditions 1 and 3 is due to G. Elliott [5]. The dynamical interpretation and the equivalence of 2 with 1 and 3 was subsequently given by Krieger [18]. As stated above and given Theorem 2.9, the implications $1 \Rightarrow 2 \Rightarrow 3$ are trivial. The implication $3 \Rightarrow 1$ is due to Elliott.

2.6 Dimension groups

In this section, we assemble a number of results concerning minimal AF-equivalence relations and the structure of our invariant in this case. In addition to being important background information, a number of these will be needed in the proof of our main result later.

We begin with a definition which, from what we saw in the last subsection, should be natural.

Definition 2.11. *A* dimension group *G is any ordered abelian group which is the limit of a sequence of groups of the form $(\mathbb{Z}^n, (\mathbb{Z}^+)^n)$ and positive group homomorphisms.*

If G is a dimension group, then the inductive system of the definition above can be used to produce a graph, (V,E), which gives a combinatorial description of the group. This may, however, have sources and sinks. We leave it as an exercise to show that by removing the vertices v in V_n with the property that $[v,n]=0$, one may construct another diagram having no sinks with limit G. The source issue is slightly more subtle. The group $G = \oplus_{n=1}^{\infty}\mathbb{Z}$ and $G^+ = \oplus_{n=1}^{\infty}\mathbb{Z}^+$ cannot be written without having an infinite collection of sources. Note, however, this group has no order unit. Given an order unit, we can arrange for a single source, v_0, which represents that order unit (see Corollary 3.18 of [14]). We omit the details, but state the result for future reference. The first part of the second statement follows at once from the first. The last part is a consequence of the Elliott-Krieger Theorem.

Theorem 2.12. *Let G be a dimension group with order unit.*

1. *There exists a Bratteli diagram (V,E) such that $K^0(V,E) \cong G$ as ordered abelian groups with order unit.*

2. *There exists an AF-equivalence relation (X,R) with $K^0(X,R) \cong G$ as ordered abelian groups with order unit. Moreover, X is finite if and only if G is cyclic.*

The definition of dimension group is not very practical in the sense that, if one is given some ordered group, it may not be very obvious how to present it as an inductive limit of the type above. This is remedied by the Theorem of Effros, Handelman and Shen (see [4, 3, 14]).

Theorem 2.13 (Effros-Handelman-Shen)**.** *A countable, ordered abelian group G is a dimension group if and only if*

1. *it is unperforated: if g is in G and $k \geq 1$ satisfy kg is in G^+, then g is in G^+, and*

2. *it satisfies Riesz interpolation: if a_1, a_2, b_1, b_2 are in G and $a_i \leq b_j$ for $i, j = 1, 2$, then there exists c in G such that $a_i \leq c \leq b_j$, for all $i, j = 1, 2$.*

Notice in particular, that being unperforated implies the group is torsion free. It is fairly easy to show the only if direction: the conditions are satisfied by the groups $(\mathbb{Z}^n, (\mathbb{Z}^+)^n)$ are preserved under inductive limits. The if direction is highly non-trivial.

There is a little ambiguity in the literature concerning the term dimension group. It was introduced by Elliott [5] as we have stated. Since then, others (see [14]) have preferred the alternate characterization as provided by the result of Effros, Handelman and Shen.

An *order ideal* (Chapter 14 of [14]) in an ordered group G is a subgroup I such that $I \cap G^+$ generates I as a group and, for any a in G^+ and b in $I \cap G^+$, if $a \leq b$ then a is in I. An ordered abelian group G is *simple* if the only order ideals are 0 and G. For example, if (V, E) is a Bratteli diagram, let v be a vertex in V_n, for some $n \geq 0$, and define I_v to be the set of all elements a such that $-k[v, n] \leq a \leq k[v, n]$, for some positive integer k. It is easy to see that I_v is an order ideal. Next, suppose that I is any non-zero order ideal in $K^0(V, E)$. It follows that I contains a non-zero positive element, say a. This must be represented by a strictly positive element in some $\mathbb{Z}V_n$ and if we let v be any vertex in V_n whose coefficient in the expression for a is positive, we have $0 \leq [v, n] \leq a$, and hence $[v, n]$ is also in I. It follows that $I_v \subset I$. We have shown that if $K^0(V, E)$ contains a non-trivial order ideal, then it contains a non-trivial order ideal of the form I_v. On the other hand, it is fairly easy to see that if the diagram is simple, then $I_v = K^0(V, E)$, for any v. It is now fairly easy to prove the following.

Theorem 2.14. *Let (V, E) be a Bratteli diagram. It is simple as a Bratteli diagram if and only if the dimension group $K^0(V, E)$ is simple as an ordered abelian group.*

Let G be an ordered abelian group with order unit u. A *state* on G is a group homomorphism $\phi : G \to \mathbb{R}$ such that $\phi(G^+) \subset [0, \infty)$ and $\phi(u) = 1$. The first part of the following is originally due to S. Kerov [17]. See also Herman, Putnam and Skau [15]. The proof of the second part is an easy exercise we leave to the reader.

Theorem 2.15. *Let (V, E) be a Bratteli diagram.*

1. *There is a bijective correspondence between $M(X(V, E), R(V, E))$ and the set of states on $K^0(X(V, E), R(V, E))$ which associates to a measure μ the state ϕ_μ defined by*

$$\phi_\mu([f]) = \int_X f d\mu.$$

for any f in $C(X(V, E), \mathbb{Z})$.

2. *There is a bijective correspondence between the set of states on $K^0(V, E)$ and the set*

$$\{\psi : V \to [0, 1] \mid \psi(v_0) = 1, \psi(v) = \sum_{i(e)=v} \psi(t(e)), \text{ for all } v \in V\},$$

which associates to a state ϕ the function

$$\psi(v) = \phi([v, n]),$$

for any v in V_n, $n \geq 0$..

We next state two fundamental results regarding the order structure of $K^0(V, E)$, for simple Bratteli diagrams, (V, E). See Chapter 4 of [3] or Chapter 14 of [14]. The first asserts that the order structure is completely described by the states.

Theorem 2.16. *Let (V, E) be a simple Bratteli diagram. An element a of $K^0(V, E)$ is strictly positive if and only if $\phi(a) > 0$, for every state ϕ on $K^0(V, E)$. Moreover, every strictly positive element is an order unit.*

The second result is an immediate consequence of the first, but its form will be quite useful.

Theorem 2.17. *Let (V, E) be a simple Bratteli diagram and let a be an element of $K^0(V, E)$. The following two conditions are equivalent.*

1. *For any integer k and any order unit b in $K^0(V, E)^+$, $b \geq ka$.*
2. *For any state ϕ on $K^0(V, E)$, we have $\phi(a) = 0$.*

An element of an ordered abelian group is called *infinitesimal* if it satisfies the first condition of the last Theorem. We let $Inf(K^0(V, E))$ denote the set of all infinitesimal elements, which is a subgroup. Moreover, in the case that $K^0(V, E)$ is simple, the quotient group $K^0(V, E)/H$ carries a natural order provided that H is a subgroup of $Inf(K^0(V, E))$, as follows. First, suppose that a is a strictly positive element of $K^0(V, E)$. It follows from Theorem 2.16 and the second part of Theorem 2.17 that the set $a + H$ is all strictly positive in $K^0(V, E)$. We define $(K^0(V, E)/H)^+$ to be the identity coset H and all cosets consisting of strictly positive elements of $K^0(V, E)$. As a simple consequence of Theorems 2.15, 2.17, the second isomorphism theorem for groups and the definitions of the order above, we have the following.

Theorem 2.18. *Let (X, R) be a minimal AF-equivalence relation. The group $K^0(X, R)/Inf(K^0(X, R))$, with its quotient order, is isomorphic to $D_m(X, R)$ as ordered abelian groups with order unit.*

There is a substantial theory devoted to studying the set of states of a dimension group and the representation of the group as affine functions on this set. We refer the reader to [14]. We will not pursue this here, but we do note the following rather concrete representation of our invariant in the case of so-called uniquely ergodic AF-equivalence relations. We leave the proof as an exercise.

Corollary 2.19. *Let (X, R) be a minimal AF-equivalence relation and suppose that $M(X, R)$ consists of a single measure, μ. Let*

$$G = \{\mu(E) \mid E \subset X, \text{ clopen } \} + \mathbb{Z} \subset \mathbb{R}.$$

Then G is an ordered abelian group with the usual order and addition from $\mathbb{R}$ and order unit 1. Moreover, $D_m(X, R)$ is isomorphic to G, as ordered abelian groups with order unit.

2.7 Minimal $\mathbb{Z}$-actions

We note the following result which shows that minimal AF-equivalence relations are very closely linked with minimal $\mathbb{Z}$-actions.

Theorem 2.20. *Let φ be a minimal action of $\mathbb{Z}$ on the Cantor set X and let y be in X. There exists an open subequivalence relation $R \subset R_\varphi$ such that*

1. *R is a minimal AF-equivalence relation,*
2. *R_φ is generated as an equivalence relation by R and $(y, \varphi(y))$,*
3. *$K^0(X, R_\varphi)$ is isomorphic to $K^0(X, R)$, as ordered abelian groups with order unit.*

We give a short sketch of the main idea, since it is quite simple. Begin with the easy observation that R_φ is the equivalence relation generated by $\{(x, \varphi(x)) \mid x \in X\}$. Choose a sequence of clopen sets $Y_1 \supset Y_2 \supset \cdots$ with intersection $\{y\}$. For each $N \geq 1$, let R_N be the equivalence relation generated by $\{(x, \varphi(x)) \mid x \in X - Y_N\}$. From the minimality of φ and the fact that Y_N is open, for each x in X, the integers k for which $\varphi^k(x)$ is in Y_N form a relatively dense set [1]. These points serve to divide the φ-orbit into (uniformly bounded) finite intervals which are the equivalence classes of R_N. A little more careful analysis shows that R_N is both compact and open. Hence the union of the R_N, denoted by R, is an AF-equivalence relation. Moreover, it is the equivalence relation generated by $\{(x, \varphi(x)) \mid x \neq y\}$. Its equivalence classes are exactly the same as those in R_φ, except that the φ-orbit of y is divided into two classes in R, namely the forward half-orbit of $\varphi(y)$ and the backward half-orbit of y. The first two parts are now fairly clear. The proof of the last statement, (or rather, a more general version) was first shown in [21]. The terminology there involves some C^*-algebra theory. A purely dynamical (and even more thorough) version is given in [13].

2.8 The absorption theorem

We finish with the statement of the main technical result which will be needed and we refer to as the absorption theorem. In fact, three different versions exist in the literature [11, 7, 19]; this can be attributed to the rather technical nature of the result. The result in [19] is the most general. We will use the version from [7] since it fits most naturally in our proof, but we must discuss some more terminology before stating the result.

Let (X, R) be an étale equivalence relation. A closed set $Y \subset X$ is said to be R-étale if the equivalence relation $R|_Y = R \cap (Y \times Y)$, with its relative topology from R, is an étale equivalence relation. It is said to be R-thin if $\mu(Y) = 0$, for every measure μ in $M(X, R)$.

In the case of AF-equivalence relations, we may provide examples of R-étale sets as follows. Let (V, E) be a Bratteli diagram. Suppose that F is a subset of E with $i(F) = t(F) \cup \{v_0\}$. Let $W = i(F)$. We call (W, F) (or just F) a *subdiagram* of (V, E). It is clear that the path space $X(W, F)$ is a subset of $X(V, E)$ and it is fairly easy to see that it is both closed and R-étale. (In fact, there is a converse of this result: if (X, R) is an AF-equivalence relation and Y is a closed, R-étale subset of X, then (X, R) and Y may be represented by a Bratteli diagram (V, E) and a subdiagram (W, F) as above. As we will not need this result, see [10], Theorem 3.11 for a precise statement.

As far as the property of being R-thin is concerned, we have the following result. It will provide an effective tool for verifying thinness. As it has not before appeared explicitly in this form, we provide a short proof.

Theorem 2.21. *Let (V, E) be a Bratteli diagram and (W, F) be a subdiagram. Suppose that there exists a positive constant M and $N \geq 1$ such that, for all $w \in W_{N-1}$, $w' \in W_N$, we have*

$$M\#F_N(w, w') \leq \#E_N(w, w'),$$

then

$$\mu(X(W, F)) \leq M^{-1},$$

for all μ in $M(X(V, E), R(V, E))$.

Proof. For the moment, fix w in W_{N-1} and w' in W_N. As a consequence of the hypothesis, we may find functions $a_m : F_N(w, w') \to E_N(w, w')$, for $m = 1, 2, \ldots, M$ which are injective and have pairwise disjoint ranges. Simply taking the union over all w, w', we have $a_m : F_N \to E_N$, for $m = 1, 2, \ldots, M$, with the same properties.

Now we define M functions $\alpha_m : X(W, F) \to X(V, E)$ by

$$\alpha_m(x_1, x_2, \ldots) = (x_1, x_2, \ldots, a_m(x_n), x_{n+1}, \ldots),$$

for $(x_1, x_2, \ldots)$ in $X(W, F)$, and $m = 1, 2, \ldots, M$.

Note first that $\{(x, \alpha_m(x)) \mid x \in X(W, F)\}$ is an $R(V, E)$-set for each m. It follows that $\mu(X(W, F)) = \mu(\alpha_m(X(W, F)))$, for every $R(V, E)$-invariant measure μ. Moreover, the sets $\alpha_m(X(W, F)), m = 1, 2, \ldots, M$ are pairwise disjoint. Thus we have

$$\begin{aligned} 1 &= \mu(X(V, E)) \\ &\geq \mu(\cup_m \alpha_m(X(W, F))) \\ &= \sum_m \mu(\alpha_m(X(W, F))) \\ &= \sum_m \mu(X(W, F)) \\ &= M\mu(X(W, F)). \end{aligned}$$

□

Let R and S be two equivalence relations on X. We define $R \times_X S = \{((x, y), (y, z)) \mid (x, y) \in R, (y, z) \in S\}$. We define $r, s : R \times_X S \to X$ by $r((x, y), (y, z)) = x, s((x, y), (y, z)) = z$. We also set $r \times s((x, y), (y, z)) = (x, z)$. We say that R and S are *transverse* if

1. $R \cap S = \{(x, x) \mid x \in X\}$,
2. there exists a homeomorphism $h : R \times_X S \to S \times_X R$ such that $r \circ h = r$ and $s \circ h = s$. That is, for each (x, y) in R and (y, z) in S, there is a unique y' with (x, y') in S and (y', z) in R (with continuity conditions on y').

As an example (and an important one since it is the situation we will encounter in our proof), suppose that (X, R) is an étale equivalence relation and $\alpha : X \to X$ is a homeomorphism such that $\alpha \times \alpha(R) = R$ and $\alpha \times \alpha : R \to R$ is a homeomorphism. Also suppose that $\alpha^2(x) = x, (x, \alpha(x)) \notin R$, for all x in X. Let $S = \{(x, \alpha(x)) \mid x \in X\}$, which is an equivalence relation. We give it the relative topology from $X \times X$. It is étale and transverse to R; the map sending $((x, y), (y, \alpha(y)))$ in $R \times_X S$ to $((x, \alpha(x)), (\alpha(x), \alpha(y))$ satisfies the desired conditions.

Theorem 2.22. *Let (X, R) be a minimal AF-equivalence relation. Let Y be a closed R-étale and R-thin subset of X and K be a compact étale equivalence relation on Y which is transverse to $R|_Y$. Then there is a homeomorphism $h : X \to X$ such that*

1. *$h \times h(R \vee K) = R$, where $R \vee K$ is the equivalence relation generated by R and K,*

2. *$h(Y)$ is R-étale and R-thin,*

3. *$h|_Y \times h|_Y : R|_Y \times K \to R|_{h(Y)}$ is a homeomorphism.*

In particular, $R \vee K$ is orbit equivalent to R.

In fact, we will only make use of the final statement of the conclusion.

We note the following interesting consequence.

Corollary 2.23. *Let φ be a minimal action of the group $\mathbb{Z}$ on a Cantor set X. The orbit relation (X, R_φ) is orbit equivalent to an AF-equivalence relation (X, R).*

Proof. Let R be the relation described after Theorem 2.20. As explained there, R_φ is generated by R and $(y, \varphi(y))$, for some point y in X. Let $Y = \{y, \varphi(y)\}, K = Y \times Y$. It is trivial to check that these satisfy the hypothesis of Theorem 2.22 and the conclusion follows at once. □

3 Statements of the results

Let us set the stage for the main result, which at first glance seems somewhat technical. If we look at the situation that $(X, \tilde{R})$ is a minimal AF-equivalence relation and $R \subset \tilde{R}$ is an open subequivalence relation, it follows that R is also AF (see 3.12 of [11]). It is clear from the definition 2.6 that $B(X, R) \subset B(X, \tilde{R})$ and hence $K^0(X, \tilde{R})$ is a quotient of $K^0(X, R)$. If we add the hypothesis that R and $\tilde{R}$ have the same invariant measures, then the kernel of the quotient map is contained in the infinitesimals of $K^0(X, R)$.

It is then natural to ask the question: If we are given two simple dimension groups (with order units) related in this fashion, do they arise from such a pair? More specifically, if G is a simple dimension group and H is a subgroup of $Inf(G)$, do there exist minimal AF-equivalence relations (X, R) and $(X, \tilde{R})$ with $R \subset \tilde{R}$ such that $K^0(X, R) \cong G$, $K^0(X, \tilde{R}) \cong G/H$ and the following diagram commutes:

$$\begin{array}{ccc} G & \xrightarrow{q} & G/H \\ \cong\downarrow & & \cong\downarrow \\ K^0(X, R) & \longrightarrow & K^0(X, \tilde{R}) \end{array}$$

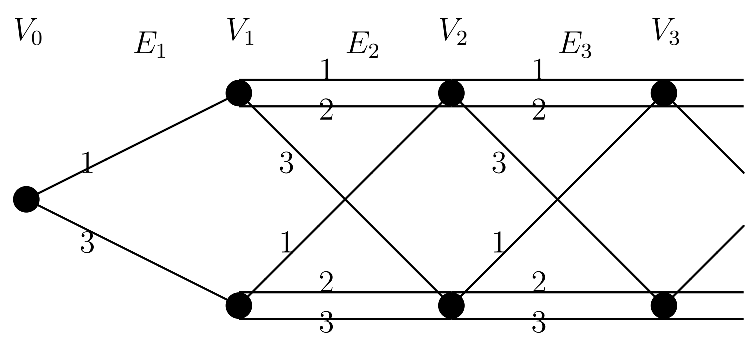
By 2.12, there exists a AF-equivalence relation, (X, R), such that $K^0(X, R) \cong G$. It is minimal by Theorems 2.14 and 2.5. It is clearly necessary that the quotient G/H be torsion free. Our main theorem asserts under these conditions, the AF-equivalence relation $\tilde{R}$ exists. In fact, much more is true; it can be realized as a small extension of R in the sense of the absorption theorem and, in particular, is orbit equivalent to R. We note that the commutativity of the diagram above will be a result of our construction, but as we do not use this, we do not make it part of the statement.

Theorem 3.1. *Let (X, R) be a minimal AF-equivalence relation. Suppose that H is a subgroup of $Inf(K^0(X, R))$ such that the quotient group $K^0(X, R)/H$ is torsion free. Then there exists a minimal AF-equivalence relation, $\tilde{R}$, on X, containing R, a closed set $Y \subset X$ and a compact étale equivalence relation K on Y such that*

1. *$(X, R), Y, K$ satisfy the hypotheses of the absorption theorem 2.22,*
2. *$\tilde{R} = R \vee K$,*
3. *$K^0(X, \tilde{R}) \cong K^0(X, R)/H$, as ordered abelian groups with distinguished order units.*

In particular, (X, R) and $(X, \tilde{R})$ are orbit equivalent.

The proof is quite long and it will be done in the next section. It will probably be useful to have an example. Consider the following Bratteli diagram



The reader will note that we have added labels to the edges of the diagram. This is a matter of convenience, for we can now see that the path space

$X(V,E)$ is homeomorphic to $\{1,3\} \times \{1,2,3\}^{\mathbb{N}}$ and the map is just reading the labels on the edges. Suppose that $V_n = \{v_n, v'_n\}$, for any $n \geq 1$. There is a unique $R(V,E)$-invariant measure which corresponds, in the sense of Theorem 2.15, to the function $f(v_n) = f(v'_n) = 2^{-1} \cdot 3^{1-n}$, for $n \geq 1$. The subgroup of infinitesimals in $K^0(V,E)$ is isomorphic to $\mathbb{Z}$; its generator is $[v_n - v'_n, n]$, for any $n \geq 1$. It is fairly easy to check that $K^0(V,E)/Inf(K^0(V,E)) \cong \mathbb{Z}[1/3]$. Suppose we would like to apply Theorem 3.1 with $H = Inf(K^0(V,E))$. The relation $\tilde{R}$ is just tail equivalence on the sequences of $\{1,3\} \times \{1,2,3\}^{\mathbb{N}}$. The closed set Y consists of the two sequences $(1,2,2,\ldots)$ and $(3,2,2,\ldots)$ and $K = Y \times Y$. The conditions that Y is closed, $R(V,E)$-étale and $R(V,E)$-thin are trivially satisfied and we leave the reader with the amusing tasks of proving K is transverse to $R(V,E)$ and $\tilde{R} = R(V,E) \vee K$.

We proceed to discuss the consequences of Theorem 3.1, which include the classification of minimal AF-relations, up to orbit equivalence, as in [10].

Corollary 3.2. *Two minimal AF-equivalence relations* (X_1, R_1) *and* (X_2, R_2) *are orbit equivalent if and only if* $K^0(X_1,R_1)/Inf(K^0(X_1,R_1))$ *and* $K^0(X_2,R_2)/Inf(K^0(X_2,R_2))$ *are isomorphic as ordered abelian groups with distinguished order units.*

Proof. The 'only if' statement follows from Theorem 2.8 and we are left to prove the 'if' direction. We note that $H = Inf(K^0(X,R))$ satisfies the hypothesis of Theorem 3.1 as follows. If a is in $K^0(X,R)$ and not in $Inf(K^0(X,R))$, then by Theorem 2.17 there is a state ϕ on $K^0(X,R)$ such that $\phi(a) \neq 0$. It follows that for any non-zero integer n, $\phi(na) \neq 0$ and hence, na is not in $Inf(K^0(X,R))$. We apply Theorem 3.1 (twice) in the special case of $H = Inf(K^0(X,R))$ to find AF-relations $\tilde{R}_1 \supset R_1$ on X_1 and $\tilde{R}_2 \supset R_2$ on X_2 such that

$$K^0(X_i, \tilde{R}_i) \cong K^0(X_i,R_i)/Inf(K^0(X_i,R_i)),$$

as ordered abelian groups with distinguished order unit, for $i = 1, 2$. It follows from Theorem 3.1 that $\tilde{R}_1 \sim R_1$ and $\tilde{R}_2 \sim R_2$. Moreover, from the hypothesis and the Elliott-Krieger Theorem 2.10 we have $\tilde{R}_1 \cong \tilde{R}_2$. This completes the proof. □

From this point, it becomes an easy matter to extend the classification up to orbit equivalence to include $\mathbb{Z}$-actions. We recall the result from [10]. At this point, it is an immediate consequence of 3.2 and 2.23.

Corollary 3.3. *Let (X_1, R_1) and (X_2, R_2) be either minimal AF-equivalence relations or arise from minimal actions of $\mathbb{Z}$ on a Cantor set. They are orbit equivalent if and only if $K^0(X_1, R_1)/Inf(K^0(X_1, R_1))$ and $K^0(X_2, R_2)/Inf(K^0(X_2, R_2))$ are isomorphic as ordered abelian groups with distinguished order units.*

4 Proof of the main result

In this section, we give the proof of 3.1. In view of Theorem 2.4, we may find a Bratteli diagram (V, E) such that

$$(X(V, E), R(V, E)) \cong (X, R).$$

We will suppress the isomorphism in our notation and we will also implicitly identify $K^0(V, E)$ and $K^0(X, R)$, as in Theorem 2.9.

Proposition 4.1. *Let G be a simple dimension group and H be a subgroup of $Inf(G)$ such that G/H is torsion free. Then G/H with the quotient ordering as in section 2.6 is also a simple dimension group. Moreover, an element x in G is strictly positive if and only if $x + H$ is strictly positive in G/H.*

Proof. We let q denote the quotient map from G to G/H. The hypotheses that G/H is torsion free implies that the quotient group is unperforated as follows. Suppose x is in G and for some $n \geq 2$, $nx + H$ is positive. If $nx + H = H$, then nx is in H and since the quotient is torsion free, x is in H and so $x + H$ is positive. Otherwise, nx is in $G^+ - \{0\}$ and this implies that x is in $G^+ - \{0\}$ and so $x + H$ is positive. It is also easy to verify that G/H satisfies Riesz interpolation and so it is a dimension group. If $I \neq \{0\}$ is an order ideal in this group then $q^{-1}(I)$ is a non-zero order ideal in G and therefore equals G. It follows that I itself is trivial and so we see that G/H is simple. Finally, we note that from Theorem 2.16 and the second part of Theorem 2.17 that H does not contain a strictly positive element. It follows that an element of G is strictly positive if and only if its image under q is also. □

We apply the result to the case $G = K^0(X, R)$ and we continue to let q denote the quotient map from $K^0(X, R)$ to $K^0(X, R)/H$.

By the Effros-Handelman-Shen Theorem 2.13, we may find a Bratteli diagram, (W, F), such that

$$K^0(W, F) \cong K^0(X, R)/H$$

as ordered abelian groups with order unit. Moreover, as $K^0(X, R)/H$ is simple, the diagram (W, F) is also simple by Theorem 2.14.

As earlier in section 2.6, we use $\varepsilon_{m,m'}$ to denote the natural group homomorphism from $\mathbb{Z}V_m$ to $\mathbb{Z}V_{m'}$ induced by E, for each $0 \leq m < m'$. We use $\theta_{n,n'}$ to denote the natural group homomorphism from $\mathbb{Z}W_n$ to $\mathbb{Z}W_{n'}$ induced by F, for each $0 \leq n < n'$. As described following Theorem 2.5, we may assume the number of vertices in W_n is strictly increasing with n and hence tends to infinity.

Let us provide some motivation for the proof. It comes from the rather pleasant example given just after the statement of Theorem 3.1. Although we did not state it there, it is fairly evident that the Bratteli diagram (W, F) in that case consists of one vertex at every level and three edges (except for E_1 which has two edges). Its path space is just the sequence space $\{1, 3\} \times \{1, 2, 3\}^{\mathbb{N}}$. In this example, what is very special is that the generators of the group H are presented at each level in the form $v_n - v'_n$. This leads directly to the fact that the quotient map from $K^0(V, E)$ to $K^0(W, F)$ can be presented in a very nice way by simply drawing one edge from each vertex of V_n to the single vertex of W_n. This also allows us to identify the path spaces of the two diagrams. Our first difficulty in proving the general result is that there is no reason that the elements of H should have such a nice form. Our strategy is to construct a replacement for the diagram (V, E), which we call $(\bar{V}, \bar{E})$, which resembles that in the example. The diagram (W, F) will not need to be changed.

Before we go further, let us recall some notation. If A, B are finite sets and $h : \mathbb{Z}A \to \mathbb{Z}B$ is a group homomorphism, then for every a in A and b in $h(a, b)$ is the unique integer such that

$$h(a) = \sum_{b \in B} h(a, b)b,$$

for every a in A.

We inductively define non-negative integers $0 = n_0 = m_0 < n_1 < m_1 < \cdots$, and positive group homomorphisms $q_k : \mathbb{Z}V_{m_{k-1}} \to \mathbb{Z}W_{n_k}$ and $\rho_k : \mathbb{Z}W_{n_k} \to \mathbb{Z}V_{m_k}$, for $k \geq 1$. The q_k's are local representatives for the quotient map q. The maps ρ_k will serve as local liftings for q, although there may be no global lifting as a group homomorphism from $K^0(X,R)/H$ to $K^0(X,R)$. More precisely, these will satisfy the following conditions:

1. $[q_k(a), n_k] = q[a, m_{k-1}]$, for all $k \geq 1, a \in \mathbb{Z}V_{m_{k-1}}$,
2. $q_k(v, w) \geq 2$, for all $v \in V_{m_{k-1}}, w \in W_{n_k}$, for all $k \geq 1$,
3. if $q[a, m_{k-1}] = 0$, for some $a \in \mathbb{Z}V_{m_{k-1}}$, then $q_k(a) = 0$, for all $k \geq 1$,
4. $q_k \circ \varepsilon_{m_{k-2},m_{k-1}} = \theta_{n_{k-1},n_k} \circ q_{k-1}$, for all $k \geq 2$,
5. $q_k \circ \rho_{k-1} = \theta_{n_{k-1},n_k}$, for all $k \geq 2$,
6. $\#W_{n_k} > \#V_{m_{k-1}}$, for all $k \geq 1$,
7. $q[\rho_k(b), m_k] = [b, n_k]$, for all $k \geq 1, b \in \mathbb{Z}W_{n_k}$,
8. $\pm 2(k+1)(\rho_k \circ q_k(v) - \varepsilon_{m_{k-1},m_k}(v)) \leq \rho_k(w)$, for all $k \geq 1$, v in $V_{m_{k-1}}$ and all w in W_{n_k}.
9. $2(k+1) \leq \rho_k(w, v')$, for all $k \geq 1$, $w \in W_{n_k}$ and $v' \in V_{m_k}$.

We begin with $n_0 = m_0 = 0$. Assume now that n_{k-1}, m_{k-1} have been defined for some $k > 0$, as well as q_{k-1} and ρ_{k-1}, if $k > 1$. We will first define q_k and n_k and then move on to ρ_k and m_k. A comment is in order before we start. Consider any of the first five conditions. Suppose that we have a map $q_k : \mathbb{Z}V_{m_{k-1}} \to \mathbb{Z}W_n$ satisfying the condition. If $n' > n$, then $\theta_{n,n'} \circ q_k : \mathbb{Z}V_{m_{k-1}} \to \mathbb{Z}W_{n'}$ also satisfies the same condition. In fact, a little more is true. If the condition is satisfied for a particular element of $\mathbb{Z}V_{m_{k-1}}$, then this second map will also satisfy it on that element. We will say that the condition continues to hold 'after increasing n'.

First, we consider the map sending a in $\mathbb{Z}V_{m_{k-1}}$ to $q[a, m_{k-1}]$ in $K^0(X,R)/H$. For each v in $V_{m_{k-1}}$, we may find $n > m_{k-1}$ and $q_k(v) \in \mathbb{Z}^+W_n$ such that $[q_k(v), n] = q[v, m_{k-1}]$. As the set $V_{m_{k-1}}$ is finite, we may choose the same n (as the largest of the n's for the individual v's) for all v. This has a unique extension to a positive group homomorphism, also denoted by q_k, into $\mathbb{Z}W_n$. We have $[q_k(a), n] = q[a, m_{k-1}]$, for all a in $\mathbb{Z}V_{m_{k-1}}$, so q_k satisfies the first of our conditions.

Let v be in $V_{m_{k-1}}$. As $[v, m_{k-1}]$ is strictly positive, it is not in the kernel of q and so $[q_k(v), n]$ is strictly positive. Using the fact that $K^0(X,R)/H$

is simple and the comments following Theorem 2.5, we may replace q_k by $\theta_{n,n'} \circ q_k$ to have $q_k(v,w) \geq 2$, for all $v \in V_{m_{k-1}}$ and $w \in W_n$. This does not change the first condition and it now satisfies the second as well.

Next, we consider the kernel of the map sending a in $\mathbb{Z}V_{m_{k-1}}$ to $q[a, m_{k-1}]$ in $K^0(X,R)/H$. Since it is a subgroup of a finitely generated group, it is itself finitely generated. If a is a generator, we know that $0 = q[a, m_{k-1}] = [q_k(a), n]$, and this implies that there exists $n' \geq n$ such that $\theta_{n',n}(q_k(a)) = 0$. By increasing n, we may assume that for all a in $\mathbb{Z}V_{m_{k-1}}$ with $q[a, m_{k-1}] = 0$, we have $q_k(a) = 0$. Again, this does not adversely affect the first two conditions. Next, by increasing n, we may assume also that $\#W_n > \#V_{m_{k-1}}$. If $k = 1$, that is all we need; we set $n_1 = n$ and q_1 as above and proceed to the construction of m_1 and ρ_1.

We now suppose that $k \geq 2$. For each a in $\mathbb{Z}V_{m_{k-2}}$, we have

$$\begin{aligned}
[q_k \circ \varepsilon_{m_{k-2},m_{k-1}}(a), n] &= q[\varepsilon_{m_{k-2},m_{k-1}}(a), m_{k-1}] \\
&= q[a, m_{k-2}] \\
&= [q_{k-1}(a), n_{k-1}] \\
&= [\theta_{n_{k-1},n} \circ q_{k-1}(a), n]
\end{aligned}$$

Since the group $\mathbb{Z}V_{m_{k-2}}$ is finitely generated, we may find n' sufficiently large so that

$$\theta_{n,n'} \circ q_k \circ \varepsilon_{m_{k-2},m_{k-1}}(a) = \theta_{n_{k-1},n'} \circ q_{k-1}(a),$$

for each generator a and hence for all a in $\mathbb{Z}V_{m_{k-2}}$. This does not change the earlier conditions and we now have the first four.

We also know that, for all b in $\mathbb{Z}W_{n_{k-1}}$,

$$[q_k \circ \rho_{k-1}(b), n] = q[\rho_{k-1}(b), m_{k-1}] = [b, n_{k-1}] = [\theta_{n_{k-1},n}(b), n],$$

where we have used the seventh condition, which holds for ρ_{k-1} by induction hypothesis, for the second equality. Exactly the same argument as we used a moment ago means that by increasing n we may assume that $q_k \circ \rho_{k-1} = \theta_{n_{k-1},n}$. Again this does not affect the earlier conditions, but it establishes the fifth of the desired properties for q_k and n. This value of n we denote by n_k.

Next, we begin to define ρ_k and m_k as follows. For each w in W_{n_k}, we may find a strictly positive element of $K^0(X,R)$ whose image under q is $[w, n_k]$. Each of these in turn, may be represented by a positive element in $\mathbb{Z}V_l$, for some l. As W_{n_k} is finite, we can assume these are all located in the same $\mathbb{Z}V_l$ and $l > n_k$. This yields a function from W_{n_k} to $\mathbb{Z}^+V_l$ which has a unique extension to a positive group homomorphism, $\lambda_k : \mathbb{Z}W_{n_k} \to \mathbb{Z}V_l$, satisfying

$$q[\lambda_k(w), l] = [w, n_k], \text{ for all } w \in W_{n_k}.$$

For any v in $V_{m_{k-1}}$, we compute

$$q[\lambda_k \circ q_k(v), l] = [q_k(v), n_k] = q[v, m_{k-1}].$$

It follows that $[(\varepsilon_{m_{k-1},l} - \lambda_k \circ q_k)(v), l]$ is in the kernel of q which is H and hence is infinitesimal. On the other hand, for each w in W_{n_k}, $\lambda_k(w)$ is strictly positive in $\mathbb{Z}V_l$. It follows that we may find $m \geq l$ such that

$$\pm 2(k+1)\varepsilon_{l,m}(\varepsilon_{m_{k-1},l}(v) - \lambda_k \circ q_k(v)) \leq \varepsilon_{l,m} \circ \lambda_k(w),$$

for all v in $V_{m_{k-1}}$ and all w in W_{n_k}. In addition, since the diagram (V, E) is simple and the elements $\lambda_k(w), w \in W_{n_k}$, are all strictly positive, we may find m sufficiently large so that

$$2(k+1) \leq \varepsilon_{l,m} \circ \lambda_k(w, v'),$$

for all $w \in W_{n_k}$ and $v' \in V_m$. We define m_k to be this value of m and set $\rho_k = \varepsilon_{l,m} \circ \lambda_k(w)$. Conditions 8 and 9 follow at once. As for condition 7, we have

$$q[\rho_k(b), m_k] = q[\varepsilon_{l,m} \circ \lambda_k(b), m_k] = q[\lambda_k(b), l] = [b, n_k],$$

for all b in $\mathbb{Z}W_{n_k}$. This completes the inductive definition of m_k, ρ_k.

For convenience, we define

$$\delta_k = \varepsilon_{m_{k-1},m_k} - \rho_k \circ q_k : \mathbb{Z}V_{m_{k-1}} \to \mathbb{Z}V_{m_k}.$$

For $v \in V_{m_{k-1}}, v' \in V_{m_k}$, let

$$\begin{aligned} \delta_k^0(v, v') &= 1 + \max\{0, \delta_k(v, v')\}, \\ \delta_k^1(v, v') &= 1 + \max\{0, -\delta_k(v, v')\}, \end{aligned}$$

and δ^0 and δ^1 be the associated group homomorphisms. In terms of our new notation, we have

$$\begin{aligned} \delta_k &= \delta_k^0 - \delta_k^1, \\ 1 &\leq \delta_k^0(v, v'), \delta_k^1(v, v'), \\ 2(k+1)(\delta_k^0(v, v') - 1) &\leq \rho_k(w, v'), \\ 2(k+1)(\delta_k^1(v, v') - 1) &\leq \rho_k(w, v'), \end{aligned}$$

for all v in $V_{m_{k-1}}$, v' in V_{m_k} and w in W_{n_k}. Adding the equation in condition 9 to each of the last two and dividing by 2, we have the following.

Lemma 4.2. *For all v in $V_{m_{k-1}}$, v' in V_{m_k} and w in W_{n_k}, we have*

$$\begin{aligned}(k+1)\delta_k^1(v,v') &\leq \rho_k(w,v'),\\ (k+1)\delta_k^0(v,v') &\leq \rho_k(w,v').\end{aligned}$$

Lemma 4.3. *For all $k \geq 1$, we have*

$$q_{k+1} \circ \delta_k = q_{k+1} \circ (\varepsilon_{m_{k-1},m_k} - \rho_k \circ q_k) = 0.$$

Proof. First, for any x in $\mathbb{Z}V_{m_{k-1}}$, we compute

$$\begin{aligned} q[\delta_k(x), m_k] &= q[(\varepsilon_{m_{k-1},m_k} - \rho_k \circ q_k)(x), m_k] \\ &= q[\varepsilon_{m_{k-1},m_k}(x), m_k] - q[\rho_k \circ q_k(x), m_k] \\ &= q[x, m_{k-1}] - [q_k(x), n_k] \\ &= 0. \end{aligned}$$

The conclusion now follows from the third condition. □

The next step is for notational convenience; we telescope our original diagrams (V,E) and (W,F) to the sequences $m_k, k \geq 0$, and $n_k, k \geq 0$, respectively. The effect on our notation is simply to make $m_k = k = n_k$, for all $k \geq 0$.

With our new indexing, we have $\#W_k > \#V_{k-1}$, for all $k \geq 1$ by condition 6. Choose any injective map $j_k : V_{k-1} \to W_k$, for each $k \geq 1$.

We now define a new Bratteli diagram, $(\bar{V}, \bar{E})$, as follows. We set $\bar{V}_0 = V_0$ and

$$\bar{V}_k = W_k \cup V_{k-1},$$

for all $k \geq 1$, where the union is considered as disjoint. Next, we define group homomorphisms $\xi_k : \mathbb{Z}V_{k-1} \to \mathbb{Z}\bar{V}_k$ and $\eta_k : \mathbb{Z}\bar{V}_k \to \mathbb{Z}V_k$ by

$$\begin{aligned} \xi_k(v) &= q_k(v) - j_k(v) + v, & v \in V_{k-1},\\ \eta_k(w) &= \rho_k(w), & w \in W_k,\\ \eta_k(v) &= \delta_k(v) + \rho_k(j_k(v)), & v \in V_{k-1}. \end{aligned}$$

We recall that $q_k(v,w) \geq 2$, for all choices of v and w and this ensures that ξ_k is positive. In addition, it follows from the conditions of Lemma 4.2

that η_k is also positive. Consider the composition $\xi_k \circ \eta_{k-1} : \mathbb{Z}\bar{V}_{k-1} \to \mathbb{Z}\bar{V}_k$, which is a positive homomorphism. We let $\bar{E}_k$ be the edge set whose induced homomorphism, denoted $\bar{\varepsilon}_{k-1,k}$, equals $\xi_k \circ \eta_{k-1}$. This completes our definition of the Bratteli diagram $(\bar{V}, \bar{E})$.

Lemma 4.4. *For all $k \geq 1$, we have*

$$\eta_k \circ \xi_k = \varepsilon_{k-1,k}.$$

The proof is by direct computation using the definitions and we omit it.

At this point, we could define edge sets going from V_{k-1} to $\bar{V}_k$ and from $\bar{V}_k$ to V_k which induce the maps ξ_k and η_k. The result would be an intertwining Bratteli diagram whose odd levels are $\bar{V}$ and even levels are V. The point of the last Lemma is that if we were to telescope this diagram to its even levels, we would obtain (V, E). Contracting to the odd levels gives $(\bar{V}, \bar{E})$, simply from the definition of $\bar{E}$. We conclude that

$$(X, R) \cong (X(V, E), R(V, E)) \cong (X(\bar{V}, \bar{E}), R(\bar{V}, \bar{E})).$$

While it is not crucial that we have an explicit notation for this isomorphism, it is important for us to note that the isomorphism which it induces between the associated K^0-groups is presented as follows. The proof is routine and we omit the details.

Lemma 4.5. *We have $(X(V, E), R(V, E)) \cong (X(\bar{V}, \bar{E}), R(\bar{V}, \bar{E}))$. Moreover, the isomorphism and its inverse induces the maps $\xi : K^0(V, E) \to K^0(\bar{V}, \bar{E})$ and $\eta : K^0(\bar{V}, \bar{E}) \to K^0(V, E)$, respectively, defined by*

$$\begin{aligned} \xi[x, k-1] &= [\xi_k(x), k], \quad x \in \mathbb{Z}V_{k-1}, \\ \eta[y, k] &= [\eta_k(y), k], \quad y \in \mathbb{Z}\bar{V}_k \end{aligned}$$

We now have a better diagram for the AF-relation (X, R) and the dimension group $K^0(X, R)$. We give a formula for the quotient map from $K^0(X, R)$ to $K^0(X, R)/H$, in terms of this new diagram.

Lemma 4.6. *For each $k \geq 1$, let $\bar{q}_k : \mathbb{Z}\bar{V}_k \to \mathbb{Z}W_k$ be defined by*

$$\begin{aligned} \bar{q}_k(w) &= w, w \in W_k, \\ \bar{q}_k(v) &= j_k(v), v \in V_{k-1} \end{aligned}$$

Then we have $\bar{q}_k \circ \xi_k = q_k$ and hence $q \circ \eta[x, k] = [\bar{q}_k(x), k]$, for all $[x, k]$ in $K^0(X, R)$.

The proof is an easy computation using the definitions and we omit it.

Lemma 4.7. *For all $k \geq 0$, we have*

$$\bar{q}_{k+1} \circ \bar{\varepsilon}_{k,k+1} = \theta_{k,k+1} \circ \bar{q}_k.$$

Again, the proof is a simple computation, first applying both sides to w in W_k and then to v in V_{k-1}. We omit the details.

The next computation will actually be the key technical step in our construction, but for the moment, it is a little difficult to motivate.

Lemma 4.8. *For all $k \geq 1$ and v in V_{k-1}, we have*

$$\bar{\varepsilon}_{k,k+1}(v - j_k(v)) = \delta_k(v) - j_{k+1}(\delta_k(v)).$$

Proof. We compute directly using the definitions and Lemma 4.3

$$\begin{aligned}
\bar{\varepsilon}_{k,k+1}(v - j_k(v)) &= \xi_{k+1} \circ \eta_k(v - j_k(v)) \\
&= \xi_{k+1}(\eta_k(v) - \eta_k(j_k(v))) \\
&= \xi_{k+1}(\delta_k(v) + \rho_k(j_k(v)) - \rho_k(j_k(v))) \\
&= \xi_{k+1}(\delta_k(v)) \\
&= q_{k+1}(\delta_k(v)) - j_{k+1}(\delta_k(v)) + \delta_k(v) \\
&= 0 - j_{k+1}(\delta_k(v)) + \delta_k(v) \\
&= \delta_k(v) - j_{k+1}(\delta_k(v)).
\end{aligned}$$

□

The relationship between the new diagram $(\bar{V}, \bar{E})$ and (W, F) becomes fairly simple. The details are contained in the next two results.

Lemma 4.9. *Let $k \geq 0$.*

1. *For w in W_k and w' in $W_{k+1} - j_{k+1}(V_k)$, we have*

$$\bar{\varepsilon}_{k,k+1}(w, w') = \theta_{k,k+1}(w, w').$$

2. *For v in V_{k-1} and w' in $W_{k+1} - j_{k+1}(V_k)$, we have*

$$\bar{\varepsilon}_{k,k+1}(v, w') = \theta_{k,k+1}(j_k(v), w').$$

3. For w in W_k and v' in V_k, we have

$$\bar{\varepsilon}_{k,k+1}(w, v') + \bar{\varepsilon}_{k,k+1}(w, j_{k+1}(v')) = \theta_{k,k+1}(w, j_{k+1}(v')).$$

4. For v in V_{k-1} and v' in V_k, we have

$$\bar{\varepsilon}_{k,k+1}(v, v') + \bar{\varepsilon}_{k,k+1}(v, j_{k+1}(v')) = \theta_{k,k+1}(j_k(v), j_{k+1}(v')).$$

Proof. We work from the equation $\theta_{k,k+1} \circ \bar{q}_k = \bar{q}_{k+1} \circ \bar{\varepsilon}_{k,k+1}$ established in Lemma 4.7 and the fact that, for w in W_k and v in $\bar{V}_k$, $\bar{q}_k(v, w) = 1$ if $w = v$ or if $w = j_k(v)$ and is zero otherwise.

If v is in $\bar{V}_k$ and w' is in $W_{k+1} - j_{k+1}(V_k)$, then

$$(\bar{q}_{k+1} \circ \bar{\varepsilon}_{k,k+1})(v, w') = \sum_{v' \in \bar{V}_{k+1}} \bar{q}_{k+1}(v', w')\bar{\varepsilon}_{k,k+1}(v, v') = \bar{\varepsilon}_{k,k+1}(v, w').$$

The first two parts follow at once as they are simply the cases of v in W_k and v in V_{k-1}, respectively.

If v is in $\bar{V}_k$ and v' is in V_k, then

$$\begin{aligned} (\bar{q}_{k+1} \circ \bar{\varepsilon}_{k,k+1})(v, j_k(v')) &= \sum_{v'' \in \bar{V}_{k+1}} \bar{q}_{k+1}(v', v'')\bar{\varepsilon}_{k,k+1}(v, v'') \\ &= \bar{\varepsilon}_{k,k+1}(v, v') + \bar{\varepsilon}_{k,k+1}(v, j_{k+1}(v')). \end{aligned}$$

Moreover, we have

$$\theta_{k,k+1} \circ \bar{q}_k(v, j_{k+1}(v')) = \theta_{k,k+1}(\bar{q}_k(v), j_{k+1}(v')) = \theta_{k,k+1}(j_k(v), j_{k+1}(v')).$$

The last two parts follow at once. □

Lemma 4.10. *For v in V_{k-1} and v' in V_{k-1}, we have*

$$\begin{aligned} \bar{\varepsilon}_{k,k+1}(v, v') - \delta^0_k(v, v') &= \bar{\varepsilon}_{k,k+1}(j_k(v), v') - \delta^1_k(v, v'), \\ \bar{\varepsilon}_{k,k+1}(v, j_{k+1}(v')) - \delta^1_k(v, v') &= \bar{\varepsilon}_{k,k+1}(j_k(v), j_{k+1}(v')) - \delta^0_k(v, v') \end{aligned}$$

Proof. We begin from the result of Lemma 4.8:

$$\bar{\varepsilon}_{k,k+1}(v - j_k(v)) = \delta_k(v) - j_{k+1}(\delta_k(v)).$$

We consider the coefficient of v' on each side. On the left, we have

$$\bar{\varepsilon}_{k,k+1}(v, v') - \bar{\varepsilon}_{k,k+1}(j_k(v), v').$$

On the right, we observe that the term $j_{k+1}(\delta_k(v))$ is a combination of the generators $j_{k+1}(V_k)$ and hence, all of its V_k terms have zero coefficient. That means the coefficient of v' on the right hand side is

$$\delta_k(v, v') = \delta_k^0(v, v') - \delta_k^1(v, v').$$

Equating the two sides and re-arranging the terms yields the first result.

For the second statement, we compare the coefficients of $j_{k+1}(v')$ on each side. On the left we have

$$\bar{\varepsilon}_{k,k+1}(v, j_{k+1}(v')) - \bar{\varepsilon}_{k,k+1}(j_k(v), j_{k+1}(v')).$$

On the right, $\delta_k(v)$ is a combination of the generators V_k and so all of its $j_{k+1}(V_k)$ coefficients are zero. Moreover, the coefficient of $j_{k+1}(v')$ in $j_{k+1}(\delta_k(v))$ is $\delta_k(v, v')$. Putting these together, the coefficient of $j_{k+1}(v')$ on the right hand side is

$$-\delta_k(v, v') = \delta_k^1(v, v') - \delta_k^0(v, v').$$

Equating the two sides and re-arranging the terms yields the second result. □

We next need the following estimates.

Lemma 4.11. *For $k \geq 1, v \in V_{k-1}$ and $v' \in V_k$, we have*

$$\begin{array}{rcl} k\delta_k^0(v, v') & \leq & \bar{\varepsilon}_{k,k+1}(v, v'), \\ k\delta_k^0(v, v') & \leq & \bar{\varepsilon}_{k,k+1}(j_k(v), j_{k+1}(v')), \\ k\delta_k^1(v, v') & \leq & \bar{\varepsilon}_{k,k+1}(j_k(v), v'), \\ k\delta_k^1(v, v') & \leq & \bar{\varepsilon}_{k,k+1}(v, j_{k+1}(v')). \end{array}$$

Proof. First, recall that $\bar{\varepsilon}_{k,k+1} = \xi_{k+1} \circ \eta_k$ and notice from the definition of ξ_{k+1}, that for any $v'', v' \in V_k$ in we have $\xi_{k+1}(v'', v') = 1$ if $v'' = v'$ and is zero otherwise. In addition, we also have

$$\xi_{k+1}(v', j_{k+1}(v')) = q_{k+1}(v', j_{k+1}(v')) - 1 \geq 2 - 1 = 1.$$

We begin with the first statement. By Lemma 4.10, we have

$$\begin{array}{rcl} \bar{\varepsilon}_{k,k+1}(v, v') & = & \displaystyle\sum_{v'' \in V_k} \eta_k(v, v'')\xi_{k+1}(v'', v') \\ & = & \eta_k(v, v') \\ & = & \rho_k(j_k(v), v') + \delta_k(v, v'). \end{array}$$

If $\delta_k(v,v') \leq 0$, then $\delta_k^0(v,v') = 1$ and $\delta_k(v,v') = 1 - \delta_k^1(v,v')$ and we have

$$\begin{aligned}
\bar{\varepsilon}_{k,k+1}(v,v') &= \rho_k(j_k(v),v') + (1 - \delta_k^1(v,v')) \\
&\geq (k+1)\delta_k^1(v,v') + (1 - \delta_k^1(v,v')) \\
&\geq k\delta_k^1(v,v') \\
&\geq k \\
&= k\delta_k^0(v,v').
\end{aligned}$$

If $\delta_k(v,v') \geq 0$, then the conclusion follows since $\rho_k(j_k(v),v') \geq (k+1)\delta_k^0(v,v')$.

For the second inequality, we have

$$\begin{aligned}
\bar{\varepsilon}_{k,k+1}(j_k(v),j_{k+1}(v')) &= \sum_{v'' \in V_k} \eta_k(j_k(v),v'')\xi_{k+1}(v'',j_{k+1}(v')) \\
&\geq \eta_k(j_k(v),v')\xi_{k+1}(v',j_{k+1}(v')) \\
&\geq \eta_k(j_k(v),v') \\
&= \rho_k(j_k(v),v') \\
&\geq (k+1)\delta_k^0(v,v') \\
&\geq k\delta_k^0(v,v').
\end{aligned}$$

For the third inequality, we have

$$\begin{aligned}
\bar{\varepsilon}_{k,k+1}(j_k(v),v') &= \sum_{v'' \in V_k} \eta_k(j_k(v),v'')\xi_{k+1}(v'',v') \\
&= \eta_k(j_k(v),v') \\
&= \rho_k(j_k(v),v') \\
&\geq (k+1)\delta_k^1(v,v') \\
&\geq k\delta_k^1(v,v').
\end{aligned}$$

Finally, we have

$$\begin{aligned}
\bar{\varepsilon}_{k,k+1}(v,j_{k+1}(v')) &= \sum_{v'' \in V_k} \eta_k(v,v'')\xi_{k+1}(v'',j_{k+1}(v')) \\
&\geq \eta_k(v,v')\xi_{k+1}(v',j_{k+1}(v')) \\
&\geq \eta_k(v,v') \\
&= \rho_k(j_k(v),v') + \delta_k(v,v').
\end{aligned}$$

If $\delta_k(v,v') \geq 0$, then the conclusion follows since $\rho_k(j_k(v),v') \geq (k+1)\delta_k^1(v,v')$. If $\delta_k(v,v') \leq 0$, then $\delta_k^0(v,v') = 1$ and $\delta_k(v,v') = 1 - \delta_k^1(v,v')$ and we have

$$\begin{aligned} \rho_k(j_k(v),v') + \delta_k(v,v') &= \rho_k(j_k(v),v') + (1 - \delta_k^1(v,v')) \\ &\geq (k+1)\delta_k^1(v,v') + 1 - \delta_k^1(v,v') \\ &\geq k\delta_k^1(v,v'). \end{aligned}$$

□

We are now ready to move on to dynamical aspects. We know that the diagram $(\bar{V},\bar{E})$ gives us a presentation of our AF-relation (X,R). The new relation $\tilde{R}$ will be presented by (W,F). Of course, the first difficulty is to establish that the path spaces for $(\bar{V},\bar{E})$ and (W,F) are the same. More precisely, we define a homeomorphism h between them. This is done by first defining a graph homomorphism from $(\bar{V},\bar{E})$ to (W,F). It will cause no confusion to denote this by h also.

For $k \geq 0$, we define $h : \bar{V}_k \to W_k$ by $h(w) = w, w \in W_k \subset \bar{V}_k$ and $h(v) = j_k(v), v \in V_{k-1}$. (The reader will note that this is the same as the definition of $\bar{q}_k$, but we prefer to regard $\bar{q}_k$ as a group homomorphism and h as a function between vertex sets.) It is worth noting that, since j_k is injective, for any w in W_k, $h^{-1}\{w\}$ is either simply $\{w\}$ if w is not in the image of j_k or else $\{v, j_k(v)\}$, if $w = j_k(v)$.

Let w be in W_k and w' be in $W_{k+1} - j_{k+1}(V_k)$. By virtue of part 1 of Lemma 4.9, we may find a bijection

$$h : \bar{E}_{k+1}(w,w') \to F_{k+1}(w,w'),$$

Similarly, for v in V_{k-1} and w' in $W_{k+1} - j_{k+1}(V_k)$, we may find a bijection

$$h : \bar{E}_{k+1}(v,w') \to F_{k+1}(j_k(v),w'),$$

using part 2 of Lemma 4.9.

Next, we consider w in W_k and v' in V_k . From part 3 of Lemma 4.9, we may find a bijection

$$h : \bar{E}_{k+1}(w,v') \cup \bar{E}_{k+1}(w, j_{k+1}(v')) \to F_{k+1}(w, j_{k+1}(v')).$$

Finally, we consider v in V_{k-1} and v' in V_k. From part 4 of Lemma 4.9, we may find a bijection

$$h : \bar{E}_{k+1}(v,v') \cup \bar{E}_{k+1}(v, j_{k+1}(v')) \to F_{k+1}(j_k(v), j_{k+1}(v')).$$

These last two maps will have to satisfy extra conditions when $k \geq 1$, as we describe below. We summarize the properties of the map h (so far) as follows.

Lemma 4.12. *1. The map h is a graph homomorphism from $(\bar{V}, \bar{E})$ to (W, F); that is, $i(h(e)) = h(i(e)), t(h(e)) = h(t(e))$, for all e in $\bar{E}$.*

2. For each v in $\bar{V}$, h is a bijection

$$h : \{e \in \bar{E} \mid i(e) = v\} \to \{f \in F \mid i(f) = h(v)\}.$$

Proof. The proof of the first part follows immediately from the definitions and we omit the details. The second part is also immediate from the definitions and the observation that the pre-image under h of a vertex w not in the image of j_k is just itself and the pre-image under h of $j_k(v)$ is $\{v, j_k(v)\}$. Again, we omit the details. □

As we mentioned above, we will require extra properties of h as follows. Let $k \geq 1$, v be in V_k and v' be in V_{k+1}. In view of Lemma 4.11, we may find subsets

$$\begin{array}{rcl} \Delta_k(v, v') & \subset & \bar{E}_k(v, v') \\ \Delta_k(j_k(v), j_{k+1}(v')) & \subset & \bar{E}_k(j_k(v), j_{k+1}(v')) \\ \Delta_k(v, j_{k+1}(v')) & \subset & \bar{E}_k(v, j_{k+1}(v')) \\ \Delta_k(j_k(v), v') & \subset & \bar{E}_k(j_k(v), v') \end{array}$$

such that

$$\begin{array}{rcccl} \#\Delta_k(v, v') & = & \#\Delta_k(j_k(v), j_{k+1}(v')) & = & \delta_k^0(v, v') \\ \#\Delta_k(j_k(v), v') & = & \#\Delta_k(v, j_{k+1}(v')) & = & \delta_k^1(v, v'). \end{array}$$

So we require that our map h satisfies

$$\begin{array}{rcccl} h(\Delta_k(v, v')) & = & h(\Delta_k(j_k(v), j_{k+1}(v'))) & \subset & F_k(j_k(v), j_{k+1}(v')) \\ h(\Delta_k(j_k(v), v')) & = & h(\Delta_k(v, j_{k+1}(v'))) & \subset & F_k(j_k(v), j_{k+1}(v')) \end{array}$$

Moreover, since h is assumed to be injective on the union of $E_k(v, v') \cup E_k(v, j_{k+1}(v'))$, these two images above are disjoint.

We need to introduce a little notation which will be useful later. Let $\alpha : \Delta_k(v, v') \to \Delta_k(j_k(v), j_{k+1}(v'))$ be the unique function such that $h(\alpha(e)) =$

$h(e)$, for all e in $\Delta_k(v, v')$. Also denote by α the inverse of this map so α is a permutation of the union of these sets and α^2 is the identity. Similarly, we define α on $\Delta_k(j_k(v), v')) \cup \Delta_k(v, j_{k+1}(v'))$ such that $h(\alpha(e)) = h(e)$ and $\alpha^2(e) = e$, for all e in the union.

Now consider the sets

$$\bar{E}_k(v, v') - \Delta_k(v, v'), \quad \bar{E}_k(j_k(v), v') - \Delta_k(j_k(v), v').$$

In view of Lemma 4.10, these have the same number of elements. So we may assume that their images under h are equal in $F_k(j_k(v), j_{k+1}(v'))$. Once h is determined to satisfy this, it then follows that the images under h of

$$\bar{E}_k(v, j_{k+1}(v')) - \Delta_k(v, j_{k+1}(v')), \quad \bar{E}_k(j_k(v), j_{k+1}(v')) - \Delta_k(j_k(v), j_{k+1}(v'))$$

are also equal. To summarize the situation, we have

$$\begin{array}{rcl} h(\Delta_k(v, v')) & = & h(\Delta_k(j_k(v), j_{k+1}(v'))) \\ h(\Delta_k(v, j_{k+1}(v'))) & = & h(\Delta_k(j_k(v), v')) \\ h(\bar{E}_k(v, v') & - & \Delta_k(v, v')) \\ & = & h(\bar{E}_k(j_k(v), v') - \Delta_k(j_k(v), v')) \\ h(\bar{E}_k(v, j_{k+1}(v')) & - & \Delta_k(v, j_{k+1}(v'))) \\ & = & h(\bar{E}_k(j_k(v), j_{k+1}(v')) - \Delta_k(j_k(v), j_{k+1}(v'))) \end{array}$$

and $F_k(j_k(v), j_{k+1}(v'))$ is the disjoint union of these four sets.

Finally, we define $\Delta_1(w_0, v_0)$ and $\Delta_1(w_0, j_1(v_0))$ to have a single edge from $\bar{E}_1(w_0, v_0)$ and $\bar{E}_1(w_0, j_1(v_0))$, respectively. Consider the subdiagram of $(\bar{V}, \bar{E})$ whose vertex set at level zero is W_0 and for $k \geq 1$ is $j_k(V_{k-1}) \cup V_{k-1}$. Its edge set at level $k \geq 1$ is the union of $\Delta_k(v, v')$, $\Delta_k(j_k(v), j_{k+1}(v'))$, $\Delta_k(j_k(v), v')$ and $\Delta_k(v, j_{k+1}(v'))$, over all choices of v, v'.

Lemma 4.13. *If e, e' are distinct edges of $\bar{E}_k$ such that $h(e) = h(e')$, then for some v in V_{k-1}, $i(e) = v$ and $i(e') = j_k(v)$ (or vice verse). In addition, $t(e) \neq t(e')$ if and only if e, e' are in Δ_k and in this case $e' = \alpha(e)$.*

Proof. If $i(e) = i(e')$ and $h(e) = h(e')$, then it follows from part 2 of Lemma 4.12 that $e = e'$. As e and e' are assumed to be distinct and $h(e) = h(e')$, we must have $i(e) \neq i(e')$. On the other hand, we have $h(i(e)) = i(h(e)) = i(h(e')) = h(i(e'))$, from part 1 of 4.12. It follows that, for some v in some V_{k-1}, $i(e) = v$ and $i(e') = j_k(v)$ (or vice verse). Now consider $t(h(e)) = h(t(e))$. If it is in $W_{k+1} - j_{k+1}(V_k)$, then $h(t(e)) = h(t(e'))$ implies that $t(e) = t(e')$, since such a vertex has only one pre-image under h.

We are left to consider the case $t(h(e)) = j_{k+1}(v')$, for some v'. In this case, $h(e) = h(e')$ is in $F_k(j_k(v), j_{k+1}(v'))$ and so it is in exactly one of the four sets

$$\begin{array}{rcl} h(\Delta_k(v,v')) & = & h(\Delta_k(j_k(v), j_{k+1}(v'))) \\ h(\Delta_k(v, j_{k+1}(v'))) & = & h(\Delta_k(j_k(v), v')) \\ h(\bar{E}_k(v,v') & - & \Delta_k(v,v')) \\ & = & h(\bar{E}_k(j_k(v), v') - \Delta_k(j_k(v), v')) \\ h(\bar{E}_k(v, j_{k+1}(v')) & - & \Delta_k(v, j_{k+1}(v'))) \\ & = & h(\bar{E}_k(j_k(v), j_{k+1}(v')) - \Delta_k(j_k(v), j_{k+1}(v'))) \end{array}$$

As $i(e) = v$ and $i(e') = j_k(v)$, e must be in one of the sets on the left-hand side before applying h, while e' is in one of the sets on the right-hand side before applying h. In either of the first two cases, we know then that $h(e) = h(e')$ means $e' = \alpha(e)$. In either of the last two cases, we have $t(e) = t(e')$. This completes the proof. □

Theorem 4.14. *Define the map* $h : X(\bar{V}, \bar{E}) \to X(W, F)$ *by*

$$h(x_1, x_2, \ldots) = (h(x_1), h(x_2), \ldots),$$

for $(x_1, x_2, \ldots)$ *in* $X(\bar{V}, \bar{E})$*. Then* h *is a homeomorphism. Moreover,*

$$h \times h : R(\bar{V}, \bar{E})) \to R(W, F)$$

is continuous and open and its image is an open subequivalence relation.

Proof. We know from Lemma 4.12 that $h : (\bar{V}, \bar{E}) \to (W, F)$ is a graph homomorphism and so the map between the path spaces is well-defined and continuous. In addition, we know from 4.12 that

$$h : \{e \in \bar{E} \mid i(e) = v\} \to \{f \in F \mid i(f) = h(v)\}$$

is a bijection, for each vertex v in $\bar{V}$. The fact that h is a bijection from $X(\bar{V}, \bar{E})$ to $X(W, F)$ follows from this; we give a proof of injectivity. Suppose that x, x' are two paths in $X(\bar{V}, \bar{E})$ such that $h(x) = h(x')$. Clearly, we have $i(x_1) = w_0 = i(x'_1)$. Now suppose that, for some $k \geq 1$, $i(x_k) = i(x'_k)$. From this, the fact that $h(x_k) = h(x'_k)$ and h being i-bijective, it follows that $x_k = x'_k$. In consequence, we have $i(x_{k+1}) = t(x_k) = t(x'_k) = i(x'_{k+1})$. Repeating this argument shows that $x_k = x'_k$ for all k and hence $x = x'$. The argument for surjectivity is similar and we omit the details.

The last statement is clear from the definitions. □

Let Y denote the path space of the diagram Δ. Observe that α induces a homeomorphism of Y, also denoted by α, which satisfies $\alpha^2(y) = y$, for all y in Y. Finally, we let K be the equivalence relation on Y generated by α:

$$K = \{(y, y), (y, \alpha(y)) \mid y \in Y\}.$$

Theorem 4.15. *The set Y is a closed, $R(\bar{V}, \bar{E})$-étale, $R(\bar{V}, \bar{E})$-thin subset of $X(\bar{V}, \bar{E})$ and the equivalence relation K is étale and transverse to $R(\bar{V}, \bar{E})|_Y$. Moreover, we have*

$$h \times h(R(\bar{V}, \bar{E}) \vee K) = R(W, F).$$

Proof. Since Y is specified by a subdiagram, it is clear that it is closed and $R(\bar{V}, \bar{E})$-étale. The fact that it is $R(\bar{V}, \bar{E})$-thin follows from Theorem 2.21 and Lemma 4.11.

Since α fixes no edge of Δ, α is a free action of the group with two elements and hence K is étale. To show K is transverse to $R(\bar{V}, \bar{E})|_Y$, we first show that, for any y in Y, $(y, \alpha(y))$ is not in $R(\bar{V}, \bar{E})$. It suffices to notice that, if e is any edge in Δ, then $t(\alpha(e)) \neq t(e)$.

It is easy to see that α is an automorphism of the diagram Δ and so it induces an automorphism of $R(\bar{V}, \bar{E})|_Y$. The fact that it is transverse follows from the discussion preceding Theorem 2.22.

Next, we claim that $h \times h(R(\bar{V}, \bar{E}) \vee K) = R(W, F)$. We know already $h \times h(R(\bar{V}, \bar{E})) \subset R(W, F)$. If e is any edge of Δ_k, it follows from the definitions of h and α that $h(\alpha(e)) = h(e)$, provided $k \geq 2$. It then follows that $h \times h(K) \subset R(W, F)$. We have therefore established the containment $h \times h(R(\bar{V}, \bar{E}) \vee K) \subset R(W, F)$.

Let x, x' be in $X(\bar{V}, \bar{E})$ and suppose that $(h(x), h(x'))$ is in $R(W, F)$. We may find k_0 such that $h(x_k) = h(x'_k)$, for all $k \geq k_0$ and hence $h(i(x_k)) = h(i(x'_k))$. If $i(x_k) = i(x'_k)$, for any value of $k \geq k_0$, then by part 2 of Lemma 4.12, we know that $x_k = x'_k$. Then $i(x_{k+1}) = t(x_k) = t(x'_k) = i(x'_{k+1})$ and the same argument shows $x_{k+1} = x'_{k+1}$. Continuing in this way we see that $x_{k'} = x'_{k'}$, for all $k' \geq k$ and so (x, x') is in $R(\bar{V}, \bar{E})$.

We are left to consider the case that $i(x_k) \neq i(x'_k)$, for all values of $k \geq k_0$. It follows from Lemma 4.12 that either x_k, x'_k are in Δ and $\alpha(x_k) = x'_k$ or else $t(x_k) = t(x'_k)$. But the latter implies that $i(x_{k+1}) = i(x'_{k+1})$, a contradiction. We conclude that x_k, x'_k are in Δ and $\alpha(x_k) = x'_k$, for $k \geq 1$. As $\Delta_k(v, v')$ is non-empty for every v in V_k and v' in V_{k+1}, we may find a path, p, in Δ

from w_0 to $i(x_{k_0})$. Let

$$y = (p_1, \ldots, p_{k_0-1}, x_{k_0}, x_{k_0+1}, \ldots), y' = (\alpha(p_1), \ldots, \alpha(p_{k_0-1}), x'_{k_0}, x'_{k_0+1}, \ldots).$$

Then (x, y) and (y', x') are in $R(\bar{V}, \bar{E})$ while (y, y') is in K. This completes the proof. □

To complete the proof of Theorem 3.1, we proceed as follows. We have a Bratteli diagram, $(\bar{V}, \bar{E})$ such that $(X(\bar{V}, \bar{E}), R(\bar{V}, \bar{E}))$ is isomorphic to the given (X, R). Suppressing this isomorphism in our notation, we let Y and K be exactly as above. The first of the three conclusions is satisfied. We let $\tilde{R} = (h \times h)^{-1}(R(W, F))$; its topology is just the usual topology on $R(W, F)$ moved by $(h \times h)^{-1}$ so it is indeed AF and the second condition is satisfied. The third condition is simply a combination of Theorem 2.9, Lemma 4.5, Lemma 4.6 and the fact that $\bar{q}_k(v) = h(v)$, for any v in $\bar{V}_k$.

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