Ring and module structures on dimension groups associated with a shift of finite type

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Abstract

We study invariants for shifts of finite type obtained as the K-theory of various C^*-algebras associated with them. These invariants have been studied intensely over the past thirty years since their introduction by Wolfgang Krieger. They may be given quite concrete descriptions as inductive limits of simplicially ordered free abelian groups. Shifts of finite type are special cases of Smale spaces and, in earlier work, the second author has shown that the hyperbolic structure of the dynamics in a Smale space induces natural ring and module structures on certain of these K-groups. Here, we restrict our attention to the special case of shifts of finite type and obtain explicit descriptions in terms of the inductive limits.

1 Introduction

We provide a brief overview of our results. Precise versions of the definitions and results will follow in the later sections.

A Smale space, as defined by David Ruelle [14], is a compact metric space, X, together with a homeomorphism, φ, which is hyperbolic. These include the basic sets of Smale's Axiom A systems. Another special case of great interest are the shifts of finite type [2], [8] where the space, here usually denoted Σ, is the path space of a finite directed graph and the homeomorphism, σ, is the left shift.

Ruelle [15] later showed how one may construct various C^*-algebras from a Smale space (X, φ). These are best described as the groupoid C^*-algebras associated with stable, unstable and homoclinic equivalence and we denote them by $S(X, \varphi, P), U(X, \varphi, P)$ and $H(X, \varphi)$, respectively, where the first two depend on a choice of a set of periodic points P. This extended earlier work of Krieger [7] who considered the case of

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shifts of finite type. For shifts of finite type, these C^*-algebras can be computed quite explicitly and are AF or approximately finite dimensional C^*-algebras. As a consequence, their K-theory groups can be easily computed as inductive limits of simplicially ordered free abelian groups. The invariant $K_0(S(\Sigma, \sigma, P))$ (or $K_0(U(\Sigma, \sigma, P)))$ has been studied intensely. It may be computed from the adjacency matrix of the underlying graph, here denoted A. The matrix A is $K \times K$, where K is the number of vertices of the graph. The group $K_0(S(\Sigma, \sigma, P))$ is usually called the *dimension group* of A, as presented in section 7.5 of [8].

The original map, φ, in a Smale space provides automorphisms of the stable, unstable and homoclinic equivalence relations and hence automorphisms of the associated C^*-algebras, which we denote by α. In [10], the second author showed that the hyperbolic nature of the dynamics implies that the automorphism α of the homoclinic algebra $H(X, \varphi)$ is asymptotically abelian. From this, various elements in the E-theory of Connes and Higson were constructed in [10]. In doing so, it is necessary to pass from the discrete parameter (α is an action of the group of integers) to a continuous one. Thus, we are lead to consider the mapping cylinder for the automorphism $(H(X, \varphi), \alpha)$, which we denote by $C(H, \alpha)$, which now has an asymptotically abelian action of the group $\mathbb{R}$. In consequence, the K-theory of $C(H, \alpha)$ obtains a natural product structure which makes it an ordered ring. The K-groups of $S(X, \varphi, P)$ and $U(X, \varphi, P)$ are modules over this ring.

The main objective of this paper is to investigate these ring and module structures for shifts of finite type. The goal is two-fold. First, it seems evident from the additional structures on the C^*-algebras that these invariants, which go beyond the usual dimension groups that have been studied, are quite fundamental to the dynamics and should be the natural ones to be considered. Of course, the result in section 5 showing that they are invariants of shift equivalence suggests that they contain no more information. However, our second motivation is the possibility that the action of the automorphism group of a shift of finite type may act on the invariant in an interesting way, not detected by the dimension group alone. Given the importance of this automorphism group in the resolution of the shift equivalence verses strong shift equivalence conjecture [6] and the difficulty of understanding it thoroughly, it seems worthwhile to consider this possibility. We plan to investigate this direction in future work.

In the second section, we provide some background information and explain the nature of the products on K-theory in the setting of general Smale spaces.

Turning our attention to shifts of finite type in the third section, our first task is to describe $K_*(C(H, \alpha))$. Recalling that A is the adjacency matrix of the graph, we let $C(A)$ denote the centralizer of A in the ring $M_K(\mathbb{Z})$ of $K \times K$ integer matrices and let $B(A)$ denote the subgroup of all matrices of the form $[A, X] = AX - XA$, where X is in $M_K(\mathbb{Z})$. We prove in Theorem 3.3 that $K_0(C(H, \alpha))$ is the inductive limit of the stationary sequence using the group $C(A)$ with endomorphism sending X to AXA. An analogous description of $K_1(C(H, \alpha))$ is also given in 3.3 involving $B(A)$.

We then turn to the issue of finding an explicit formula for the product. It should be noted that this product is on $K_*(C(H, \alpha)) = K_0(C(H, \alpha)) \oplus K_1(C(H, \alpha))$, as a $\mathbb{Z}_2$-graded group, so there are several cases to consider. The results are in theorems 3.9, 3.18, and 3.26. Section 3.5 gives analogous formulae for the module structures.

In the fourth section, we provide a kind of duality result between the groups $K_0(S(\Sigma, \sigma, P))$ and $K_0(U(\Sigma, \sigma, P))$. Specifically, we identify a commutative subring R of $K_0(C(H, \alpha))$ such that

$$\begin{aligned} Hom_R(K_0(S(\Sigma, \sigma, P)), R) &\cong K_0(U(\Sigma, \sigma, P)), \\ Hom_R(K_0(U(\Sigma, \sigma, P)), R) &\cong K_0(S(\Sigma, \sigma, P)), \end{aligned}$$

as R-modules.

In the fifth section, we show that the ring and module structures are invariants of shift equivalence. In the sixth section, we show how Smale's spectral decomposition result may be used to extend the results from mixing SFTs to irreducible SFTs. Finally, in the last section, we present some examples, including one where the ring is non-commutative. To the best of our knowledge, this is the first example of a C^*-algebra whose K-theory has a natural ring structure which is not commutative.

2 Smale spaces

2.1 Dynamics

Smale spaces were defined by Ruelle in [14], based on the Axiom A systems studied by Smale in [16]. For a precise definition and many results relevant to this paper, we refer the reader to [10].

Roughly speaking, a Smale space is a topological dynamical system (X, φ) in which X is a compact metric space with distance function d, and φ is a homeomorphism. The structure of (X, φ) is such that each point $x \in X$ has two local sets associated to it: A set, $V^s(x, \epsilon)$, on which the map φ is (exponentially) contracting; and a set, $V^u(x, \epsilon)$, on which the map φ^{-1} is contracting. We call these sets the local stable and unstable sets for x. Furthermore, x has a neighbourhood, U_x that is isomorphic to $V^u(x, \epsilon) \times V^s(x, \epsilon)$. In other words, the sets $V^u(x, \epsilon)$ and $V^s(x, \epsilon)$ provide a coordinate system for U_x such that, under application of the map φ, one coordinate contracts, and the other expands. We denote this homeomorphism by $[\cdot, \cdot] : V^u(x, \epsilon) \times V^s(x, \epsilon) \to U_x$.

We now define three equivalence relations on X. We say x and y are *stably equivalent* and write $x \stackrel{s}{\sim} y$ if

$$\lim_{n \to +\infty} d(\varphi^n(x), \varphi^n(y)) = 0.$$

We say x and y are *unstably equivalent* and write $x \stackrel{u}{\sim} y$ if

$$\lim_{n \to -\infty} d(\varphi^n(x), \varphi^n(y)) = 0.$$

Finally, we say x and y are *homoclinic* and write $x \stackrel{h}{\sim} y$ if $x \stackrel{s}{\sim} y$ and $x \stackrel{u}{\sim} y$. We denote these three equivalence relations by G^s,G^u, and G^h. We also let $V^s(x), V^u(x)$ and $V^h(x)$ denote the equivalence classes of a point x in each.

Throughout the first sections of this paper, we will assume our Smale space is mixing: for every pair of non-empty open sets, U and V, there is a constant N such that $\varphi^n(U) \cap V$ is non-empty if $n \geq N$.

2.2 C^*-algebras

In general, an equivalence relation is a groupoid and we construct groupoid C^*-algebras from stable, unstable, and homoclinic equivalence. The construction of these C^*-algebras for a given Smale space is originally due to Ruelle ([15]). In the case of a shift of finite type, these are the algebras studied by Cuntz and Krieger in [4], [7]. We summarize as follows.

The groupoid for homoclinic equivalence may be endowed with an étale topology, as in [10]. In [12], it was shown to be amenable. We let $H(X, \varphi)$ denote its C^*-algebra.

For stable and unstable equivalence, we proceed as in [12]. We first fix a finite φ-invariant subset of X, P. We note that in an irreducible Smale space, the set of periodic points is dense, so there are plenty of choices. We then consider the set of all points in X that are unstably equivalent to a point in P, call this $V^u(P)$. This set may be endowed with a natural topology in which it is locally compact and Hausdorff. The groupoid that we actually use to construct our stable algebra is then the groupoid of stable equivalence restricted to the set $V^u(P)$. As described in [12], this groupoid also has a natural étale topology and is amenable. We let $S(X, \varphi, P)$ denote the reduced C^*-algebra. For different choices of P, the groupoids are equivalent to each other in the sense of Muhly, Renault and Williams [9], and the groupoid C^*-algebras are Morita equivalent (see [12]). The construction of $U(X, \varphi, P)$ is a similar .

In [10], it is shown that $H(X, \varphi)$ is contained in the multiplier algebras of both $S(X, \varphi, P)$ and $U(X, \varphi, P)$; that is there are products $H(X, \varphi) \times S(X, \varphi, P), S(X, \varphi, P) \times H(X, \varphi) \to S(X, \varphi, P)$ and similarly for $U(X, \varphi, P)$. Although the definition of $S(X, \varphi, P)$ and $U(X, \varphi, P)$ are slightly different in [10] than as above, the same formulae still yield well-defined products.

The homeomorphism φ yields a $*$-automorphism on each of the three algebras associated to (X, φ). For f, a continuous function of compact support on any one of our groupoids, we define $\alpha(f)$ by $\alpha(f)(x, y) = f(\varphi^{-1}(x), \varphi^{-1}(y))$. There are several asymptotic commutation results that arise from α. We recall the following theorem from [10]. Let and a, b be in $H(X, \varphi)$, let c be in $S(X, \varphi, P)$ and d be in $U(X, \varphi, P)$. Then we have

1. $||[\alpha^n(a), b]|| \to 0$ as $n \to \pm\infty$,
2. $||[\alpha^n(a), c]|| \to 0$ as $n \to -\infty$,
3. $||[\alpha^n(a), d]|| \to 0$ as $n \to +\infty$.

Let

$$C(H, \alpha) = \{f : \mathbb{R} \to H(X, \varphi) \mid f \text{ cont.}, \ f(t+1) = \alpha(f(t))\}$$

and define automorphisms $\alpha_t(f)(s) = f(t+s), s \in \mathbb{R}$, for each t in $\mathbb{R}$. First, we note that, for f in $C(H, \alpha)$ and c in $S(X, \varphi, P)$, the formulae $f \times c \to f(0)c$ and $c \times f \to cf(0)$ also define products from $C(H, \alpha) \times S(X, \varphi, P), S(X, \varphi, P) \times CH \to S(X, \varphi, P)$ and similarly for $U(X, \varphi, P)$. Secondly, it is an immediate consequence of the result above that, for f, g in $C(H, \alpha)$, c in $S(X, \varphi, P)$ and d in $U(X, \varphi, P)$, we have

1. $||[\alpha_t(f), g]|| \to 0$ as $t \to \pm\infty$,
2. $||[\alpha_t(f), c]|| \to 0$ as $t \to -\infty$,
3. $||[\alpha_t(f), d]|| \to 0$ as $t \to +\infty$.

2.3 K-Theory

In passing from $H(X, \varphi)$ to $C(H, \alpha)$, we have turned the integer parameter to a real one. The significance is that these maps now define asymptotic morphisms in the sense of Connes and Higson ([3]). Specifically, the map $C(H, \alpha) \otimes C(H, \alpha) \to C(H, \alpha)$ given by $f \otimes g \mapsto \alpha_t(f)\alpha_{-t}(g)$ in an asymptotic morphism which determines an element in the group $E(C(H, \alpha) \otimes C(H, \alpha), C(H, \alpha))$. This in turn yields a map $K_*(C(H, \alpha) \otimes C(H, \alpha)) \to K_*(C(H, \alpha))$. The idea is simple: if ρ_t is an asymptotic homomorphism from a C^*-algebra A to another B, and p is a projection in A, then $\rho_t(p)$ is almost a projection in B, for large values of t. More precisely, $\rho_t(p)^2 - \rho_t(p)$ and $\rho_t(p)^* - \rho_t(p)$ are small in norm and so the spectrum of $(\rho_t(p) + \rho_t(p)^*)/2$ is concentrated near 0 and 1. We may use functional calculus with the function $\chi_{(1/2,\infty)}$ to obtain a projection in B. This depends on t, but the projections vary continuously and hence determine a unique element in the K-theory of B.

Combining this with a Kunneth Theorem (eg. Theorem 23.1.3 in [1]) gives a map $K_*(C(H, \alpha)) \otimes K_*(C(H, \alpha)) \to K_*(C(H, \alpha))$. That is, a ring structure on the group $K_*(C(H, \alpha))$. Similarly, the asymptotic morphisms $S(X, \varphi, P) \otimes C(H, \alpha) \to S(X, \varphi, P)$ defined by $c \otimes f \to cf(-t)$ and $U(X, \varphi, P) \otimes C(H, \alpha) \to U(X, \varphi, P)$ defined by $f \otimes d \to f(t)d$ give rise to right and left $C(H, \alpha)$-module structures for $S(X, \varphi, P)$ and $U(X, \varphi, P)$, respectively.

We now define the ring structure on $K_0(C(H,\alpha))$ in more detail. For any C^*-algebra A and $n \geq 1$, we let $M_n(A)$ denote the C^*-algebra of $n \times n$ matrices over A. We also let $P_n(A)$ denote the set of projections in $M_n(A)$. We let $P_\infty(A)$ denote the union of all $P_n(A)$. For $f \in M_n(C(H,\alpha))$, $g \in M_m(C(H,\alpha))$ we define $(f \times g)_t \in M_{nm}(C(H,\alpha))$ componentwise by

$$((f \times g)_t)_{(i,j)(i',j')} = (f_{i,i'} \times g_{j,j'})_t = \alpha_t(f_{i,i'})\alpha_{-t}(g_{j,j'}),$$

for $1 \leq i, i' \leq n, 1 \leq j, j' \leq m, t \geq 0$. For $p \in P_n(C(H,\alpha))$, $q \in P_m(C(H,\alpha))$, there exists $T > 0$ such that

$$\chi_{(1/2,\infty)}(p \times q)_t \in P_{nm}(C(H,\alpha))$$

for $t \geq T$. The function $t \mapsto \chi_{(1/2,\infty)}(p \times q)_t$ is continuous, so for $t \geq T$, $\chi_{(1/2,\infty)}(p \times q)_t$ forms a continuous path of projections in $P_{nm}(CH)$. Thus, for $t_1, t_2 \geq T$

$$\left[\chi_{(1/2,\infty)}(p \times q)_{t_1}\right]_0 = \left[\chi_{(1/2,\infty)}(p \times q)_{t_2}\right]_0.$$

The following proposition gives a concrete form for the product on $K_0(C(H,\alpha))$. We state the result without proof.

Proposition 2.1. *For $p, q \in P_\infty(C(H,\alpha))$ we define the product $[p]_0[q]_0 \in K_0(C(H,\alpha))$ by*

$$[p]_0[q]_0 = \lim_{t\to\infty} \left[\chi_{(1/2,\infty)}(p \times q)_t\right]_0$$

Our product will actually be defined on $K_*(C(H,\alpha)) = K_0(C(H,\alpha)) \oplus K_1(C(H,\alpha))$, as a $\mathbb{Z}_2$-graded group. To do so, we simply use the fact that $K_1(C(H,\alpha)) \cong K_0(SC(H,\alpha))$, where SA denotes the suspension of a C^*-algebra A. Then an obvious extension yields asymptotic morphisms $C(H,\alpha) \otimes SC(H,\alpha) \to SC(H,\alpha)$, $SC(H,\alpha) \otimes C(H,\alpha) \to SC(H,\alpha)$ and $SC(H,\alpha) \otimes SC(H,\alpha) \to S^2C(H,\alpha)$.

As a first step toward computing the product, we describe the K-theory of the mapping cylinder, $C(H,\alpha)$.

We have the following short exact sequence.

$$0 \longrightarrow SH \xrightarrow{\iota} C(H,\alpha) \xrightarrow{e_0} H(X,\varphi) \longrightarrow 0$$

The map e_0 is evaluation at 0, and $\iota(f)(s) = \alpha^k(f(s-k))$ for $k \leq s \leq k+1$. We thus get the following 6-term exact sequence of K groups.

$$\begin{array}{ccccc} K_0(SH) & \xrightarrow{\iota_*} & K_0(C(H,\alpha)) & \xrightarrow{(e_0)_*} & K_0(H(X,\varphi)) \\ \uparrow & & & & \downarrow \\ K_1(H(X,\varphi)) & \xleftarrow[(e_0)_*]{} & K_1(C(H,\alpha)) & \xleftarrow[\iota_*]{} & K_1(SH) \end{array}$$

For a general irreducible Smale space, it is a difficult problem to obtain a nice description of $K_*(H(X,\varphi))$, and hence difficult to describe concretely $K_*(C(H,\alpha))$. However, in section 3 we will see that in the SFT case, the above 6-term exact sequence can be used to describe $K_*(C(H,\alpha))$ in a concrete way.

3 Shifts of finite type

3.1 Dynamics

We begin with a finite directed graph G. That is, G has a vertex set $V(G)$, an edge set $E(G)$ and initial and terminal maps $i, t : E(G) \to V(G)$. We let A denote the adjacency matrix of the graph. That is, if we enumerate the vertices of $V(G) = \{v_1, v_2, \dots, v_K\}$, then A is the $K \times K$ matrix whose i, j entry is the number of edges e with $i(e) = v_i$ and $t(e) = v_j$.

We let Σ be the associated bi-infinite path space of the graph:

$$\Sigma = \{(e_n)_{n\in\mathbb{Z}} \mid e_n \in E(G), t(e_n) = i(e_{n+1}), \text{ for all } n \in \mathbb{Z}\}.$$

We also let σ denote the left shift map. See [8] for a complete treatment. The system (Σ, σ) is a Smale space [11] and is mixing precisely when the matrix A is primitive; i.e. there is a positive integer M such that A^N has no zero entries for all $N \geq M$.

3.2 C^*-Algebras

We begin by describing the algebras $H(\Sigma, \sigma)$, and $S(\Sigma, \sigma, P)$. These constructions follow (with slight notational modifications) those in [11]. We present the details here for completeness, and because some of our later proofs will use this notation. With A is primitive and M as above, fix $N \geq M$, $v_i, v_j \in V(G)$. Define

$$\Xi_{N,v_i,v_j} = \{\xi = (\xi_{-N+1}, \cdots, \xi_N) \ \mid \ t(\xi_N) = v_j, \ i(\xi_{-N+1}) = v_i\}.$$

Notice that Ξ_{N,v_i,v_j} consists of all paths of length $2N$ in G with initial vertex v_i and terminal vertex v_j, so $\#\Xi_{N,v_i,v_j} = A^{2N}_{ij} > 0$. For $\xi \in \Xi_{N,v_i,v_j}$ define

$$V_{N,v_i,v_j}(\xi) = \{x \in \Sigma \ \mid \ x_n = \xi_n \quad \forall -N+1 \leq n \leq N\}.$$

Note that for fixed N, $V_{N,v_i,v_j}(\xi)$ and $V_{N,v_i',v_j'}(\eta)$ intersect only if $\xi = \eta$, $v_i = v_i'$, and $v_j = v_j'$. Now let $\xi\ , \eta \in \Xi_{N,v_i,v_j}$. Define

$$E_{N,v_i,v_j}(\xi, \eta) = \{(x, y) \mid \{x_n\}_{-N+1}^{N} = \xi, \ \{y_n\}_{-N+1}^{N} = \eta, \ x_n = y_n \ \forall n > N, \ n < -N+1\}.$$

Then

1. $E_{N,v_i,v_j}(\xi, \eta)$ and $E_{N,v_i',v_j'}(\xi', \eta')$ intersect only if $\xi = \xi'$, $\eta = \eta'$, $v_i = v_i'$, $v_j = v_j'$.
2. The sets $E_{N,v_i,v_j}(\xi, \eta)$ for $N \geq 1$, $v_i, \ v_j \in V(G)$, $\xi, \ \eta \in \Xi_{N,v_i,v_j}$ form a neighbourhood base of compact, open sets for the topology on $G^h(\Sigma, \sigma)$.

Now let

$$e_{N,v_i,v_j}(\xi, \eta) = \chi_{E_{N,v_i,v_j}(\xi,\eta)} \in C_c(G^h(\Sigma, \sigma)).$$

Note that $\overline{span\{e_{N,v_i,v_j}(\xi, \eta)\}} = H(\Sigma, \sigma)$. Consider the product of two such functions.

$$e_{N,v_i,v_j}(\xi, \eta) * e_{N,v_i',v_j'}(\xi', \eta')(x, y) = \sum_{x \overset{h}{\sim} z} e_{N,v_i,v_j}(\xi, \eta)(x, z) e_{N,v_i',v_j'}(\xi', \eta')(z, y).$$

It is easily verified that this reduces to

$$e_{N,v_i,v_j}(\xi, \eta) * e_{N,v_i',v_j'}(\xi', \eta') = \begin{cases} e_{N,v_i,v_j}(\xi, \eta') & \text{if } \eta = \xi' \\ 0 & \text{otherwise} \end{cases}$$

Now let

$$H_{N,v_i,v_j} = span\{e_{N,v_i,v_j}(\xi, \eta) \mid \xi, \ \eta \in \Xi_{N,v_i,v_j}\} \cong M_{k(N,v_i,v_j)}(\mathbb{C}).$$

where $k(N, v_i, v_j) = \#\Xi_{N,v_i,v_j} = A^{2N}_{v_1,v_2}$. Now we define

$$H_N = span(\{e_{N,v_i,v_j}(\xi, \eta) \mid \xi, \ \eta \in \Xi_{N,v_i,v_j}; \ v_i, \ v_j \in V(G)\}),$$

and notice that

$$H_N = \bigoplus_{v_i \in V(G)} \bigoplus_{v_j \in V(G)} H_{N,v_i,v_j} = \bigoplus_{(v_i,v_j)\in V(G)\times V(G)} H_{N,v_i,v_j} \cong \bigoplus_{(v_i,v_j)} M_{k(N,v_i,v_j)}(\mathbb{C}).$$

Now $H_N \subset H_{N+1}$, and $H(\Sigma, \sigma)$ is the direct limit of the H_N's. To see how H_N is embedded in H_{N+1}, observe that

$$e_{N,v_i,v_j}(\xi, \eta) = \sum_{y_1 \in E_i} \sum_{y_2 \in E_j} e_{N+1,v_l,v_k}(y_1 \xi y_2, y_1 \eta y_2),$$

where $i(y_1) = v_l$, $t(y_2) = v_k$, $E_i = \{y \in E(G) \mid t(y) = v_i\}$, and $E_j = \{y \in E(G) \mid i(y) = v_j\}$. In particular, H_{N+1,v_l,v_k} contains $A_{li}A_{jk}$ copies of H_{N,v_i,v_j}.

We now describe the action of α on $H(\Sigma, \sigma)$.

$$\alpha(e_{N,v_i,v_j}(\xi, \eta)) = \sum_k \sum_{\xi' \in \Xi_{1,v_j,v_k}} e_{N+1,v_i,v_k}(\xi\xi', \eta\xi'),$$

and

$$\alpha^{-1}(e_{N,v_i,v_j}(\xi, \eta)) = \sum_l \sum_{\xi' \in \Xi_{1,v_l,v_i}} e_{N+1,v_l,v_j}(\xi'\xi, \xi'\eta).$$

In particular α and α^{-1} map H_N into H_{N+1}.

The construction of $S(\Sigma, \sigma, P)$ is very similar. We briefly outline the details. Fix a finite σ-invariant set $P \subset \Sigma$. Fix $N \geq M$, $v_i \in V(G)$. Define

$$\Xi_{N,v_i} = \{\xi = (\xi_{-N+1}, \cdots, \xi_N) \mid t(\xi_N) = v_i,\ i(\xi_{-N+1}) = i(p_{-N}) \text{ for some } p \in P\}.$$

Again we mention that Ξ_{N,v_i} is non-empty, as A^{2N} is strictly positive. For $\xi \in \Xi_{N,v_i}$ we can extend ξ backwards by setting $\xi_{-n} = p_{-n}$ for $n > N - 1$. Now for $\xi \in \Xi_{N,v_i}$ we define

$$V_{N,v_i}(\xi) = \{x \in \Sigma \mid x_n = \xi_n \quad \forall n \leq N\}.$$

Note that for fixed N, $V_{N,v_i}(\xi)$ and $V_{N,v_j}(\eta)$ intersect only if $\xi = \eta$, and $v_i = v_j$. Now let ξ , $\eta \in \Xi_{N,v_i}$. Define

$$E_{N,v_i}(\xi, \eta) = \{(x, y) \mid x_n = \xi_n,\ y_n = \eta_n\ \forall n \leq N,\ x_n = y_n\ \forall n > N\}.$$

The collection of sets $\{E_{N,v_i}(\xi, \eta)\}$ forms a clopen base for the topology on $G^s(\Sigma, \sigma, P)$, and we are left to consider functions of the form

$$e_{N,v_i}(\xi, \eta) = \chi_{E_{N,v_i}(\xi,\eta)}.$$

Proceeding as we did above for $H(\Sigma, \sigma)$, we see that for fixed N and i

$$e_{N,v_i}(\xi, \eta) * e_{N,v_i}(\xi', \eta') = \begin{cases} e_{N,v_i}(\xi, \eta') & \text{if } \eta = \xi', \\ 0 & \text{if } \eta \neq \xi'. \end{cases}$$

As above, we let $S_{N,v_i} = span\{e_{N,v_i}(\xi, \eta) \mid \xi, \eta \in \Xi_{N,v_i}\}$ and notice that

$$S_{N,v_i} \cong M_{k(N,v_i)}(\mathbb{C}),$$

where $k(N, v_i)$ is the number of paths of length $2N$ starting at a vertex of $p \in P$ and ending at v_i.

$$S_N = \bigoplus_{v_i \in V(G)} S_{N,v_i} \cong \bigoplus M_{k(N,v_i)}(\mathbb{C}).$$

Finally we notice that $S_N \subset S_{N+1}$ and let $S(\Sigma, \sigma, P)$ be the direct limit of the S_N's. Similar to the above,

$$e_{N,v_i}(\xi, \eta) = \sum_{y \in S} e_{N+1,v_k}(\xi y, \eta y),$$

where $t(y) = v_k$ and $S = \{y \in E(G) \mid i(y) = v_i\}$. So we see that S_{N,v_k} contains A_{ik} copies of S_{N,v_i}.

Similar to the $H(\Sigma, \sigma)$ case we see that

$$\alpha(e_{N,v_i}(\xi, \eta)) = \sum_k \sum_{\xi' \in \Xi_{1,v_i,v_k}} e_{N+1,v_k}(\xi\xi', \eta\xi'),$$

and

$$\alpha^{-1}(e_{N,v_i,v_j}(\xi, \eta)) = e_{N+1,v_i,v_j}(\xi, \eta).$$

3.3 K-Theory

We now describe the K-theory for $H(\Sigma, \sigma)$ and $S(\Sigma, \sigma, P)$ in the case that (Σ, σ) is mixing. The irreducible case is handled in section 6. The computation of the K-theory follows easily from the description as AF-algebras in section 3.2. These results also appear in [10]. We begin with $H(\Sigma, \sigma)$. This was first computed by Krieger in [7].

As $H(\Sigma, \sigma)$ is AF, $K_1(H(\Sigma, \sigma)) = 0$. $K_0(H_N) = \bigoplus_{V \times V} \mathbb{Z} \cong M_K(\mathbb{Z})$, and $K_0(H(\Sigma, \sigma))$ is the inductive limit of the following system.

$$M_K(\mathbb{Z}) \xrightarrow{X \mapsto AXA} M_K(\mathbb{Z}) \xrightarrow{X \mapsto AXA} M_K(\mathbb{Z}) \longrightarrow \cdots$$

The inductive limit group can be described as

$$K_0(H) \cong (M_K(\mathbb{Z}) \times \mathbb{N}) / \sim .$$

Where, for $n \leq k$, $(X, n) \sim (Y, k)$ if and only if $A^{k-n+l} X A^{k-n+l} = A^l Y A^l$ for some $l \in \mathbb{N}$. We denote the equivalence class of (X, N) under $\sim$ by $[X, N]$.

Recall the automorphism $\alpha : H \to H$. We now wish to describe $\alpha_* : K_0(H) \to K_0(H)$. Referring back to section 3.2, it is another straightforward calculation to see that for $[X, N] \in K_0(H)$ we have $\alpha([X, N]) = [XA^2, N+1]$. Similarly, $\alpha^{-1}([X, N]) = [A^2 X, N+1]$.

We now briefly outline the computation of $K_*(S(\Sigma, \sigma, P))$. As in the case of $H(\Sigma, \sigma)$, $S(\Sigma, \sigma, P)$ is AF and hence $K_1(S(\Sigma, \sigma, P)) = 0$, and $K_0(S(\Sigma, \sigma, P))$ is the direct limit of the following system.

$$\mathbb{Z}^K \xrightarrow{v \mapsto vA} \mathbb{Z}^K \xrightarrow{v \mapsto vA} \mathbb{Z}^K \xrightarrow{v \mapsto vA} \cdots .$$

We can therefore write

$$K_0(S(\Sigma, \sigma, P)) \cong (\mathbb{Z}^K \times \mathbb{N})/\sim,$$

where, for $n \leq m$, $(v, n) \sim (w, m)$ if and only if there exists $k \in \mathbb{N}$ such that $vA^{k+m-n} = wA^k$. We write $[v, n]$ for the equivalence class under $\sim$.

Once again proceeding as in the case of $H(\Sigma, \sigma)$ we can show that $\alpha_*[v, N] = [vA^2, N+1] = [vA, N]$ and $\alpha_*^{-1}[v, N] = [v, N+1]$.

We are now ready to describe $K_*(C(H, \alpha))$ in the SFT case. Once again, this follows [10]. As $K_1(H(\Sigma, \sigma)) \cong K_0(SH) = 0$ the six term exact sequence from section 2.3 becomes

$$\begin{array}{ccccc} 0 & \longrightarrow & K_0(C(H,\alpha)) & \xrightarrow{(e_0)_*} & K_0(H(\Sigma,\sigma)) \\ & & & & \big\downarrow {\scriptstyle (id-\alpha_*)} \\ 0 & \longleftarrow & K_1(C(H,\alpha)) & \xleftarrow[\iota_*]{} & K_0(H(\Sigma,\sigma)) \end{array}$$

We see immediately that

$$K_0(C(H,\alpha)) \cong ker(id - \alpha_*),$$

and

$$K_1(C(H,\alpha)) \cong coker(id - \alpha_*).$$

The following well known Lemma will be of use in describing $K_1(C(H,\alpha))$.

Lemma 3.1. *Let G be an abelian group, $\psi : G \to G$ an endomorphism, and $H < G$ a ψ-invariant subgroup, and consider the following diagram*

$$\begin{array}{ccccc} H & \xrightarrow{\psi} & H & \xrightarrow{\psi} & \cdots \\ \downarrow{\scriptstyle\iota} & & \downarrow{\scriptstyle\iota} & & \\ G & \xrightarrow{\psi} & G & \xrightarrow{\psi} & \cdots \end{array}$$

where the vertical maps ι are given by inclusion, $\iota(x) = x$. Then $\lim_{\to} H < \lim_{\to} G$ and

$$\lim_{\to} G / \lim_{\to} H \cong \lim_{\to}(G/H)$$

The following definition will be useful.

Definition 3.2. *Let A be a $K \times K$ integer matrix. We define*

$$C(A) = \{X \in M_K(\mathbb{Z}) \mid AX = XA\},$$

and

$$B(A) = \{X \in M_K(\mathbb{Z}) \mid X = AY - YA \textit{ for some } Y \in M_K(\mathbb{Z})\}.$$

The following theorem gives a nice description of $K_0(C(H,\alpha))$ and $K_1(C(H,\alpha))$ in terms of inductive limits.

Theorem 3.3. *Let (Σ, σ) be a mixing Smale space with $K \times K$ adjacency matrix A and $C(H,\alpha)$ the corresponding mapping cylinder.*

Then $K_0(C(H,\alpha))$ is isomorphic to the inductive limit of the following system

$$C(A) \xrightarrow{X \mapsto AXA} C(A) \xrightarrow{X \mapsto AXA} C(A) \longrightarrow \cdots$$

Similarly, $K_1(C(H,\alpha))$ is isomorphic to the inductive limit of the following system

$$M_K(\mathbb{Z})/B(A) \xrightarrow{X \mapsto AXA} M_K(\mathbb{Z})/B(A) \xrightarrow{X \mapsto AXA} M_K(\mathbb{Z})/B(A) \longrightarrow \cdots$$

Proof: Consider the following diagram:

$$\begin{array}{ccccc} C(A) & \xrightarrow{X \mapsto AXA} & C(A) & \longrightarrow & \cdots \\ \downarrow{\scriptstyle\iota} & & \downarrow{\scriptstyle\iota} & & \\ M_n(\mathbb{Z}) & \xrightarrow{X \mapsto AXA} & M_n(\mathbb{Z}) & \longrightarrow & \cdots \end{array}$$

Where the vertical maps are given by $\iota(X) = X$. The diagram clearly commutes, so ι induces a well defined group homomorphism $\iota : \lim C(A) \to K_0(H)$. It is also clear that $im(\iota) \subset ker(id - \alpha_*)$. We are left to show that ι is injective and that the image of ι is $ker(id - \alpha_*) \subset K_0(H(\Sigma, \sigma))$.

We start by showing injectivity. Let $[X,m] \in \lim C(A)$ be such that $\iota([X,m]) = [0, m+k] \in K_0(H(\Sigma,\sigma))$ for some k, then $A^{k+l}XA^{k+l} = A^l 0 A^l$ for some l, ie $A^j X A^j = 0$ for some $j \geq n$. But then $[X,m] = [0, j+m] \in \lim C(A)$, and ι is injective.

Now suppose $[X,k] \in ker(id - \alpha_*) \subset K_0(H)$. Then $(id - \alpha*)[X,k] = [0,m] \in K_0(H)$ for some $m > k$, ie $[AXA - XA^2, k+1] = [0,k]$ or $A^{m-k+l}XA^{m-k+l} - A^{m-k+l-1}XA^{m-k+l+1} = 0$ for some l. Letting $j = m-k+l$ we get $A^j X A^j = A^{j-1}XA^{j+1}$ or after multiplying on the left by A, $A^{j+1}XA^j = A^j X A^{j+1}$. So $Y = A^j X A^j \in C(A)$ and $\iota([Y, j+k]) = [X,k]$. So $\iota : \lim C(A) \to ker(id - \alpha_*)$ is an isomorphism.

We now show that $im(id - \alpha_*)$ is isomorphic to the inductive limit of

$$B(A) \xrightarrow{X \mapsto AXA} B(A) \xrightarrow{X \mapsto AXA} B(A) \xrightarrow{X \mapsto AXA} \cdots$$

Consider the diagram

$$\begin{array}{ccccc} B(A) & \xrightarrow{X \mapsto AXA} & B(A) & \longrightarrow & \cdots \\ \downarrow \iota & & \downarrow \iota & & \\ M_K(\mathbb{Z}) & \xrightarrow{X \mapsto AXA} & M_K(\mathbb{Z}) & \longrightarrow & \cdots \end{array}$$

where the vertical maps, ι, are given by inclusion $\iota(X) = (X)$. Clearly the above diagram commutes, so ι extends to a well defined map on the inductive limit groups:

$$\iota : \lim_{\to} B(A) \longrightarrow \lim_{\to} M_K(\mathbb{Z}) \quad (\cong K_0(H)).$$

We first show that $im(\iota) \subset im(id - \alpha_*)$. Suppose $[X,k] \in (B(A) \times \mathbb{N})/\sim$, then there exists Y such that $X = YA - AY$, so

$$\iota[X,k] = \iota[YA - AY, k] = [YA - AY, k] \in (M_K(\mathbb{Z}) \times \mathbb{N})/\sim .$$

Now

$$[YA - AY, k] = [A(YA - AY)A, k+1] = [(AY)A^2 - A(AY)A, k+1] = (id - \alpha_*)[-YA, k],$$

so

$$\iota(\lim_{\to} B(A)) \subset im(id - \alpha_*).$$

A straightforward argument now shows that ι is $1-1$ and onto, so

$$\iota : \lim_{\to} B(A) \to im(id - \alpha_*)$$

is an isomorphism. Now using lemma 3.1 we see immediately that $K_1(C(H,\alpha)) \cong coker(id - \alpha_*) \cong K_0(H(\Sigma,\sigma))/im(id - \alpha_*)$ is the limit of the inductive system

$$M_K(\mathbb{Z})/B(A) \xrightarrow{X \mapsto AXA} M_K(\mathbb{Z})/B(A) \xrightarrow{X \mapsto AXA} M_K(\mathbb{Z})/B(A) \longrightarrow \cdots$$

□

3.4 The Product on $K_*(C(H,\alpha))$

We wish to describe the ring structure on $K_*(C(H,\alpha))$ in terms of the matrix characterization of $K_*(C(H,\alpha))$ obtained in section 3.3. We start by determining the ring structure on the subring $K_0(C(H,\alpha))$.

We begin with a couple of observations.

1. The real parameter t in Prop. 2.1 can clearly be replaced with the integer parameter n. This will be helpful as in the SFT case $K_0(C(H,\alpha)) \subset K_0(H(\Sigma,\sigma))$, and on $K_0(H(\Sigma,\sigma))$ α_*^n is defined, but $(\alpha_t)_*$ is not.
2. In the case of a SFT, we can use a slightly simpler, though equivalent, definition for the product on $K_0(C(H,\alpha))$. See Prop. 3.5.

Consider the following $*$-subalgebra of $H(\Sigma,\sigma)$.

$$\mathcal{H}(\Sigma,\sigma) = span\{e_{N,v_i,v_j}(\xi,\eta)\}.$$

Notice that $\mathcal{H}(\Sigma,\sigma)$ is dense in $H(\Sigma,\sigma)$ and that, for each $p \in M_\infty(H(\Sigma,\sigma))$, there exists $q \in M_\infty(\mathcal{H}(\Sigma,\sigma))$ such that $[p]_0 = [q]_0$. We also consider the following $*$-subalgebra of $C(H,\alpha)$, $\mathcal{CH} = \{f \in C(H,\alpha) \mid f(0) \in \mathcal{H}\}$. We again notice that $\mathcal{CH}$ is dense in $C(H,\alpha)$ and, for each $p \in M_\infty(C(H,\alpha))$, there exists $q \in M_\infty(\mathcal{CH})$ such that $[p]_0 = [q]_0$.

The following lemma will be useful in computing the product on $K_0(C(H,\alpha))$.

Lemma 3.4. *Let $e_{N,v_i,v_j}(\xi,\xi)$, $e_{N,v_l,v_k}(\eta,\eta) \in C_c(G^h(\Sigma,\sigma))$ be as in section 3.2. For $n \geq N$ we have*

$$\alpha^n(e_{N,v_i,v_j}(\xi,\xi))\alpha^{-n}(e_{Nv_lv_k}(\eta,\eta)) = \sum_{\xi_1 \in S} e_{N+n,v_i,v_l}(\xi\xi_1\eta, \xi\xi_1\eta),$$

where

$$S = \{\xi_1 \mid |\xi_1| = 2n-2N,\ i(\xi_1) = t(\xi),\ t(\xi_1) = i(\eta).\}$$

Proof: For notational convenience, let $a = e_{Nv_iv_j}(\xi,\xi)$, $b = e_{Nv_lv_k}(\eta,\eta)$. Then

$$\alpha^n(a) = \sum_{v \in V(G)} \sum_{\xi' \in \Xi_{2n,v_j,v}} e_{N+n,v_i,v}(\xi\xi',\xi\xi')$$

and

$$\alpha^{-n}(b) = \sum_{v \in V(G)} \sum_{\eta' \in \Xi_{2n,v,v_k}} e_{N+n,v,v_k}(\eta'\eta,\eta'\eta)$$

so

$$\alpha^n(a)\alpha^{-n}(b) = \sum_{v \in V(G)} \sum_{\xi'} \sum_{\tilde{v} \in V(G)} \sum_{\eta'} e_{N+n,v_i,v}(\xi\xi',\xi\xi')e_{N+n,v,v_k}(\eta'\eta,\eta'\eta).$$

Where the sum is over ξ' such that $i\xi' = t\xi$ and η' such that $t(\eta') = i(\eta)$. Furthermore, each summand is 0 unless $\xi\xi' = \eta'\eta$. Write $\xi' = \xi_1\xi_2$, $\eta' = \eta_1\eta_2$ where $|\xi_2| = |\eta| = |\xi| = |\eta_1| = 2N$ and $|\xi_1| = |\eta_2| = 2n-2N$. Now $\xi\xi_1\xi_2 = \eta_1\eta_2\eta$ implies $\xi = \eta_1$, $\xi_2 = \eta$, and $\xi_1 = \eta_2$, which in turn imply $v_i = \tilde{v}$, $v = v_l$, $i(\xi_1) = i(\eta_2) = t(\xi) = v_j$, and $t(\xi_1) = t(\eta_2) = i(\eta) = v_k$. So the sum becomes

$$\alpha^n(a)\alpha^{-n}(b) = \sum_{\xi_1 \in S} e_{N+n,v_i,v_l}(\xi\xi_1\eta),$$

where

$$S = \{\xi_1 \mid |\xi_1| = 2n-2N,\ i(\xi_1) = t(\xi),\ t(\xi_1) = i(\eta).\}$$

□

Proposition 3.5. *Let (Σ,σ) be an irreducible SFT, $p,q \in P_\infty(\mathcal{CH})$. By identifying $K_0(C(H,\alpha))$ with a subgroup of $K_0(H(\Sigma,\sigma))$, the product can be written*

$$[p(0)]_0[q(0)]_0 = \lim_{n\to\infty}[\alpha^n(p(0))\alpha^{-n}(q(0))]_0.$$

Proof: From section 3.3 we know that $K_0(C(H,\alpha)) \cong ker(id-\alpha_*) \subset K_0(H(\Sigma,\sigma))$ where the isomorphism is given by evaluation at 0. So we have

$$\begin{aligned}[p(0)]_0[q(0)]_0 &= \lim_{t\to\infty} \left[\chi_{(1/2,\infty)}(p\times q)_t(0)\right]_0 \\ &= \lim_{t\to\infty} \left[\chi_{(1/2,\infty)}(p(0)\times q(0))_t\right]_0 \\ &= \lim_{n\to\infty} \left[\chi_{(1/2,\infty)}(p(0)\times q(0))_n\right]_0 .\end{aligned}$$

We now look more closely at

$$(p(0)\times q(0))_n = \frac{\alpha^n(p(0))\alpha^{-n}(q(0)) + \alpha^{-n}(q(0))\alpha^n(p(0))}{2}.$$

Since $p(0), q(0) \in M_\infty(\mathcal{H}(\Sigma,\sigma))$, it will suffice to consider the product $\alpha^n(a)\alpha^{-n}(b)$, where $a = e_{Nv_iv_j}(\xi,\xi)$ and $b = e_{Nv_lv_k}(\eta,\eta)$. Fix $n \geq N$, then from Lemma 3.4

$$\alpha^n(a)\alpha^{-n}(b) = \sum_{\xi_1\in S} e_{N+n,v_i,v_l}(\xi\xi_1\eta).$$

A similar computation yields

$$\alpha^{-n}(b)\alpha^n(a) = \sum_{\eta_2\in S} e_{N+n,v_i,v_l}(\xi\eta_2\eta),$$

so these two sums are equal. This implies

$$\alpha^n(a)\alpha^{-n}(b) = \alpha^{-n}(b)\alpha^n(a)$$

is a projection. Now for p, q as above, there exists N such that for $n > N$

$$\alpha^n(p(0))\alpha^{-n}(q(0)) = \alpha^n(q(0))\alpha^{-n}(p(0)),$$

and this is a projection. Thus we have

$$[p(0)]_0[q(0)]_0 = \lim_{n\to\infty}[\alpha^n(p(0))\alpha^{-n}(q(0))]_0.$$

□

Remark 3.6. *The preceding proposition gives the product on positive elements in K_0. The product extends to all of K_0 by linearity and it is clear that it is positive.*

Under the isomorphism of Theorem 3.3, every element of $K_0(C(H,\alpha))$ is equal to some $[X,N] \in \lim C(A)$. As each such X is a linear combination of matrices of the form e_{ij}, we start with two matrices of this form, their corresponding projections in $H(\Sigma,\sigma)$ and multiply according to Prop. 3.5.

Remark 3.7. *As $[e_{ij},N] \in K_0(H(\Sigma,\sigma))$ need not be an element of the subgroup $K_0(C(H,\alpha))$, the formula we derive for the product of two such elements will not be well defined in general. Ie. the element of $K_0(H(\Sigma,\sigma))$, $[\alpha^n(a)\alpha^{-n}(b)]_0$, in the following Lemma depends (in general) on the integer n and thus $\lim_{n\to\infty}[\alpha^n(a)\alpha^{-n}(b)]_0$ need not exist. However, if we apply the formula to linear combinations of such elements, $[X,N]$, $[Y,M]$ which are in $K_0(C(H,\alpha))$ the product is well defined.*

Lemma 3.8. *Let $a,b \in P_\infty(H(\Sigma,\sigma))$. If $[a]_0 = [X,N]$, $[b]_0 = [Y,N]$, then for $n \geq N$*

$$[\alpha^n(a)\alpha^{-n}(b)]_0 = [XA^{2n-2N}Y, N+n].$$

Proof: For $a, b \in P_\infty(H(\Sigma,\sigma))$, we can find $\bar{a}, \bar{b} \in P_\infty(\mathcal{H}(\Sigma,\sigma))$ such that $[a]_0 = [\bar{a}]_0$, $[b]_0 = [\bar{b}]_0$. It therefore suffices to prove the result for rank one projections $\bar{a} = e_{Nv_iv_j}(\xi,\xi)$ and $\bar{b} = e_{Nv_lv_k}(\eta,\eta)$. Then $[\bar{a}]_0 = [e_{ij}, N], [\bar{b}]_0 = [e_{kl}, N] \in K_0(H(\Sigma,\sigma))$. Fix $n \geq N$, then using Lemmas 3.4 and 3.5 we see that

$$\alpha^n(\bar{a})\alpha^{-n}(\bar{b}) = \sum_{\xi_1} e_{N+n,v_i,v_l}(\xi\xi_1\eta)$$

is a projection. The number of summands is thus the number of paths ξ_1 of length $2n-2N$ from v_j to v_k, ie A^{2n-2N}_{jk}. Noticing that $A^{2n-2N}_{jk} e_{il} = e_{ij}A^{2n-2N}e_{kl}$, we have

$$[\alpha^n(\bar{a})\alpha^{-n}(\bar{b})]_0 = [e_{ij}A^{2n-2N}e_{kl}, N+n].$$

□

Theorem 3.9. *Let X, Y be in $C(A)$ and $N, M \geq 0$. The product of $[X,N]$ and $[Y,M]$ in $(C(A)\times\mathbb{N})/\sim \cong K_0(C(H,\alpha))$ is given by*

$$[X,N] * [Y,M] = [XY, N+M].$$

Proof: Let $p,\ q \in P_\infty(\mathcal{CH})$, with $[p(0)]_0 = [X,N]$, $[q(0)]_0 = [Z,N]$ (under the isomorphism in Prop. 3.3). Note here that we have chosen representatives of each equivalence class such that the integer N is equal. From Prop. 3.5 we know the product is given by

$$[p(0)]_0[q(0)]_0 = \lim_{n\to+\infty}[\alpha^n(p(0))\alpha^{-n}(q(0))]_0.$$

However, we know that for $n \geq N$ the sequence $[\alpha^n(p(0))\alpha^{-n}(q(0))]_0$ is constant, so we can write

$$[p]_0[q]_0 = \lim_{n\to+\infty}[\alpha^n(p(0))\alpha^{-n}(q(0))]_0 = [\alpha^N(p(0))\alpha^{-N}(q(0))]_0$$

By Lemma 3.8 this is

$$[X,N] * [Z,N] = \lim_{n\to+\infty}[XA^{2n-2N}Z, N+n] = [XZ, 2N].$$

Now suppose $[Y,M] \in K_0(C(H,\alpha))$. Let $M_1 = \max\{N, M\}$.

$$\begin{aligned}
[X,N] * [Y,M] &= [A^{M_1-N}XA^{M_1-N}, M_1] * [A^{M_1-M}YA^{M_1-M}, M_1]\\
&= [A^{M_1-N}XA^{M_1-N}A^{M_1-M}YA^{M_1-M}, 2M_1]\\
&= [A^{M_1-N}A^{M_1-M}XYA^{M_1-M}A^{M_1-N}, N+M+|M-N|]\\
&= [A^{|M-N|}XYA^{|M-N|}, N+M+|M-N|]\\
&= [XY, N+M]
\end{aligned}$$

□

Corollary 3.10. *The multiplicative identity $[I,0]$ in the ordered group $K_0(C(H,\alpha))$ is in fact an order unit.*

Proof: That $[I,0]$ is the multiplicative identity is immediate from Theorem 3.9. As (Σ,σ) is mixing, A is primitive, so there exists K such that A^{2k} is strictly positive for all $k > K$. Let $[X,k] \in K_0(C(H,\alpha))$, wolog $k > K$. Now there exists an integer l such that $lA^{2k} > X$ entry-wise. In other words $l[I,0] = l[A^{2k}, k] > [X,k]$. □

In [10] the first author proved that integration against a certain measure gave a trace on $H(X,\varphi)$ (for any mixing Smale space (X,φ), and that this easily extended to a trace on $C(H,\alpha)$. Moreover, it was proved that the trace induced a ring homomorphism from $K_0(C(H,\alpha))$ to $\mathbb{R}$. In the case of a SFT we can write down explicitly what the trace is in terms of the inductive systems and see concretely the ring homomorphism.

Corollary 3.11 (Corollary to Thms. 3.3, 4.3 in [10])**.** *Let τ^{CH} be the trace on $C(H,\alpha)$. Then for $[X,K] \in K_0(C(H,\alpha))$, $\tau_*^{CH}[X,K] = \lambda^{-2k} u_l X u_r$. Where λ is the Perron-Frobenius eigenvalue of A and u_l, u_r are the left and right Perron-Frobenius eigenvectors of A normalized so that $u_l u_r = 1$.*

Remark 3.12. *With the trace as in Cor. 3.11 it is a straightforward computation to verify that it is a ring homomorphism.*

We now calculate the product of an element of $K_0(C(H,\alpha))$ with an element of $K_1(C(H,\alpha))$. First recall the 6-term exact sequence:

$$\begin{array}{ccccccc} 0 & \longrightarrow & K_0(C(H,\alpha)) & \xrightarrow{(e_0)_*} & K_0(H(\Sigma,\sigma)) & & \\ & & & & & \searrow^{id-\alpha_*} & \\ 0 & \longleftarrow & K_1(C(H,\alpha)) & \xleftarrow{\iota_*} & K_1(SH) & \xleftarrow{\cong} & K_0(H(\Sigma,\sigma)) \end{array}$$

Hence every element of $K_1(C(H,\alpha))$ is the image under ι_* of some element in $K_1(SH) \cong K_0(H(\Sigma,\sigma))$. Also recall that $K_0(H)$ is generated by rank one projections in $P_1(H(\Sigma,\sigma))$. We will proceed as follows. Starting with a $p \in P_1(H(\Sigma,\sigma))$, and $q \in P_m(C(H,\alpha))$. We find $u_p \in U_1(\widetilde{SH})$ (corresponding to the isomorphism $K_1(SH) \cong K_0(H(\Sigma,\sigma)))$. We then find $\iota_*(u_p) \in U_1(\widetilde{C(H,\alpha)})$ and $\tilde{p} \in P_2(\widetilde{SC(H,\alpha)})$ (corresponding to the isomorphism $K_1(C(H,\alpha)) \cong K_0(SC(H,\alpha)))$. We then simply embed $\tilde{p}$ and q in $P_\infty(C(S^1, C(H,\alpha)))$ and multiply according to Lemma 3.21.

The following Lemmas are standard results in K-theory, see for example Theorems 10.1.3 and 11.1.2 in [13].

Lemma 3.13. *Let $p \in P_m(H)$, then under the isomorphism $K_0(H) \cong K_1(SH)$, $p \mapsto u_p \in U_m(\widetilde{SH})$ where*

$$u_p(s) = e^{2\pi i s} p + (1-p), \text{ for } 0 \leq s \leq 1$$

Lemma 3.14. *Let $u_p \in U_m(\widetilde{C(H,\alpha)})$, then under the isomorphism $K_1(C(H,\alpha)) \cong K_0(\widetilde{SC(H,\alpha)})$, $u_p \mapsto \tilde{p} \in P_{2m}(\widetilde{SC(H,\alpha)})$ where*

$$\tilde{p} = v_p \begin{bmatrix} I_m & 0_m \\ 0_m & 0_m \end{bmatrix} v_p^*$$

and

$$v_p(t) = \overline{R_t} \begin{bmatrix} u_p & 0 \\ 0 & I_m \end{bmatrix} \overline{R_t}^* \begin{bmatrix} I_m & 0 \\ 0 & u_p^* \end{bmatrix}$$

where

$$\overline{R_t} = \begin{bmatrix} \cos(\frac{\pi}{2}t) I_m & -\sin(\frac{\pi}{2}t) I_m \\ \sin(\frac{\pi}{2}t) I_m & \cos(\frac{\pi}{2}t) I_m \end{bmatrix},$$

so that

$$v_p(0) = \begin{bmatrix} u_p & 0 \\ 0 & u_p^* \end{bmatrix}, \quad v_p(1) = \begin{bmatrix} I_m & 0 \\ 0 & I_m \end{bmatrix}.$$

We state and prove one more Lemma before we compute the product of an element of $K_0(C(H,\alpha))$ with an element of $K_1(C(H,\alpha))$.

Lemma 3.15. *For $p \in P_1(\mathcal{H}(\Sigma,\sigma))$, $q \in P_\infty(\mathcal{CH})$, there exists $N \in \mathbb{N}$ such that, for $n,\ m \geq N$, the matrix $(p \times q)_n$ with (i,j) entry given by $\alpha^n(p)\alpha^{-n}(q_{ij}(0))$ is in $P_\infty(H(\Sigma,\sigma))$ and*

$$[(p \times q)_n]_0 - [(p \times q)_m]_0 \in Im(id - \alpha_*).$$

Proof: The existence of N such that $(p \times q)_n \in P_\infty(H)$ for all $n \geq N$ follows immediately from 3.5. Now fix $m > n \geq N$ and consider

$$\begin{aligned}\alpha_*[\alpha^n(p)\alpha^{-n}(q_{ij}(0))]_0 &= [\alpha^{n+1}(p)\alpha^{-n+1}(q_{ij}(0))]_0 \\ &= [\alpha^{n+1}(p)\alpha^{-n+1}(q_{ij}(-2))]_0 \text{ by homotopy invariance} \\ &= [\alpha^{n+1}(p)\alpha^{-n-1}(q_{ij}(0))]_0.\end{aligned}$$

So

$$(id - \alpha_*)[\alpha^n(p)\alpha^{-n}(q_{ij}(0))]_0 = [\alpha^n(p)\alpha^{-n}(q_{ij}(0))]_0 - [\alpha^{n+1}(p)\alpha^{-n-1}(q_{ij}(0))]_0,$$

and hence, by induction

$$[\alpha^n(p)\alpha^{-n}(q_{ij}(0))]_0 - [\alpha^m(p)\alpha^{-m}(q_{ij}(0))]_0 \in Im(id - \alpha_*).$$

□

Remark 3.16. *The same result clearly holds for the product $(q \times p)_n$ with entries $\alpha^n(q_{ij}(0))\alpha^{-n}(p)$.*

Definition 3.17. *For $p \in P_1(\mathcal{H}(\Sigma, \sigma))$, $q \in P_\infty(\mathcal{CH})$, $n \geq N$ define $(p \times q)_n$ as in Lemma 3.15. With a slight abuse of notation, we will often drop the n and write $(p \times q)$.*

We are now ready to state a main result.

Theorem 3.18. *Let X be in $C(A)$, let Y be in $M_K(\mathbb{Z})$ and let $M, N \geq 0$. The product of $[X, N]$ in $K_0(C(H, \alpha))$ and $[Y + B(A), M]$ in $K_1(C(H, \alpha))$ is*

$$[X, N] * [Y + B(A), M] = [XY + B(A), N + M] \in K_1(C(H, \alpha))$$

and

$$[Y + B(A), M] * [X, N] = [YX + B(A), N + M] \in K_1(C(H, \alpha)).$$

The proof of this Theorem is quite long, so we will break the proof down into a series of Lemmas.

Lemma 3.19. *Let $p \in P_1(\mathcal{H}(\Sigma, \sigma))$, $q \in P_m(\mathcal{CH})$ for some m. Let $u_p \in U_1(\widetilde{C(H, \alpha)})$ be*

$$u_p(s) = e^{2\pi i s}\alpha^n(p) + (1 - \alpha^n(p)) \quad \text{for } n \leq s \leq n + 1,$$

Then for n large enough (as in Lemma 3.15), the matrix $(u_p \times q)_n$ is given by

$$((u_p \times q)_n)_{ij} = \alpha_n(u_p)\alpha_{-n}(q_{ij}) = (u_{p \times q})_{ij} - (I_m - \alpha_{-n}(q_{ij})).$$

in other words

$$(u_p \times q)_n = u_{p \times q} - (I_m - \alpha_{-n}(q)) = u_{p \times q}\alpha_{-n}(q).$$

Proof:

$$\begin{aligned}(\alpha_n(u_p)\alpha_{-n}(q_{ij}))(s) &= u_p(s + n)q_{ij}(s - n) \\ &= \alpha_s((u_p(n))(q_{ij}(-n))) \\ &= \alpha_s\left(\left(e^{2\pi i n}\alpha^n(p) + (1 - \alpha^n(p))\right)\alpha^{-n}(q_{ij}(0))\right) \\ &= \alpha_s\left(e^{2\pi i n}\alpha^n(p)\alpha^{-n}(q_{ij}(0)) - \alpha^n(p)\alpha^{-n}(q_{ij}(0)) + \alpha^{-n}(q_{ij}(0))\right) \\ &= \alpha_s\left(e^{2\pi i n}((p \times q)_n)_{ij} - ((p \times q)_n)_{ij} + \alpha^{-n}(q_{ij}(0))\right) \\ &= \alpha_s\left(e^{2\pi i n}(p \times q)_n + (I_m - (p \times q)_n) - (I_m - \alpha^{-n}(q(0)))\right)_{ij} \\ &= \alpha_s\left(u_{(p \times q)}(0) - (I_m - \alpha^{-n}(q(0)))\right)_{ij} \\ &= \left(u_{(p \times q)}(s) - (I_m - \alpha^{-n}(q(s)))\right)_{ij}\end{aligned}$$

so

$$(u_p \times q)_n = u_{p\times q} - (I_m - \alpha_{-n}(q)).$$

Now

$$\begin{aligned}\alpha_n(u_p)\alpha_{-n}(q) &= \alpha_n(u_p)\alpha_{-n}(q)^2 \\ &= (u_{p\times q} - (I_m - \alpha_{-n}(q)))\,\alpha_{-n}(q) \\ &= u_{p\times q}\alpha_{-n}(q).\end{aligned}$$

□

Lemma 3.20. *Let p and q be as in Lemma 3.19, $u_{p\times q}$ and $v_{p\times q}$ as in Lemmas 3.13 and 3.14. Then*

$$u_{p\times q}(I_m - \alpha_{-n}(q)) = I_m - \alpha_{-n}(q)$$

and

$$v_{p\times q}\begin{bmatrix} I_m - \alpha_{-n}(q) & 0 \\ 0 & I_m - \alpha_{-n}(q)\end{bmatrix} = \begin{bmatrix} I_m - \alpha_{-n}(q) & 0 \\ 0 & I_m - \alpha_{-n}(q)\end{bmatrix}$$

Proof: Straightforward calculations from the definitions of $u_{p\times q}$, $v_{p\times q}$ and the result of Lemma 3.19. □

The following Lemma allows us to use a slightly simplified form of the product, similar to Prop. 3.5.

Lemma 3.21. *Let $p \in P_1(\mathcal{H}(\Sigma,\sigma))$, $q \in P_m(\mathcal{CH})$. Let $\tilde{p} \in P_2(C(S^1, C(H,\alpha)))$, then there exists $T \in \mathbb{R}$ such that for $t > T$*

$$\alpha_t(\tilde{p})_{(ij)}\alpha_{-t}(q_{kl}) = \alpha_{-t}(q_{kl})\alpha_t(\tilde{p}_{(ij)}).$$

Moreover, if we define $(\tilde{p}\times q)_t$ componentwise by

$$((\tilde{p}\times q)_t)_{(ik)(jl)} = \alpha_t(\tilde{p}_{(ij)})\alpha_{-t}(q_{kl})$$

then for $t > T$ $(\tilde{p}\times q)_t$ is a projection in $P_{2m}(C(S^1, C(H,\alpha)))$ and

$$[\tilde{p}]_0[q]_0 = \lim_{t\to\infty}[(\tilde{p}\times q)_t]_0$$

Proof: The product $(\tilde{p}\times q)_n$ defined in the statement of the Lemma is the same as regular matrix multiplication between the following two $2m\times 2m$ matrices.

$$(\tilde{p}\times q)_n = \begin{bmatrix} \alpha_n\tilde{p}_{11}I_m & \alpha_n\tilde{p}_{12}I_m \\ \alpha_n\tilde{p}_{21}I_m & \alpha_n\tilde{p}_{22}I_m\end{bmatrix}\begin{bmatrix} \alpha_{-n}(q) & 0 \\ 0 & \alpha_{-n}(q)\end{bmatrix}.$$

For the 2×2 matrix Y we will denote by $\bar{Y}$ the $2m\times 2m$ with $m\times m$ block entries $Y_{ij}I_m$. Also notice that $\overline{XY} = \bar{X}\bar{Y}$. We now have

$$\begin{aligned}(\tilde{p}\times q)_n &= \alpha_n\Big(\bar{v_p}\begin{bmatrix} I_m & 0 \\ 0 & 0\end{bmatrix}\bar{v_p^*}\Big)\alpha_{-n}\begin{bmatrix} q & 0 \\ 0 & q\end{bmatrix} \\ &= \overline{\alpha_n(v_p)}\begin{bmatrix} I_m & 0 \\ 0 & 0\end{bmatrix}\overline{\alpha_n(v_p^*)}\begin{bmatrix} \alpha_{-n}(q) & 0 \\ 0 & \alpha_{-n}(q)\end{bmatrix}.\end{aligned}$$

From the proof of Lemma 3.19 and Prop. 3.5 we see that

$$\alpha_n(u_p^*)\alpha_{-n}(q) = \alpha_{-n}(q)\alpha_n(u_p^*).$$

From the definition of (v_p^*), and the fact that R_t is a matrix of scalars, we can show that

$$\begin{bmatrix} \alpha_{-n}(q) & 0 \\ 0 & \alpha_{-n}(q)\end{bmatrix}$$

commutes with both $\overline{\alpha_n(v_p)}$ and $\overline{\alpha_n(v_p^*)}$.

So

$$\begin{aligned}(\tilde{p}\times q)_n &= \overline{\alpha_n(v_p)}\begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix}\overline{\alpha_n(v_p^*)}\begin{bmatrix} \alpha_{-n}(q) & 0 \\ 0 & \alpha_{-n}(q) \end{bmatrix} \\ &= \begin{bmatrix} \alpha_{-n}(q) & 0 \\ 0 & \alpha_{-n}(q) \end{bmatrix}\overline{\alpha_n(v_p)}\begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix}\overline{\alpha_n(v_p^*)}.\end{aligned}$$

Hence for sufficiently large n $(\tilde{p}\times q)_n \in P_{2m}(C(S^1, C(H,\alpha)))$ and

$$[\tilde{p}]_0[q]_0 = \lim_{n\to\infty}[(\tilde{p}\times q)_n]_0$$

□

The following Lemma contains the bulk of the calculation involved in proving Prop. 3.18.

Lemma 3.22. *Let $p \in P_1(\mathcal{H}(\Sigma,\sigma))$, $q \in P_m(\mathcal{CH})$. Let $u_p \in U_1(\widetilde{C(H,\alpha)})$ be*

$$u_p(s) = e^{2\pi i s}\alpha^n(p) + (1-\alpha^n(p)) \quad \text{for } n \le s \le n+1.$$

Let $\tilde{p} \in P_2(\widetilde{SC(H,\alpha)})$ be as in Lemma 3.14. Then

$$[\tilde{p}]_0[q]_0 = [\widetilde{p\times q}]_0 - [I_m - q]_0$$

Proof: As $\tilde{p} \in P_2(C(\widetilde{S^1, CH}))$, $q \in P_m(C(S^1\tilde{,} CH))$, by Lemma 3.21 the product is (for sufficiently large n)

$$[\tilde{p}]_0[q]_0 = [(\tilde{p}\times q)_n]_0.$$

Where $(\tilde{p}\times q)_n \in P_{2m}(C(\widetilde{S^1, CH}))$ is as defined in Lemma 3.21. Now as in the proof of Lemma 3.21,

$$(\tilde{p}\times q)_n = \overline{\alpha_n(v_p)}\begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix}\overline{\alpha_n(v_p^*)}\begin{bmatrix} \alpha_{-n}(q) & 0 \\ 0 & \alpha_{-n}(q) \end{bmatrix},$$

and

$$\begin{aligned}\overline{\alpha_n(v_p^*)}\begin{bmatrix} \alpha_{-n}(q) & 0 \\ 0 & \alpha_{-n}(q) \end{bmatrix} &= \begin{bmatrix} I_m & 0 \\ 0 & \alpha_n(u_p)I_m \end{bmatrix}\bar{R}_t\begin{bmatrix} \alpha_n(u_p^*)\alpha_{-n}(q) & 0 \\ 0 & \alpha_{-n}(q) \end{bmatrix}\bar{R}_t^* \\ &= \begin{bmatrix} I_m & 0 \\ 0 & \alpha_n(u_p)I_m \end{bmatrix}\bar{R}_t\begin{bmatrix} u_{p\times q}^*\alpha_{-n}(q) & 0 \\ 0 & \alpha_{-n}(q) \end{bmatrix}\bar{R}_t^* \\ &= \begin{bmatrix} \alpha_{-n}(q) & 0 \\ 0 & u_{p\times q}\alpha_{-n}(q) \end{bmatrix}\bar{R}_t\begin{bmatrix} u_{p\times q}^* & 0 \\ 0 & I_m \end{bmatrix}\bar{R}_t^* \\ &= \begin{bmatrix} \alpha_{-n}(q) & 0 \\ 0 & \alpha_{-n}(q) \end{bmatrix}v_{p\times q}^*.\end{aligned}$$

So we have

$$\begin{aligned}(\tilde{p}\times q)_n &= \overline{\alpha_n(v_p)}\begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix}\overline{\alpha_n(v_p^*)}\begin{bmatrix} \alpha_{-n}(q) & 0 \\ 0 & \alpha_{-n}(q) \end{bmatrix} \\ &= \overline{\alpha_n(v_p)}\begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \alpha_{-n}(q) & 0 \\ 0 & \alpha_{-n}(q) \end{bmatrix}v_{p\times q}^* \\ &= v_{p\times q}\begin{bmatrix} \alpha_{-n}(q) & 0 \\ 0 & 0 \end{bmatrix}v_{p\times q}^* \\ &= v_{p\times q}\begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix}v_{p\times q}^* - v_{p\times q}\begin{bmatrix} I_m-\alpha_{-n}(q) & 0 \\ 0 & 0 \end{bmatrix}v_{p\times q}^* \\ &= \widetilde{p\times q} - \begin{bmatrix} I_m-\alpha_{-n}(q) & 0 \\ 0 & 0 \end{bmatrix}.\end{aligned}$$

So,

$$[\tilde{p}]_0[q]_0 = [\tilde{p} \times q]_0 = [\widetilde{p \times q}]_0 - [I_m - \alpha_{-n}(q)]_0 = [\widetilde{p \times q}]_0 - [I_m - q]_0$$

□

In the above, $[\tilde{p}]_0$ is an element of $K_0(\widetilde{SC(H,\alpha)})$. We wish to consider $K_0(SC(H,\alpha))$, for which it suffices to consider elements of the form $[\tilde{p}]_0 - [\tilde{p}(0)]_0$.

Lemma 3.23. *Let $p \in P_1(\mathcal{H}(\Sigma,\sigma))$, $q \in P_m(\mathcal{CH})$ so that $[\tilde{p}]_0 - [\tilde{p}(0)]_0 \in K_0(SC(H,\alpha))$, $[q]_0 \in K_0(C(H,\alpha))$. Then*

$$([\tilde{p}]_0 - [\tilde{p}(0)]_0)[q]_0 = [\widetilde{p \times q}]_0 - [\widetilde{p \times q}(0)]_0.$$

Proof:

$$\begin{aligned}
([\tilde{p}]_0 - [\tilde{p}(0)]_0)[q]_0 &= ([\tilde{p}]_0 - [I_1]_0)[q]_0 \\
&= [\tilde{p} \times q]_0 - [q]_0 \\
&= [\widetilde{p \times q}]_0 - [I_m - q]_0 - [q]_0 \quad \text{Lemma 3.22} \\
&= [\widetilde{p \times q}]_0 - [I_m]_0 \\
&= [\widetilde{p \times q}]_0 - [\widetilde{p \times q}(0)]_0.
\end{aligned}$$

□

Remark 3.24. *The map $\phi : K_0(H) \cong K_1(SH) \hookrightarrow K_1(C(H,\alpha)) \cong K_0(SC(H,\alpha))$ is given by*

$$\phi([p]_0) = [\tilde{p}]_0 - [\tilde{p}(0)]_0.$$

Where, for $\tilde{p} \in P_{2n}(SC(H,\alpha))$, $\tilde{p}(0) = I_n$. So Lemma 3.23 simply says that to multiply $\phi([p]_0)$ by $[q]_0$, we can first multiply $[p]_0$ and $[q]_0$ in $K_0(H)$, which by Lemma 3.15 is well defined modulo $Im(id - \alpha_*)$, and then apply ϕ. As $Im(id - \alpha_*)$ is exactly the kernel of the map $K_1(SH) \hookrightarrow K_1(C(H,\alpha))$ everything is well defined.

Remark 3.25. *Completely analogous calculations to those above show that*

$$[q]_0([\tilde{p}]_0 - [\tilde{p}(0)]_0) = [\widetilde{q \times p}]_0 - [\widetilde{q \times p}(0)]_0.$$

The proof of Theorem 3.18 is now immediate.

Proof of Theorem 3.18 Let $[X,N] \in K_0(C(H,\alpha))$, $[Y + B(A), M] \in K_1(C(H,\alpha))$. By Lemma 3.23 and the remarks following it, the product

$$[X,N] * [Y + B(A), M] \in K_1(C(H,\alpha)).$$

Is obtained by multiplying $[X,N]$ and $[Y,M]$ in $K_0(H)$, and projecting into

$$K_0(H)/Im(id - \alpha_*) \cong K_1(C(H,\alpha)).$$

Now the product of $[X,N]$ and $[Y,M]$ is computed in the same way as in the proof of Theorem 3.9, so we have

$$[X,N] * [Y + B(A), M] = [XY + B(A), N + M],$$

and similarly

$$[Y + B(A), M] * [X,N] = [YX + B(A), N + M].$$

□

Finally, we turn to the product $K_1(C(H,\alpha)) \times K_1(C(H,\alpha)) \to K_0(C(H,\alpha))$.

Theorem 3.26. *For any a, b in $K_1(C(H,\alpha))$, their product is zero in $K_0(C(H,\alpha))$.*

Proof We know that SH is an ideal in $C(H, \alpha)$ and the map induced by inclusion is surjective on K_1. Secondly, it is clear that for integer values of the parameter t, the asymptotic homomorphism on a pair from SH takes values again in SH. Hence, we conclude that the range of the product must lie in the image of $K_0(SH)$ in $K_0(C(H, \alpha))$ under the inclusion map. On the other hand, $K_0(SH) = 0$. This completes the proof. □

3.5 $K_*(C(H, \alpha))$-Modules

We now turn to the issue of computing the module structures for $K_0(S(\Sigma, \sigma, P))$ and $K_0(U(\Sigma, \sigma, P))$ over the ring $K_0(C(H, \alpha))$. (Note that since both $S(\Sigma, \sigma, P)$ and $U(\Sigma, \sigma, P)$ are AF-algebras and have trivial K_1 groups, this is the only non-trivial case to consider.)

Recall the description of $K_0(S(\Sigma, \sigma, P))$ from section 3.2.

The construction of $U(\Sigma, \sigma, P)$ and $K_*(U(\Sigma, \sigma, P))$ follow immediately from the observation that the unstable algebra associated to a SFT with adjacency matrix A is simply the stable algebra associated to the SFT with adjacency matrix A^T. So $K_0(U(\Sigma, \sigma, P))$ is the inductive limit of

$$\mathbb{Z}^K \xrightarrow{v \mapsto vA^T} \mathbb{Z}^K \xrightarrow{v \mapsto vA^T} \mathbb{Z}^K \longrightarrow \cdots$$

or thinking of $\mathbb{Z}^K$ as column vectors

$$\mathbb{Z}^K \xrightarrow{v \mapsto Av} \mathbb{Z}^K \xrightarrow{v \mapsto Av} \mathbb{Z}^K \longrightarrow \cdots$$

In the later case, the maps α_* and α_*^{-1} are given by $\alpha_*[v, N] = [v, N+1]$ and $\alpha_*^{-1}[v, N] = [A^2 v, N+1]$.

Computing the module structure in much the same way as the product structure on $K_0(C(H, \alpha))$ above, we arrive at the following theorem.

Theorem 3.27. *Let (Σ, σ) be a mixing SFT with $K \times K$ adjacency matrix A.*

1. *For v in $\mathbb{Z}$, considered as a row vector, X in $C(A)$, $M, N \geq 0$, the product of $[v, N]$ in $K_0(S(\Sigma, \sigma, P))$ with $[X, M]$ in $K_0(C(H, \alpha))$ is*
$$[v, N] * [X, M] = [vX, N + 2M].$$
2. *For v in $\mathbb{Z}$, considered as a column vector, X in $C(A)$, $M, N \geq 0$, the product of $[X, M]$ in $K_0(C(H, \alpha))$ with $[v, N]$ in $K_0(U(\Sigma, \sigma, P))$ is*
$$[X, M] * [w, N] = [Xw, N + 2M].$$

The dimension group invariants $K_0(S(\Sigma, \sigma, P))$ and $K_0(U(\Sigma, \sigma, P))$ are usually accompanied by canonical automorphisms, namely those induced by α. In fact, these automorphisms are implemented by multiplication by canonical elements of our ring.

Proposition 3.28. *The automorphisms α_* and α_*^{-1} on $K_0(S(\Sigma, \sigma, P))$ are induced by multiplication by $[A, 0]$ and $[A, 1]$ respectively.*

Proof: For $[v, k] \in K_0(S(\Sigma, \sigma, P))$, we have $\alpha_*[v, k] = [vA, k] = [v, k] * [A, 0]$. The second statement follows from the fact $[A, 0] * [A, 1] = [A^2, 1] = [I, 0]$, which is the unit of our ring. □

The algebras $S(\Sigma, \sigma, P)$ and $U(\Sigma, \sigma, P)$ have traces defined similarly to trace on $H(\Sigma, \sigma)$ defined in [10]. We denote these by τ^s and τ^u respectively. The maps on K-theory induced by these traces can be computed in terms of the inductive systems.

Remark 3.29. *For $[v, N] \in K_0(S(\Sigma, \sigma, P))$, $[w, M] \in K_0(U(\Sigma, \sigma, P))$ we have $\tau^s_*([v, N]) = \lambda^{-N} v u_r$, $\tau^u_*([w, M]) = \lambda^{-M} u_l w$. Where λ is the Perron-Frobenius eigenvalue of A, and u_l, u_r are the left, right Perron-Frobenius eigenvectors of A. Moreover, if $[X, K] \in K_0(C(H, \alpha))$, it is straightforward to verify that*

$$\begin{aligned} \tau^{CH}_*([v, N] * [X, K]) &= \tau^s_*([v, N])\tau^{CH}_*([X, K]), \\ \tau^{CH}_*([X, K] * [w, M]) &= \tau^{CH}_*([X, K])\tau^u_*([w, M]). \end{aligned}$$

4 Duality

We now look for a duality result for $K_0(S(\Sigma, \sigma, P))$ and $K_0(U(\Sigma, \sigma, P))$ of the form $Hom_R(K_0(S(\Sigma, \sigma, P)), R) \cong K_0(U(\Sigma, \sigma, P))$ for some subring R of $K_0(C(H, \alpha))$. As $K_0(C(H, \alpha))$ is, in general, non-commutative, the subring R should lie in the center of the ring $K_0(C(H, \alpha))$.

Definition 4.1. *Let R be the subring of $K_0(C(H, \alpha))$ generated by $[A, 0]$ and $[A, 1]$.*

Let us repeat an earlier observation that $[A, 0]$ and $[A, 1]$ are inverses in $K_0(C(H, \alpha))$ since $[A, 0] * [A, 1] = [A^2, 1] = [I, 0]$. In particular, R contains the unit of $K_0(C(H, \alpha))$.

Remark 4.2. *It is clear that R is contained in the center of the ring $K_0(C(H, \alpha))$, $Z(K_0(C(H, \alpha)))$. In fact, in many, but not all, cases $R = Z(K_0(C(H, \alpha)))$. See section 7 for examples where these two subrings are not equal. Roughly speaking, the difference between R and the center of $K_0(C(H, \alpha))$, Z, is that R consists of all integer polynomials in A, whereas Z consists of all integer matrices which are rational polynomials in A.*

Theorem 4.3. *Let (Σ, σ) be a mixing SFT, and R the subring of $K_0(C(H, \alpha))$ as above. Then*

$$Hom_R(K_0(S(\Sigma, \sigma, P)), R) \cong K_0(U(\Sigma, \sigma, P))$$

as left R-modules.

Let A have minimal polynomial $m_A(x) = x^l(x^k + a_{k-1}x^{k-1} + \cdots + a_0)$ with $a_0 \neq 0$ (so m_A has degree $k + l$, and l is the multiplicity of 0 as a root). Further, we let $p_A(x) = x^k + a_{k-1}x^{k-1} + \cdots + a_0$, so that $m_A(x) = p_A(x)x^l$.

The following Lemma sharpens our description of R.

Lemma 4.4. *Let $\mathbb{Z}_{k-1}[x]$ denote the set of integral polynomials with degree less than k (the degree of $p_A(x)$). Then we have*

$$R = \{[p(A), N] \in K_0(C(H, \alpha)) \mid p \in \mathbb{Z}_{k-1}[x], N \geq 0\}.$$

Moreover, for $p \in \mathbb{Z}_{k-1}[x]$ and $N \geq 0$, $[p(A), N] = 0$ if and only if $p = 0$.

Proof: The containment $\supset$ follows from the observation that, for any $i \geq 0$ and $N \geq 0$, we have

$$[A^i, N] = [A^{i+2j}, N + j] = [A, 0]^{i+j-N} * [A, 1]^{N+j},$$

where j is chosen so that $i + j - N > 0$.

For the reverse inclusion, it is clear that any element of R must be of the form $[p(A), N]$, where p is some integer polynomial and $N \geq 0$. Let j be the degree of p. Of course, if $j < k$, then we are done, so suppose that $j \geq k$. Consider $p(x)x^{2l}$ and divide it by $m_A(x)x^l = p_A(x)x^{2l}$ and write the result:

$$p(x)x^{2l} = p_A(x)x^{2l}d(x) + q(x),$$

where $q(x)$ has degree less than that of $p_A(x)x^{2l}$, namely $2l+k$. It follows that $p(A)A^{2l} = q(A)$. Also, x^{2l} clearly divides both $p(x)x^{2l}$ and $p_A(x)x^{2l}d(x)$, so it also divides $q(x)$. We write $q(x) = q_0(x)x^{2l}$. Now $q_0(x)$ has degree less than k and

$$[p(A), N] = [p(A)A^{2l}, N+l] = [q(A), N+l] = [q_0(A)A^{2l}, N+l] = [q_0(A), N].$$

This completes the proof of the first statement.

For the second statement, the "if" part is clear. Now suppose $[c_{k-1}A^{k-1} + \cdots + c_0 I, N] = [0, 0]$. In other words, there exists m such that $(c_{k-1}A^{k-1} + \cdots + c_0 I)A^{2m} = 0$. But since l is the multiplicity of 0 as a root of the minimal polynomial, it must be true that $(c_{k-1}A^{k-1} + \cdots + c_0 I)A^l = 0$. So we have $c_{k-l-1}A^{k+l-1} + \cdots + c_0 A^l = 0$. We recall that the minimal polynomial of A has degree $k+l$, so $\{A^{k+l-1}, A^{k+l-2}, \ldots, A^l\}$ is a linearly independent set, and we see that all of the c_i's equal 0. □

Lemma 4.5. *Let $\varphi \in Hom_R(K_0(S), R)$, there exists $z \in \mathbb{Z}^K$ (considered as a column vector) and $N \in \mathbb{N}$ such that, for each $[v, n] \in K_0(S)$,*

$$\varphi[v,n] = [vA^n z A^{k-1} + vA^n(A + a_{k-1}I)zA^{k-2} + \cdots + vA^n(A^{k-1} + a_{k-1}A^{k-2} + \cdots + a_1 I)zI, N+n].$$

We denote this homomorphism by $\varphi_{(z,N)}$.

Proof: Let $\{v_i\}_{i=1}^K$ be the standard basis for $\mathbb{Z}^K$ and fix $\varphi \in Hom_R(K_0(S), R)$. For each i consider $\varphi[v_i, 0] = [X_i, N_i]$ where N_i is the least integer in the equivalence class. That is to say, if $[Y, M] = [X_i, N_i]$ then $N_i \le M$. Now define

$$N = \max\{N_i \mid 1 \le i \le K\}.$$

So for all $v \in \mathbb{Z}^K$ we can write $\varphi[v, 0] = [X_v, N]$ for some $X_v \in S(A)$. Now recall that $[v, n] * [A, 1] = [v, n+1]$, and $[X_v, N] * [A, 1] = [X_v A, N+1]$, and φ is an R-module homomorphism. So

$$\varphi[v,n] = \varphi\left([v,0] * [A,1]^n\right) = \varphi[v,0] * [A,1]^n = [X_v, N] \otimes [A,1]^n = [X_v A^n, N+n].$$

This shows that the number N is important data in describing φ and that φ can be described completely by its restriction to terms of the form $[p(A), 0]$, where p is in $\mathbb{Z}_{k-1}[x]$. Also, for any $[v, n] \in K_0(S)$, $\varphi[v, 0]$ must be of the form

$$\varphi[v_i, 0] = [(x_{k-1})_i A^{k-1} + (x_{k-2})_i A^{k-2} + \cdots + (x_0)_i I, N],$$

for some integers $(x_{k-1})_i, \ldots, (x_0)_i$. It then follows from linearity that for $v \in \mathbb{Z}^K$ we have

$$\varphi[v, 0] = [vx_{k-1}A^{k-1} + vx_{k-2}A^{k-2} + \cdots + vx_0 I, N],$$

where $x_j \in \mathbb{Z}^K$ is a column vector for each $0 \le j \le k-1$.

We now use the fact that φ is a module homomorphism to impose conditions on the x_i. First, notice that $[v, 0] * [A, 0] = [vA, 0]$, and $[X, N] * [A, 0] = [XA, N]$. Now, from above

$$\varphi[vA, 0] = [vAx_{k-1}A^{k-1} + vAx_{k-2}A^{k-2} + \cdots + vAx_0 I, N]$$

but, since $[vA, 0] = [v, 0] * [A, 0]$ and φ is a module homomorphism we have

$$\begin{aligned}
\varphi[vA, 0] &= \varphi([v,0] * [A,0]) \\
&= \varphi([v,0]) * [A,0] \\
&= [vx_{k-1}A^{k-1} + vx_{k-2}A^{k-2} + \cdots + vx_0 I, N] * [A, 0] \\
&= [vx_{k-1}A^k + vx_{k-2}A^{k-1} + \cdots + vx_0 A, N] \\
&= [vx_{k-1}(-a_{k-l}A^{k-1} - \cdots - a_0 I) + vx_{k-2}A^{k-1} + \cdots + vx_0 A, N] \\
&= [v(x_{k-2} - a_{k-1}x_{k-1})A^{k-1} + \cdots + v(x_0 - a_1 x_{k-1})A - a_0 v x_{k-1} I, N].
\end{aligned}$$

Comparing coefficients of like powers of A in these two expressions for $\varphi[vA,0]$, and noting that $v \in \mathbb{Z}^K$ was arbitrary, we see (in light of Lemma 4.4) that

$$\begin{aligned} x_{k-2} - a_{k-1}x_{k-1} &= Ax_{k-1} \\ x_{k-3} - a_{k-2}x_{k-1} &= Ax_{k-2} \\ &\vdots \\ x_0 - a_1 x_{k-1} &= Ax_1 \\ -a_0 x_{k-1} &= Ax_0. \end{aligned}$$

So if we let $z = x_{k-1}$, then all other x_j's are determined, and we have

$$\varphi[v,0] = [vzA^{k-1} + v(A + a_{k-1}I)zA^{k-2} + \cdots + v(A^{k-1} + a_{k-1}A^{k-2} + \cdots + a_1 I)zI, N].$$

Now consider

$$\begin{aligned} \varphi[v,1] &= \varphi([v,0] * [A,1]) \\ &= [vzA^{k-1} + v(A + a_{k-1}I)zA^{k-2} + \cdots \\ &\quad + v(A^{k-1} + a_{k-1}A^{k-2} + \cdots + a_1 I)zI, N] * [A,1] \\ &= [vzA^k + v(A + a_{k-1}I)zA^{k-1} + \cdots + v(A^{k-1} + a_{k-1}A^{k-2} + \cdots + a_1 I)zA, N+1]. \end{aligned}$$

Expanding this expression using $[A^k, N+1] = [-a_{k-1}A^{k-1} - \cdots - a_0 I, N+1]$ and simplifying we are left with

$$\begin{aligned} \varphi[v,1] = [&vAzA^{k-1} + vA(A + a_{k-1}I)zA^{k-2} + \cdots \\ &+ vA(A^{k-1} + a_{k-1}A^{k-2} + \cdots + a_1 I)zI, N+1], \end{aligned}$$

Similarly, we can show inductively that

$$\begin{aligned} \varphi[v,n] = [&vA^n zA^{k-1} + vA^n(A + a_{k-1}I)zA^{k-2} + \cdots \\ &+ vA^n(A^{k-1} + a_{k-1}A^{k-2} + \cdots + a_1 I)zI, N+n]. \end{aligned}$$

We denote this homomorphism $\varphi_{(z,N)}$. □

Lemma 4.5 shows that each $\varphi \in Hom_R(K_0(S), R)$ is of the form $\varphi_{(z,N)}$ for some $(z,N) \in \mathbb{Z}^K \times \mathbb{N}$. It is also clear that each $(z,N) \in \mathbb{Z}^K \times \mathbb{N}$ gives rise to $\varphi_{(z,N)} \in Hom_R(K_0(S), R)$ and that $\varphi_{(z,N)} + \varphi_{(w,N)} = \varphi_{(z+w,N)}$. It remains to be be determined when $\varphi_{(z,N)} = \varphi_{(w,M)}$.

Lemma 4.6. *Let z, $w \in \mathbb{Z}^K$, $N \le M \in \mathbb{N}$. $\varphi_{(z,N)} = \varphi_{(w,M)}$ if and only if there exists $m \in \mathbb{N}$ such that $A^{2(m+M-N)}z = A^{2m}w$.*

Proof: First suppose there exists $m \in \mathbb{N}$ such that $A^{2(m+M-N)}z = A^{2m}w$. Let $k = m + M - N$, then for all $[v,n] \in K_0(S)$

$$\begin{aligned} \varphi_{(z,N)}[v,n] &= [vA^n zA^{k-1} + \cdots + vA^n(A^{k-1} + a_{k-1}A^{k-2} + \cdots + a_1 I)zI, N+n] \\ &= [A^{2k}(vA^n zA^{k-1} + \cdots \\ &\quad + vA^n(A^{k-1} + a_{k-1}A^{k-2} + \cdots + a_1 I)zI), M+m+n] \\ &= [A^{2m}(vA^n wA^{k-1} + \cdots \\ &\quad + vA^n(A^{k-1} + a_{k-1}A^{k-2} + \cdots + a_1 I)wI), M+m+n] \\ &= [vA^n wA^{k-1} + \cdots + vA^n(A^{k-1} + a_{k-1}A^{k-2} + \cdots + a_1 I)wI, M+n] \\ &= \varphi_{(w,M)}[v,n]. \end{aligned}$$

Now suppose $\varphi_{(z,N)} = \varphi_{(w,M)}$, for each $[v,n] \in K_0(S)$ there exists $m \in \mathbb{N}$ such that

$$A^{2(m+M-N)}(vA^n zA^{k-1} + \cdots + vA^n(A^{k-1} + a_{k-1}A^{k-2} + \cdots + a_1 I)zI) \\ = A^{2m}(vA^n wA^{k-1} + \cdots + vA^n(A^{k-1} + a_{k-1}A^{k-2} + \cdots + a_1 I)wI),$$

However, as l is the multiplicity of 0 as a root to the minimal polynomial of A, $A^{l+j}X = A^{l+j}Y$ if and only if $A^l X = A^l Y$, so we can replace m by l to get an expression which is valid for any $[v,n]$. This becomes

$$[vA^n A^{2(l+M-N)}zA^{k-1} + \cdots + vA^n(A^{k-1} + a_{k-1}A^{k-2} + \cdots + a_1 I)A^{2(l+M-N)}zI, M+l+n] \\ = [vA^n A^{2l}wA^{k-1} + \cdots + vA^n(A^{k-1} + a_{k-1}A^{k-2} + \cdots + a_1 I)A^{2l}wI, M+l+n].$$

Comparing the coefficients of A^{k-1}, in light of Lemma 4.4, we see that

$$vA^n A^{2(l+M-N)}z = vA^n A^{2l}w$$

for any $v \in \mathbb{Z}^{\#V(G)}$, so

$$A^n A^{2(l+M-N)}z = A^n A^{2l}w.$$

□

Proposition 4.7. *$Hom_R(K_0(S), R)$ is equal to the limit of the following inductive system*

$$\mathbb{Z}^K \xrightarrow{z \mapsto A^2 z} \mathbb{Z}^K \xrightarrow{z \mapsto A^2 z} \mathbb{Z}^K \xrightarrow{z \mapsto A^2 z} \cdots$$

In other words

$$Hom_R(K_0(S), R) \cong (\mathbb{Z}^K \times \mathbb{N})/\sim_2$$

where, for $N \leq M$, $(z,N) \sim_2 (w,M)$ if and only if there exists $m \in \mathbb{N}$ such that $A^{2(m+M-N)}z = A^{2m}w$.

Proof: Follows immediately from Lemma 4.6 and the comments following Lemma 4.5. □

We are now ready to prove our main result.

Proof of Theorem 4.3 To prove that $Hom_R(K_0(S), R) \cong K_0(U)$ as left R-modules we use the characterizations

$$Hom_R(K_0(S), R) \cong (\mathbb{Z}^K \times \mathbb{N})/\sim_2$$

and

$$K_0(U) \cong (\mathbb{Z}^K \times \mathbb{N})/\sim_1$$

where, for $N \leq M$, $(z,N) \sim_1 (w,M)$ if and only if there exists $m \in \mathbb{N}$ such that $A^{m+M-N}z = A^m w$. For $(z,N) \in \mathbb{Z}^K \times \mathbb{N}$ we denote the equivalence class under $\sim_2$ by $[z,N]_2$, and the equivalence class under $\sim_1$ by $[z,N]_1$.

Let $[z,N]_2 \in Hom_R(K_0(S), R)$, $[w,M]_1 \in K_0(U)$. Consider the maps

$$\begin{aligned} \phi : Hom_R(K_0(S), R) &\rightarrow K_0(U), \text{ and} \\ \psi : K_0(U) &\rightarrow Hom_R(K_0(S), R) \end{aligned}$$

given by

$$\begin{aligned} \phi[z,N]_2 &= [z,2N]_1, \\ \psi[w,M]_1 &= [A^M w, M]_2 \end{aligned}$$

It is straight forward to check that ϕ is a group isomorphism and $\psi = \phi^{-1}$. We must check that ϕ respects the R-module structure. As R is generated by $[A,1]$ and $[A,0]$ is suffices to check only these two elements of R. Fix $\varphi_{[z,N]_2} \in Hom_R(K_0(S), R)$, then for $[v,n] \in K_0(S)$ straightforward calculations show that

$$\begin{aligned} [A,1] * \varphi_{[z,N]_2}[v,n] &= \varphi_{[z,N]_2}([v,n] * [A,1]) \\ &= \varphi_{[Az,N+1]_2}[v,n] \end{aligned}$$

and

$$\begin{aligned} [A,0] * \varphi_{[z,N]_2}[v,n] &= \varphi_{[z,N]_2}([v,n] * [A,0]) \\ &= \varphi_{[Az,N]_2}[v,n]. \end{aligned}$$

So on $Hom_R(K_0(S), R) \cong (\mathbb{Z}^{\#V(G)} \times \mathbb{N})/\sim_2$ the R-module structure is given by

$$\begin{aligned} [A,1] * [z,N]_2 &= [Az, N+1]_2, \\ [A,0] * [z,N]_2 &= [Az, N]_2. \end{aligned}$$

Now

$$[A,1] * \phi[z,N]_2 = [A,1] * [z,2N]_1 = [Az,2N+2]_1 = \phi[Az,N+1]_2 = \phi([A,1] * [z,N]_2),$$

and

$$[A,0] * \phi[z,N]_2 = [A,0] * [z,2N]_1 = [Az,2N]_1 = \phi[Az,N]_2 = \phi([A,0] * [z,N]_2).$$

So ϕ is an isomorphism of R-modules. □

We conclude this section by presenting one more characterization of the ring R.

Proposition 4.8. *Let (Σ, σ) be a SFT with adjacency matrix A. Let the minimal polynomial of A be $m_A(x) = x^l(x^k + a_{k-1}x^{k-1} + \cdots + a_0)$ with $a_0 \neq 0$. Let R be the subring of $K_*(C(H,\alpha))$ described above. Then*

$$R \cong \mathbb{Z}[x, x^{-1}]/ < p_A(x) >,$$

where $p_A(x) = x^k + a_{k-1}x^{k-1} + \cdots + a_0$.

Proof: Consider the map $\phi : \mathbb{Z}[x,x^{-1}] \to R$ given by $\phi(p) = p([A,0],[A,1])$. ϕ is clearly onto, so to prove the claim we need only show that $ker(\phi) =< p_A(x) >$. First notice that

$$p_A([A,0]) = [p_A(A), 0] = [A^l p_A(A) A^l, l] = [m_A(A) A^l, l] = [0,0].$$

So $< p_A(x) > \subset ker(\phi)$.

Now suppose $p \in ker(\phi)$, where

$$p(x, x^{-1}) = \sum_{i=-m}^{n} c_i x^i.$$

$p(x,x^{-1})$ is in $ker(\phi)$ if and only if $x^m p(x,x^{-1}) = \sum_{i=0}^{n+m} c_{i-m} x^i \in ker(\phi)$. So it suffices to consider elements $q(x) \in ker(\phi)$ of the form $q(x) = c_n x^n + \cdots + c_0$. We then have $q([A,0]) = [0,0]$, or

$$[(c_n A^n + \cdots + c_0 I), 0] = [0,0].$$

Lemma 4.4 then implies that either $c_i = 0$ for all $0 \leq i \leq n$, in which case $q = 0 \in< p_A(x) >$, or $n \geq k$. Furthermore, we have that, for some m

$$A^m(c_n A^n + \cdots + c_0 I)A^m = A^m 0 A^m = 0,$$

but, as l is the multiplicity of 0 as a root of the minimal polynomial of A, we have

$$A^l(c_n A^n + \cdots + c_0 I) = 0.$$

So the minimal polynomial $m_A(x) = x^l(x^k + a_{k-1}x^{k-1} + \cdots + a_0)$ must divide the polynomial $x^l(c_n x^n + \cdots + c_0 I)$, and hence $p_A(x)|q(x)$. In other words $q(x) = f(x)p_A(x)$ for some $f \in \mathbb{Z}[x, x^{-1}]$ and $q(x) \in < p_A(x) >$. Therefore, $ker(\phi) =< p_A(x) >$ and

$$R \cong \mathbb{Z}[x, x^{-1}]/ < p_A(x) > .$$

□

5 Shift equivalence

We now discuss how this relates to shift equivalence of integer matrices. Much of the work along this vein was done by Williams (eg [17]) and Kim and Roush (eg [5], [6]). Chapter 7 of [8] presents a nice treatment of this material.

Definition 5.1. *The non-negative $n \times n$ integer matrix A and the non-negative $m \times m$ integer matrix B are said to be* **shift equivalent** *if there exist non negative integer matrices R $(n \times m)$ and S $(m \times n)$ and a positive integer k such that:*

- $RS = A^k$
- $SR = B^k$
- $AR = RB$
- $SA = BS.$

In this case we call k the **lag**. *We denote shift equivalence by $\sim_{SE}$*

Shift equivalence classifies $K_0(S(\Sigma, \sigma, P))$ up to isomorphism of the dimension triple $(K_0(S(\Sigma, \sigma, P)), K_0(S(\Sigma, \sigma, P))^+, \alpha_*)$, as shown in Theorem 7.5.8 of [8].

We now show that two SFTs with shift equivalent adjacency matrices also have isomorphic ring/module structures of K-theory.

Remark 5.2. *Suppose A and B are shift equivalent with lag k, with R, S as in Defn. 5.1. Then A and B are shift equivalent with lag $k + 1$, where the equivalence is implemented by AR and S (there are other choices that would also work, for example, R and BS).*

Theorem 5.3. *Let (Σ_A, σ_A) and (Σ_B, σ_B) be SFTs with adjacency matrices A, and B respectively. If A and B are shift equivalent, then*

$$(K_*(C(H_A, \alpha_A)), K_0(S_A), K_0(U_A)) \cong (K_*(C(H_B, \alpha_B)), K_0(S_B), K_0(U_B))$$

in the following sense: there exist

1. $\phi_H : K_*(C(H_A, \alpha_A)) \to K_*(C(H_B, \alpha_B))$ *an isomorphism of ordered rings,*
2. $\phi_S : K_0(S(\Sigma_A, \sigma_A)) \to K_0(S(\Sigma_B, \sigma_B))$ *an isomorphism of ordered groups,*
3. $\phi_U : K_0(U(\Sigma_A, \sigma_A)) \to K_0(U(\Sigma_B, \sigma_B))$ *an isomorphism of ordered groups, and*
4. *for all $h \in K_*(C(H_A, \alpha_A))$, $s \in K_0(S(\Sigma_A, \sigma_A))$, $u \in K_0(U(\Sigma_A, \sigma_A))$ we have $\phi_S(s * h) = \phi_S(s) * \phi_H(h)$ and $\phi_U(h * u) = \phi_H(h) * \phi_U(u)$.*

Proof: Suppose $A \sim_{SE} B$ with R, S and lag k as in Defn. 5.1. Without loss of generality we assume k is even (see remark above). From section 7.5 of [8] we have that ϕ_S, ϕ_U defined by

$$\phi_S[v, N] = [vR, N + \frac{k}{2}] \quad , \phi_U[w, M] = [Sw, M + \frac{k}{2}]$$

are the desired order preserving group isomorphisms. Now for $[X, N] + [Y, M + B(A)] \in K_*(C(H_A, \alpha_A))$ define

$$\phi_H([X, N] + [Y + B(A), M]) = [SXR, N + \frac{k}{2}] + [SYR + B(B), M + \frac{k}{2}].$$

It is then straightforward to verify that ϕ_H is a ring homomorphism, and that condition 4 is satisfied. □

6 Irreducible Smale space

In this section we extend the results proved in previous sections to the case that the Smale space is *irreducible*. The key is Smale's spectral decomposition (see [14], [16]).

Theorem 6.1 (Smale)**.** *Let (x, φ) be an irreducible Smale space. Then there exists a positive integer N and subsets $X_1, X_2, \ldots, X_N$ of X which are closed, open, pairwise disjoint, and whose union equals X. These sets are cyclicly permuted by φ, and $\varphi^N|_{X_i}$ is mixing for each i. These sets are unique up to (cyclic) relabelling.*

This theorem can, alternatively, be formulated as follows.

Corollary 6.2. *Let (X, φ) be an irreducible Smale space, then there exists a mixing Smale space (Y, ψ) and a positive integer N such that $X \cong Y \times \{1, \ldots, N\}$ and*

$$\varphi(y, i) = \begin{cases} (y, i+1) & \text{if } 1 \leq i \leq N-1 \\ (\psi(y), 1) & \text{if } i = N \end{cases}$$

Proof: Let $X_1, \ldots, X_N$ be as in Prop. 6.1. It suffices to show that for $1 \leq i \leq N-1$, $(X_i, \varphi^N) \cong (X_{i+1}, \varphi^N)$ with the topological conjugacy realized by the map φ.

As φ is a homeomorphism, it suffices to show that, for all $x \in X_i$

$$\varphi \circ \varphi^N(x) = \varphi^N \circ \varphi(x),$$

which is obvious. Now setting $Y = X_1$, $\psi = \varphi^N$ we have $X \cong Y \times \{1, \ldots, N\}$ and $\varphi(y, i) = (y, i+1)$ for $1 \leq i \leq N-1$. Finally, for all $1 \leq i \leq N$, we have $\varphi^N(y, i) = (\varphi^N(y), i) = (\psi(y), i)$ which implies $\varphi(y, N) = (\psi(y), 1)$. □

Let (X, φ) be a Smale space and fix $n \in \mathbb{N}$. It is easy to see that (X, φ^n) is also a Smale space with the same bracket function $[\cdot, \cdot]$. It is also easy to see that the 3 equivalence relations are unchanged by switching from φ to φ^n. For example, for $x \in X$ the set $V^s(x)$ is the same whether we consider the map φ or φ^n. In particular, for a finite φ-invariant (also φ^n-invariant) set $P \subset X$ the groupoids $G^s(X, \varphi, P)$ and $G^s(X, \varphi^n, P)$ are the same. Similarly for $G^u(X, \varphi, P)$ and $G^u(X, \varphi^n, P)$, and $G^h(X, \varphi)$ and $G^h(X, \varphi^n)$. It then follows that $S(X, \varphi, P) = S(X, \varphi^n, P)$ and similarly for the unstable and homoclinic algebras. It should be noted that while $S(X, \varphi, P) = S(X, \varphi^n, P)$, the automorphisms α_φ and α_{φ^n} are not equal.

Now suppose (X, φ) is an irreducible Smale space and (Y, ψ), $n \in \mathbb{N}$ are as in Prop. 6.2. So (Y, ψ) is mixing, $X \cong Y \times \{1, 2, \ldots, n\}$ and $\varphi(x, i) = (x, i+1)$ if $1 \leq i \leq n-1$, $\varphi(x, n) = (\psi(x), 1)$. If we consider the Smale space (X, φ^n) we still have $X \cong Y \times \{1, 2, \ldots, n\}$, and now $\varphi^n(x, i) = (\psi(x), i)$. So (X, φ^n) is a disjoint union of n copies of the mixing Smale space (Y, φ).

If we now fix a finite φ-invariant set $P \subset X \cong Y \times \{1, 2, \ldots, n\}$, and let $\tilde{P}$ be $P \cap Y \times \{1\}$ we immediately see that

$$\begin{aligned} S(X, \varphi, P) = S(X, \varphi^n, P) &\cong \bigoplus_i^n S(Y, \psi, \tilde{P}), \\ U(X, \varphi, P) = U(X, \varphi^n, P) &\cong \bigoplus_i^n U(Y, \psi, \tilde{P}), \text{ and} \\ H(X, \varphi) = H(X, \varphi^n, P) &\cong \bigoplus_i^n H(Y, \psi). \end{aligned}$$

Denote by α_φ and α_ψ the $*$-automorphisms on $S(X,\varphi,P)$ and $S(Y,\psi,\tilde{P})$ respectively. It is then straightforward to see that α_φ permutes the summands of $\bigoplus_i^n S(Y,\psi,\tilde{P})$. In particular, for $a \in S(Y,\psi,\tilde{P})$ we have

$$\alpha_\varphi(a,i) = \begin{cases} (a,i+1) & 1 \le i \le n-1 \\ (\alpha_\psi(a),1) & i = n. \end{cases}$$

The corresponding results hold for $U(X,\varphi,P)$ and $H(X,\varphi)$ similarly.

We briefly describe the mapping cylinder and its K-theory ring in the case that (X,φ) is an irreducible Smale space.

Lemma 6.3. *Let $(X,\varphi) \cong (Y,\psi) \times \{1,2,\dots,n\}$ be an irreducible Smale space as in Prop. 6.2. Then for $f \in C(H(X,\varphi),\alpha^\varphi)$ we can write*

$$f(t) = (f_1(t), f_1(t-1), \dots, f_1(t-n+1))$$

for some $f_1 \in C(\mathbb{R}, H(Y,\psi))$ such that $\alpha^\psi(f_1(t)) = f_1(t+n)$. In other words $f_1(t) = \tilde{f}_1(\frac{t}{n})$ for some $\tilde{f}_1 \in C(H(Y,\psi),\alpha^\psi)$.

Proof: We can write $f \in C(H(X,\varphi),\alpha^\varphi)$ as

$$f(t) = (f_1(t), f_2(t), \dots, f_n(t))$$

where each $f_i \in C(\mathbb{R}, H(Y,\psi))$ and recall that $\alpha^\varphi(f(t)) = f(t+1)$. Now

$$\alpha^\varphi(f(t)) = (\alpha^\psi(f_n(t)), f_1(t), f_2(t), \dots, f_{n-1}(t))$$

so, for $1 \le i \le n-1$ we have

$$f_i(t) = f_{i+1}(t+1), \text{ or } f_{i+1}(t) = f_i(t-1),$$

so for $0 \le k \le n-1$

$$f_{1+k}(t) = f_1(t-k).$$

Also,

$$f_1(t+1) = \alpha^\psi(f_n(t)) = \alpha^\psi(f_1(t-n+1)), \text{ or } f_1(t+n) = \alpha^\psi(f_1(t)).$$

□

Proposition 6.4. *Let $(X,\varphi) \cong (Y,\psi) \times \{1,2,\dots,n\}$ be an irreducible Smale space, then $C(H_X,\alpha_\varphi) \cong C(H_Y,\alpha_\psi)$.*

Proof: For $f \in C(H(X,\varphi),\alpha^\varphi)$ the map from Lemma 6.3 which sends f to $\tilde{f}_1 \in C(H(Y,\psi),\alpha^\psi)$. Has inverse

$$g(t) \mapsto (g(nt), g(nt-1), \dots, g(nt-n+1)).$$

□

Corollary 6.5. *$K_*(C(H_X,\alpha_\varphi)) \cong K_*(C(H_Y,\alpha_\psi))$ as rings.*

Remark 6.6. *It is now straightforward to write down the module structures for an irreducible Smale space. In particular, if (X,φ) is an irreducible Smale space then*

$$S(X,\varphi,P) \cong \bigoplus_1^n S(Y,\psi,\tilde{P})$$

and

$$C(H_X,\alpha_\varphi) \cong C(H_Y,\alpha_\psi).$$

So for $[b] \in K_*C(H_Y,\alpha_\psi)$, $([a_1],[a_2],\dots,[a_n]) \in S(X,\varphi,P)$ we have

$$([a_1],[a_2],\dots,[a_n])[b] = ([a_1][b],[a_2][b],\dots,[a_n][b]).$$

The corresponding results for the module structures on $U(X,\varphi,P)$ and $H(X,\varphi)$ also hold.

7 Examples

We provide examples in which $K_0(C(H,\alpha))$ in non-commutative, and in which the subring R of Theorem 4.3 is strictly contained in the center of $K_0(C(H,\alpha))$. We leave the details of the calculations to the reader.

7.1 Non-Commutative $K_0(C(H,\alpha))$

We begin with an example in which the ring $K_0(C(H,\alpha))$ is non-commutative. Consider the SFT, (Σ,σ) with adjacency matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

It is straightforward to check that the set of matrices $\{X_1,\ldots,X_5\}$ below span $C(A)$.

$$X_1 = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad X_2 = \begin{bmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad X_3 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix}$$

$$X_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \quad X_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

We now compute the product structure for $K_0(C(H,\alpha))$. As $AX_iA = X_i$ for $1 \le i \le 4$ we have $[X_i,N] = [X_i,0]$. Now for $i,j \in \{1,2,3,4\}$, we compute

$$\begin{aligned} [X_i,0]*[X_j,0] &= [X_iX_j,0] \\ [I,N]*[X_i,M] &= [X_i,N+M] = [X_i,0] \\ [I,N]*[I,M] &= [I,N+M]. \end{aligned}$$

The subgroup $< X_1,X_2,X_3,X_4 >\cong \mathbb{Z}^4$ is in fact an ideal. Computing the products X_iX_j we get the (non-commutative) multiplication table for the ideal:

	X_1	X_2	X_3	X_4
X_1	$2X_1$	$2X_2$	$-X_1$	$-X_2$
X_2	$-X_1$	$-X_2$	$2X_1$	$2X_2$
X_3	$2X_3$	$2X_4$	$-X_3$	$-X_4$
X_4	$-X_3$	$-X_4$	$2X_3$	$2X_4$

7.2 $R \ne Z(K_0(C(H,\alpha)))$

We provide two examples of SFTs in which the subring R is strictly contained in the center, Z, of $K_0(C(H,\alpha))$. In the first example, it is also true that $Hom_Z(K_0(S(\Sigma,\sigma,P)),Z) \cong K_0(U(\Sigma,\sigma,P))$ (even though $R \ne Z$). However, in the second example, we really do need to restrict to R to get the duality result. We begin by stating two results which enable us to characterize the center of $K_0(C(H,\alpha))$. The first is well known from linear algebra, and the second follows easily from the first.

Lemma 7.1. *For $A \in M_N(\mathbb{Z})$, $C(A)$ as above, the center of $C(A)$ is $\mathbb{Q}[A] \cap M_N(\mathbb{Z})$. We denote this by $Z(A)$.*

Proposition 7.2. *Let (Σ, σ) be a SFT with adjacency matrix A. The center of $K_0(C(H,\alpha))$, Z, is the limit of the following inductive system.*

$$Z(A) \overset{X\mapsto AXA}{\longrightarrow} Z(A) \overset{X\mapsto AXA}{\longrightarrow} \cdots$$

Now consider the SFT with adjacency matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$

It is easy to check that $C(A) = Z(A)$. We also notice that $\frac{1}{2}(A - I) \in Z(A)$, but not $\mathbb{Z}[A]$. Moreover,

$$A\frac{1}{2}(A-I)A = \frac{1}{2}(A^3 - A^2) = \frac{1}{2}(5A + 3I) = \frac{1}{2}(A-I) + 2A + 2I,$$

so $[\frac{1}{2}(A - I), 0] \neq [X, N]$ for any $[X, N] \in R$. I.e. $Z \neq R$.

We now show that $Hom_Z(K_0(S), Z) \cong K_0(U)$.

As is the proof of Lemma 4.5, we can show that, for $\varphi \in Hom_Z(K_0(S), Z)$ there is $z \in (\mathbb{Z}/2)^2$, $N \in \mathbb{N}$ such that

$$\varphi[v, n] = [vA^n zA + vA^n(A - 2I)zI, N + n],$$

with the extra condition on z that $vA^n z + vA^n(A - 2I)z \in \mathbb{Z}$ for all $v \in \mathbb{Z}^2$, $n \in \mathbb{N}$. However, a quick check shows that this is satisfied for any $z \in (\mathbb{Z}/2)^2$, and as in Cor. 4.7 have that $Hom_Z(K_0(S), Z)$ is the inductive limit of the following system.

$$(\mathbb{Z}/2)^2 \xrightarrow{z\mapsto zA^2} (\mathbb{Z}/2)^2 \longrightarrow \cdots$$

Now, the isomorphism $\phi : Hom_Z(K_0(S), Z) \to Hom_R(K_0(S), R)$ is simply given by

$$\phi[z, n] = [2z, n].$$

Thus

$$Hom_Z(K_0(S), Z) \cong Hom_R(K_0(S), R) \cong K_0(U).$$

In the next example $Hom_Z(K_0(S), Z) \neq Hom_R(K_0(S), R)$. Consider the SFT with adjacency matrix

$$A = \begin{bmatrix} 0 & 1 & 5 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

$$\frac{1}{2}(A^2 + A) = \begin{bmatrix} 3 & 3 & 3 \\ 1 & 1 & 3 \\ 1 & 1 & 3 \end{bmatrix}$$

and $Z(A) = span\{I, A, \frac{1}{2}(A^2 + A)\}$. Moreover

$$A\frac{1}{2}(A^2 + A) = \frac{1}{2}(A^2 + A) + 3A + 3I$$

so

$$Z = \lim_{\to} Z(A) \neq \lim_{\to} span\{I, A, A^2\} = R.$$

We now show that $Hom_Z(K_0(S), Z) \neq Hom_R(K_0(S), R)$. Let $\varphi \in Hom_Z(K_0(S), Z)$, there exists $z \in (\mathbb{Z}/2)^3$, $N \in \mathbb{N}$ such that

$$\varphi[v, n] = [vA^n zA^2 + vA^{n+1} zA + vA^n(A^2 - 7I)zI, N + n],$$

with the additional conditions on z that $vA^n z + vA^{n+1}z$, and $vA^n(A^2 - 5I)z$ are in $\mathbb{Z}$. One readily checks that the set of all such z is

$$\mathcal{Z} = span_{\mathbb{Z}}\{\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}\} = span_{\mathbb{Z}}\{\begin{bmatrix} \frac{1}{2} \\ \frac{-1}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}\}$$

and $Hom_Z(K_0(S), Z)$ is the limit of

$$\mathcal{Z} \xrightarrow{z \mapsto zA} \mathcal{Z} \longrightarrow \cdots$$

which is not isomorphic to $K_0(U(\Sigma, \sigma, P))$ as a left Z module. To see this, note that any $[z, N] \in Hom_Z(K_0(S), Z)$ can be written as $[z, N] = [z, 0] * [A^N, N]$, so we restrict to looking at elements of the form $[z, 0]$. Suppose $\phi : Hom_Z(K_0(S), Z) \to K_0(U(\Sigma, \sigma, P))$ is a module homomorphism, we show that it cannot be an isomorphism. Let u_1, u_2, u_3 denote the three vectors in the second description of $\mathcal{Z}$ above.

$$\begin{aligned} Au_1 &= -u_1, \\ Au_2 &= 3u_2, \\ Au_3 &= 3u_2 - 2u_3. \end{aligned}$$

Now suppose $(\phi[u_i, 0]) = [v_i, n]$ (we can choose equivalence class representatives such that n does not depend on i), we know that

$$[Av_1, n] = [v_1, n] * [A, 0] = (\phi[u_1, 0]) * [A, 0] = \phi[Au_1, 0] = [-v_1, n].$$

Now, as $v_1 \in \mathbb{Z}^3$, and $Av_1 = -v_1$ we must have $v_1 = [1\ -1\ 0]^T$, or an integer multiple thereof. We similarly find that $[Av_2, n] = [3v_2, n]$, which implies $v_2 = [2\ 1\ 1]$. Finally $[Av_3, n] = [3v_2 - 2v_3, n]$ from which it follows that

$$v_3 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + k \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} \text{ for some } k \in \mathbb{Z}.$$

A quick check shows that there is no choice for k such that $[1\ 0\ 0]^T \in span_{\mathbb{Z}}\{v_1, v_2, v_3\}$, which is to say

$$\left[\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, n \right] \notin image(\phi)$$

and hence ϕ is not an isomorphism.

To summarize, we have shown that the subring R generated by the action of α need not equal the center $Z = Z(K_0(C(H, \alpha)))$, and moreover, when $R \neq Z$, $Hom_Z(K_0(S(\Sigma, \sigma, P)), Z)$ is isomorphic to $K_0(U(\Sigma, \sigma, P))$ in some, but not all cases.

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