A homology theory for Smale spaces

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Preface

In the 1960's, Steven Smale began an ambitious program to study the dynamics of smooth maps on manifolds [33]. This program has since become a substantial part of the theory of dynamical systems and can be found in many basic texts. In addition, many of the fundamental ideas and principles have had a large influence throughout dynamical systems. For example, see [7, 16, 20, 30].

Smale introduced the notion of an Axiom A system: the main condition is that the map, when restricted to its set of non-wandering points, has a hyperbolic structure. Smale showed that the non-wandering of an Axiom A system can be canonically decomposed into finitely many disjoints sets, each of which is irreducible in a certain sense. Such sets are called basic sets. The study of the system then breaks down into the study of the individual basic sets and the problem of how the rest of the manifold is assembled around them. An important subtlety that Smale realized from the start was that the basic sets were not typically submanifolds, but rather some sort of fractal object. Moreover, the hyperbolic nature of the dynamics on the basic set (along with the assumption that the periodic points are dense in the non-wandering set) created the conditions which are now usually referred to as chaos. This study forms the mathematical foundations for chaos, which has had a profound impact in many areas of science and beyond.

David Ruelle introduced the notion of a Smale space in an attempt to axiomatize the dynamics of an Axiom A system, when restricted to a basic set (or the non-wandering set) [31]. This involved giving a definition of hyperbolicity for a homeomorphism of a compact metric space. From our point of view here, there are two essential differences between the non-wandering set for an Axiom A system and a Smale space. The first is that in the former, every point is non-wandering, which need not be the case for a Smale space. Indeed our constructions will involve a number of Smale spaces with wandering points. Secondly, a Smale space can be described without seeing it as a subset of a manifold (to which the dynamics extends in an appropriate manner).

Very early on, a particular class of Smale spaces took a prominent role: the shifts of finite type. As the name suggests, these are dynamical systems of a highly combinatorial nature and the spaces involved are always totally disconnected. They had been studied previously, but in this context they appeared in two essential ways. First, Smale showed how such systems could appear as basic sets for Axiom A system; the full 2-shift is a crucial feature in Smale's horseshoe. Secondly, they could be used to code more complicated systems. This idea is originally credited to Hadamard for modelling geodesic flows [19] and was pursued in work of Morse [26], Morse and Hedlund [27] and many others. For a modern treatment, see [4]. For Axiom A systems, Bowen [6], building on work of Adler-Weiss, Sinai and others, proved that every basic set (or every Smale space) is the image of a shift of finite type under a finite-to-one, continuous, equivariant surjection. The study of shifts of finite type has become a major one in dynamical systems, in part due to important connections with coding theory and information theory [23].

The zeta function of a dynamical system was introduced by Artin and Mazur [2] as a convenient tool for encoding the data of the number of periodic points (of all periods) of the system. Bowen and Landford [9] gave a simple formula for the zeta function of a shift of finite type which is the inverse of a polynomial. Following this, using Bowen's result on the coding by a shift of finite type, Manning [24] proved that the zeta function of the restriction of an Axiom A system to any basic set was a rational function. Indeed, it was written as a product of the zeta function for the coding shift and other shifts which coded the multiplicities of the factor map. Based on this, Bowen conjectured the existence of some sort of homology theory for basic sets and a Lefschetz-type formula which would link the number of periodic points with the trace of the action of the dynamics on this homology. (See section 3 of [7].) At least formally, the situation is similar to that of the Weil conjectures [17].

The first step in this direction was achieved independently by Bowen and Franks [8] and Krieger [22] by describing invariants for shifts of finite type. The two are slightly different (they appear almost to be duals of each other). (A word of warning: the invariant usually referred to as the Bowen-Franks group is *not* the one to which we refer here.) We will concentrate on Krieger's version, often referred to as the dimension group invariant. There are several reasons, primarily because it has been the focus of a great deal of research in symbolic dynamics and has proved to be a highly effective invariant.

Having this invariant at hand for shifts of finite type makes Bowen's conjectured homology theory look even more likely, but the problem has remained open until now. Here, we give a solution by presenting a homology theory for Smale spaces which is built very much along the lines indicated, using a refinement of Bowen's coding by a shift of finite type and Krieger's invariant. In fact, our theory improves on Bowen's conjecture by providing finite rank groups, instead of finite-dimensional vector spaces.

In some sense, many of the usual tools of algebraic topology, such as Cech cohomology, seem poorly suited to the study of basic sets. For example, the Cech cohomology of the underlying space of an infinite shift of finite type is not finite rank. It can be hoped that our homology theory will provide a more useful and effective invariant for basic sets in an Axiom A system.

I am grateful to Mike Boyle and to Jerry Kaminker for many helpful discussions on a wide variety of issues related to this work. I am also grateful to Trung Luu for his help regarding ordered and alternating complexes. I am grateful to David Ruelle for pointing out Corollary 6.1.2 and its significance. I am also grateful to Robin Deeley, Brady Killough and Mike Whittaker for finding a mistake in the original version of Theorem 3.5.11, and for many interesting discussions around the issue and its resolution. Finally, I am grateful to the referee for many helpful suggestions with the manuscript.

Chapter 1

Summary

The aim of this paper is to define a type of homology theory for Smale spaces, which include the basic sets for Smale's Axiom A systems. The existence of such a theory was conjectured by Rufus Bowen [7]. Our approach is based on Krieger's dimension group invariant for shifts of finite type. We will use this chapter as an introduction to the concepts and a summary of the paper, stating the main new definitions and results.

We will be concerned with Smale spaces, as defined by David Ruelle [31]; we summarize here informally, the precise definition will be given in Definition 2.1.6. We have a compact metric space (X, d) and a homeomorphism, φ, of X. Such a topological dynamical system is called a Smale space if it possesses local coordinates of contracting and expanding directions. Roughly, to any point x in X and ϵ sufficiently small, there exist subsets $X^s(x, \epsilon)$ and $X^u(x, \epsilon)$ called the local stable and unstable sets at x. Their Cartesian product is homeomorphic to a neighbourhood of x. The parameter ϵ controls the diameter of these sets and, as it varies, these product neighbourhoods form a neighbourhood base at x. The contracting/expanding nature of φ is described by the condition that there is a constant $0 < \lambda < 1$ such that

$$\begin{aligned} d(\varphi(y), \varphi(z)) &\leq \lambda d(y, z), \quad \text{for all } y, z \in X^s(x, \epsilon), \\ d(\varphi^{-1}(y), \varphi^{-1}(z)) &\leq \lambda d(y, z), \quad \text{for all } y, z \in X^u(x, \epsilon). \end{aligned}$$

There is also a condition roughly indicating that the local product structure is invariant under φ. Ruelle's precise definition involves the existence of a map $[x, y]$ defined on pairs (x, y) which are sufficiently close and with range in X. The idea is that $[x, y]$ is the unique point in the intersection of the local stable set of x and the local unstable set of y. Ruelle provided appropriate axioms for this map. Given the map $[\cdot,\cdot]$, the set $X^s(x,\epsilon)$ consists of those y with $d(x,y) \leq \epsilon$ and $[x,y]=y$, or equivalently (via the axioms) $[y,x]=x$.

Ruelle's main objective in giving such a definition was to provide an axiomatic framework for studying the basic sets for Smale's Axiom A systems [33, 7]. While Smale's original definition is in terms of smooth maps of Riemannian manifolds, the basic sets typically have no such smooth structure. In addition, there are examples of Smale spaces, some of which will play an important part in this paper, which do not appear in any obvious way as a basic set sitting in a manifold. Moreover, basic sets, by their definition, are irreducible while Smale spaces need not be. Again, there will be many Smale spaces which play an important part here which fail to be non-wandering.

In a Smale space, two points x and y are stably equivalent if

$$\lim_{n\to+\infty} d(\varphi^n(x),\varphi^n(y)) = 0.$$

This is an equivalence relation on X and the equivalence class of a point x is denoted $X^s(x)$, also called its global stable set. There is an analogous definition of unstable equivalence obtained by replacing φ with φ^{-1}. As the notation would suggest, there is a close link between the local stable (unstable) sets and the global ones. This is made precise in Proposition 2.1.11. In addition, the local stable sets provide natural topologies on the stable sets as described in Proposition 2.1.12.

For our purposes, the most important examples of Smale spaces are the shifts of finite type. Begin with a finite directed graph G. The associated shift space, Σ_G, consists of all bi-infinite paths in G and the dynamics is provided by the left shift map, σ. An excellent reference for these systems is the book of Lind and Marcus [23]. We describe these precisely in Section 2.2, including the local product structure. We define shifts of finite type as any system topologically conjugate to (Σ_G,σ), for some graph G. There a number of reasons why shifts of finite type play a crucial rôle. First of all, there are important applications in coding theory. Secondly, many dynamical problems in the general setting become combinatorial ones for shifts of finite type. Thirdly, these are precisely the Smale spaces whose underlying space is totally disconnected (Theorem 2.2.8) and we will use this fact frequently. Finally, the two most important features for us here are Krieger's dimension group invariant (which we discuss below) and Bowen's seminal result that, given any irreducible Smale space, (X,φ), there exists a shift of finite type, (Σ,σ), and a finite-to-one factor map $\pi:(\Sigma,\sigma)\to(X,\varphi)$.

In [22], motivated by R.F. Williams' notion of shift equivalence [35], Wolfgang Krieger introduced a beautiful invariant for shifts of finite type. It is an ordered abelian group having certain properties which are summarized by saying it is a dimension group. In fact, there are two such invariants associated to each shift of finite type, which Krieger called the past and the future dimension groups. Here, we use the notation for stable and unstable instead. For a shift of finite type, (Σ, σ), we will denote these two invariants by $D^s(\Sigma, \sigma)$ and $D^u(\Sigma, \sigma)$. Briefly, our aim in this paper is to extend the definition of Krieger's invariants so that they are defined for all Smale spaces. We will not quite achieve this, but we will provide a definition for all Smale spaces which are irreducible or even non-wandering (see 2.1.3), including the basic sets of Axiom A systems.

The order structure on Krieger's invariant is a very important part of the theory. It will play no part in this paper and we treat the invariant simply as an abelian group, but including this will be a goal of future investigations.

We present a precise treatment of Krieger's invariant in chapter 3. For the moment, we take the opportunity to discuss it in more general terms, giving some history and motivation. Krieger's starting point was the realization that two C^*-algebras could be constructed from a shift of finite type. This construction is quite ingenious, but since that time, more sophisticated techniques have been developed and, in modern language, these are the C^*-algebras associated with stable and unstable equivalence. Krieger also realized their structure could be described quite explicitly; each is the closure of an increasing sequence of finite dimensional subalgebras. As such, it was an easy matter to compute their K-theory groups, and these are are the invariants. See [13].

Motivated by knowing the K-theory of the C^*-algebras involved, Krieger gave a description of the invariants more directly related to the dynamics [22]. As motivation, let us consider the computation of the Cech cohomology of a compact, totally disconnected space X [14, 34, 5]. In general, to compute cohomology, one begins with a (finite) open cover of the space. It has associated a simplicial object called its nerve, coding which elements of the open cover intersect non-trivially. One computes the (co-)homology of this simplicial complex. Finally, one repeats this, refining the open covers, obtaining an inductive system of cohomology groups and takes the limit. One of the first main results is that any sequence of open covers will provide the same result, provided they generate the topology of the space. Hence, if X is totally disconnected, the simplest thing to do is to consider finite partitions of X into clopen sets. In this case, the nerve becomes trivial and after taking the inductive limit one obtains the answer as the free abelian group generated by all clopen subsets, modulo the subgroup generated by $U \cup V - U - V$, if U and V are clopen and disjoint.

With this in mind for motivation, we describe Krieger's invariant for a shift of finite type (Σ, σ) as follows. Consider all sets U which are clopen subsets of a stable equivalence class, $\Sigma^s(e)$, where, e varies over all points of Σ. The topology on $\Sigma^s(e)$ is that described in Proposition 2.1.12. There are clearly too many such sets: if we look at a rectangular neighbourhood in Σ, each stable fibre in the rectangle is homeomorphic to all of the others. A clopen set in one has natural images in the others and we should consider these as equivalent. We introduce an equivalence relation of this set of subsets, take the free abelian group on the equivalence classes and finally take the quotient by a subgroup as in the case of cohomology above. This is described precisely in section 3.3. This is a minor variation on Krieger's original definition.

It is important to note that the equivalence relation above which we are putting on the clopen subsets of the stable sets is derived naturally from unstable equivalence. It would seem then, that we are attempting to compute the Cech cohomology (in dimension zero) of the quotient space of Σ by unstable equivalence. Fortunately, this is not the case, because this space is uncountable but has only finitely many open subsets. The viewpoint of Alain Connes' program of noncommutative geometry is that such a badly behaved quotient space should be replaced by a noncommutative C^*-algebra and this is exactly what Krieger's construction accomplishes in this case.

The next point to mention in the discussion of Krieger's invariant is that it is highly computable. If we are given a specific graph G and consider its associated shift of finite type, (Σ_G, σ), the invariants $D^s(\Sigma_G, \sigma)$ and $D^s(\Sigma_G, \sigma)$ can be computed from the graph. We review this in Section 3.2 and Theorem 3.3.3.

The functorial properties of Krieger's invariant are quite a subtle matter. Put simply, when does a map between two shifts of finite type induce a map at the level of the invariants? To be more precise, a map between two dynamical systems (Y, ψ) and (X, φ) is a continuous function $\pi : Y \to X$ such that $\pi \circ \psi = \varphi \circ \pi$. Assuming both systems are Smale spaces, we say that π is s-bijective if π maps the stable equivalence class of y bijectively to the stable equivalence class of $\pi(y)$, for each y in Y. There is obviously an analogous definition of a u-bijective map. This definition is a slight variant of the notion of a resolving map, as introduced by David Fried [18], which requires only injectivity when restricted to stable or unstable sets. A consequence is that an s-bijective map is a homeomorphism between the local stable sets (Theorem 2.5.12). This notion comes into play for Krieger's invariant in a crucial way. If $\pi : (\Sigma, \sigma) \to (\Sigma', \sigma)$ is an s-bijective map between two shifts of finite type, then it induces a natural group homomorphism, denoted π^s, from $D^s(\Sigma, \sigma)$ to $D^s(\Sigma', \sigma)$ and also one, denoted π^{u*}, from $D^u(\Sigma', \sigma)$ to $D^u(\Sigma, \sigma)$. Analogous results hold for u-bijective factor maps. Versions of these results are proved in [10] but the approach there is to show the existence of specific presentations of the shifts for which the construction of the induced maps becomes quite simple. The first problem with this is that the naturality is not clear; that is, if π and ρ are two such maps (with appropriate ranges and domains), is $(\pi \circ \rho)^s = \pi^s \circ \rho^s$? By avoiding specific presentations, our treatment makes this property clear. Of course, if the shifts of finite type are presented by specific graphs, then we are faced with the problem of giving explicit computations for the induced maps. This is done in Sections 3.4 and 3.5.

As mentioned above, one of the fundamental results on Smale spaces and the reason that shifts of finite type play such an important role in the theory is Bowen's Theorem. To repeat, for any irreducible Smale space (X, φ), there is a shift of finite type (Σ, σ) and a finite-to-one factor map $\pi : (\Sigma, \sigma) \to (X, \varphi)$.

Let us speculate a little on how Bowen's result might be used to obtain an extension of Krieger's invariant to a general Smale space (X, φ). Starting from $\pi : (\Sigma, \sigma) \to (X, \varphi)$ as above, we may form

$$\Sigma_N(\pi) = \{(e_0, \ldots, e_N) \mid e_n \in \Sigma, \pi(e_0) = \pi(e_1) = \cdots = \pi(e_N)\}.$$

for every $N \geq 0$. With the obvious dynamics, this is also a shift of finite type. So for each N, we may consider the invariant $D^s(\Sigma_N(\pi), \sigma)$. Further, for each $N \geq 1$ and $0 \leq n \leq N$, there is a factor map, denoted by δ_n, from $(\Sigma_N(\pi), \sigma)$ to $(\Sigma_{N-1}(\pi), \sigma)$ which simply deletes entry n. We would like to consider the maps induced by the δ_n and sum over n, with alternating signs, to obtain a boundary map from $D^s(\Sigma_N(\pi), \sigma)$ to $D^s(\Sigma_{N-1}(\pi), \sigma)$ and in this way obtain a chain complex.

This approach is analogous to the computation of Cech cohomology for a compact manifold. Here, (X, φ) replaces the manifold. An open cover may be regarded as a surjection from another space which is the disjoint union of Euclidean balls onto the manifold. Our (Σ, σ) plays the rôle of this disjoint union of Euclidean balls. The systems $\Sigma_N(\pi)$ are playing the roles of the various intersections of the open sets in the covers. If one assumes these are also topologically Euclidean balls (i.e. this is a *good* cover [5]), then the cohomology of the manifold may be computed from the resulting complex (and the knowledge of the cohomology of a Euclidean ball). In our situation, the fact that each $\Sigma_N(\pi)$ is a shift of finite type is automatic.

The flaw in this plan comes exactly when one wants to consider the map from $D^s(\Sigma_N(\pi), \sigma)$ to $D^s(\Sigma_{N-1}(\pi), \sigma)$ induced by δ_n. Without some extra structure, there is no reason for δ_n to be s-bijective (or u-bijective). Hence, the subtle nature of the functoriality of Krieger's invariant is the obstruction.

The difficulties above are resolved *if* we begin with a factor map $\rho : (\Sigma, \sigma) \to (X, \varphi)$, where (Σ, σ) is a shift of finite type, which is either s-bijective or u-bijective. In the case the map ρ is s-bijective, the maps δ_n will also be s-bijective and we may form a chain complex from the invariants $D^s(\Sigma_N(\rho), \sigma)$. In the case the map ρ is u-bijective, the maps δ_n will be u-bijective and we may form a cochain complex. It is worth noting however, that this situation is a very special one. For example, the hypothesis that the map ρ is s-bijective implies that the local stable sets in X are totally disconnected. Nevertheless, Chapter 4 is devoted to the study of the complexes of such a map. This Chapter is rather technical and we will not review any of the results here, but they are used in critical ways in the following chapter where the homology is defined.

In this context, let us mention the work of Williams [36] who defined and studied expanding attractors, which are basic sets in which the stable sets are totally disconnected while the unstable sets are Euclidean. He gave a construction of such systems as inverse limits of branched manifolds and showed that every expanding attractor may be written in this way. More recently, Wieler [37] generalized this to Smale spaces whose stable sets are totally disconnected, without any hypothesis on the unstable sets. She provides an inverse limit construction for such Smale spaces and proves that every irreducible Smale space with totally disconnected stable sets can written in this way.

Let us return now to the general case and the failed program we outlined above which began with Bowen's Theorem. The problem with the approach really lies with Bowen's Theorem, which is not strong enough. Instead of beginning with a Smale space, (X, φ), and finding a shift of finite type which maps onto it, we instead look for two other Smale spaces, (Y, ψ) and (Z, ζ) along with factor maps $\pi_s : (Y, \psi) \to (X, \varphi)$ and $\pi_u : (Z, \zeta) \to (X, \varphi)$. The key hypotheses are that π_s and π_u are s-bijective and u-bijective, respectively, while $Y^u(y)$ and $Z^s(z)$ are totally disconnected, for every y in Y and z in Z. What this means for (Y, ψ) is that its local unstable sets are totally disconnected, while its local stable sets are homeomorphic to those of X. Bowen's Theorem gives the existence of a Smale space which covers X and which is totally disconnected. Here, we ask for such constructions which disconnect in each local coordinate separately.

Definition 2.6.2. *Let (X, φ) be a Smale space. We say that $\pi = (Y, \psi, \pi_s, Z, \zeta, \pi_u)$ is an s/u-*bijective pair *for (X, φ) if*

1. *(Y, ψ) and (Z, ψ) are Smale spaces,*
2. *$\pi_s : (Y, \psi) \to (X, \varphi)$ is an s-bijective factor map,*
3. *$Y^u(y)$ is totally disconnected, for every y in Y,*
4. *$\pi_u : (Z, \zeta) \to (X, \varphi)$ is a u-bijective factor map,*
5. *$Z^s(z)$ is totally disconnected, for every z in Z.*

The crucial result for us here is the existence of s/u-bijective pairs. This is proved (essentially) in [29] and [15]. It is worth mentioning here that the systems (Y, ψ) and (Z, ζ) are Smale spaces: they are not given as basic sets for Axiom A systems. This is the one point in our work where Smale spaces take a preferred place over basic sets.

Theorem 2.6.3. *If (X, φ) is a non-wandering Smale space, then there exists an s/u-bijective pair for (X, φ).*

Given an s/u-bijective pair $\pi = (Y, \psi, \pi_s, Z, \zeta, \pi_u)$ for a Smale space (X, φ), we may first form the fibred product of the two maps π_s and π_u as

$$\Sigma(\pi) = \{(y, z) \in Y \times Z \mid \pi_s(y) = \pi_u(z)\}.$$

It is a compact metric space, the map $\psi \times \zeta$, which we denote by σ, defines a homeomorphism of it. The two projection maps to Y and Z, denoted ρ_u and ρ_s, are u-bijective and s-bijective factor maps, respectively 2.5.13. One immediate consequence (already suggested by our notation) is that $(\Sigma(\pi), \sigma)$ is a shift of finite type. We can see now why our result on the existence of s/u-bijective pairs is a strengthening of Bowen's result because $(\Sigma(\pi), \sigma)$ together with factor map $\pi_s \circ \rho_u = \pi_u \circ \rho_s$ satisfies the conclusion of Bowen's Theorem, but with the added condition that the factor map has decompositions as given. In fact, another way of stating our result is to ask for a shift of finite type, (Σ, σ), and factor map onto (X, φ) which may be factored as $\pi_s \circ \rho_u$ and as $\pi_u \circ \rho_s$, with π_s, ρ_s being s-bijective while π_u, ρ_u are u-bijective . It is not true that any factor map has such decompositions [21].

We return to our earlier idea of using the $\Sigma_N(\pi)$ to build a chain complex from their dimension group invariants. Now, we consider a sort of two-variable version of that construction.

Definition 2.6.4. *Let $\pi = (Y, \psi, \pi_s, Z, \zeta, \pi_u)$ be an s/u-bijective pair for the Smale space (X, φ). For each $L, M \geq 0$, we define*

$$\begin{aligned}\Sigma_{L,M}(\pi) &= \{(y_0, \ldots, y_L, z_0, \ldots, z_M) \mid y_l \in Y, z_m \in Z, \\ &\quad \pi_s(y_l) = \pi_u(z_m), 0 \leq l \leq L < 0 \leq m \leq M\}.\end{aligned}$$

For convenience, we also let $\Sigma(\pi) = \Sigma_{0,0}(\pi)$, which is simply the fibred product of the spaces Y and Z. We let $\rho_u(y, z) = y$ and $\rho_s(y, z) = z$ denote the usual maps from $\Sigma(\pi)$ to Y and Z respectively.

For all, $L, M \geq 0$, we also define

$$\sigma(y_0, \ldots, y_L, z_0, \ldots, z_M) = (\psi(y_0), \ldots, \psi(y_L), \zeta(z_0), \ldots, \zeta(z_M)),$$

for all $(y_0, \ldots, y_L, z_0, \ldots, z_M)$ in $\Sigma_{L,M}(\pi)$.

Finally, for $L \geq 1$ and $0 \leq l \leq L$, we let $\delta_{l,} : \Sigma_{L,M}(\pi) \to \Sigma_{L-1,M}(\pi)$ be the map which deletes entry y_l. Similarly, for $M \geq 1$ and $0 \leq m \leq M$, we let $\delta_{,m} : \Sigma_{L,M}(\pi) \to \Sigma_{L,M-1}(\pi)$ be the map which deletes entry z_m.

As the notation would suggest, these systems are all shifts of finite type. Moreover, we have the following key fact.

Theorem 2.6.13. *Let π be an s/u-bijective pair for (X, φ).*

1. *For all $L \geq 1$, $M \geq 0$ and $0 \leq l \leq L$, the map $\delta_{l,} : \Sigma_{L,M}(\pi) \to \Sigma_{L-1,M}(\pi)$ which deletes y_l from $(y_0, \ldots, y_L, z_0, \ldots, z_M)$ is an s-bijective factor map.*

2. *For all $L \geq 0$, $M \geq 1$ and $0 \leq m \leq M$, the map $\delta_{,m} : \Sigma_{L,M}(\pi) \to \Sigma_{L,M-1}(\pi)$ which deletes z_m from $(y_0, \ldots, y_L, z_0, \ldots, z_M)$ is a u-bijective factor map.*

In Chapter 5, we assemble these groups and maps into a double complex.

Definition 5.1.1. *Let π be an s/u-bijective pair for the Smale space (X, φ).*

1. *For each $L, M \geq 0$, we define*

$$C^s(\pi)_{L,M} = D^s(\Sigma_{L,M}(\pi), \sigma).$$

We define these groups to be zero for all other integral values of L, M. We define also

$$d^s(\pi)_{L,M} = \sum_{0 \leq l \leq L} (-1)^l \delta^s_{l,} + \sum_{0 \leq m \leq M+1} (-1)^{L+m} \delta^{s*}_{,m}$$

on the summand $C^s(\pi)_{L,M}$. Notice that the first sum has range $C^s(\pi)_{L-1,M}$, while the second has range $C^s(\pi)_{L,M+1}$. The first term only appears when $L \geq 1$ and the second only for $M \geq 0$.

2. *For each $L, M \geq 0$, we define*

$$C^u(\pi)_{L,M} = D^u(\Sigma_{L,M}(\pi), \sigma).$$

We define these groups to be zero for all other integral values on L, M. We define

$$d^u(\pi)_{L,M} = \sum_{0 \leq l \leq L+1} (-1)^l \delta^{u*}_{l,} + \sum_{0 \leq m \leq M} (-1)^{L+m} \delta^u_{,m}$$

on the summand $C^u(\pi)_{L,M}$. Notice that the first sum has range $C^u(\pi)_{L+1,M}$, while the second has range $C^u(\pi)_{L,M-1}$. The first term only appears when $L \geq 0$ and the second only for $M \geq 1$.

Notice that, if we let S_N denote the permutation group on N symbols, then there is an obvious action of $S_{L+1} \times S_{M+1}$ on $\Sigma_{L,M}(\pi)$ which commutes with the dynamics σ, for every $L, M \geq 0$.

Returning to the comparison with Cech cohomology, one difference with our approach is that our systems $\Sigma_{L,M}(\pi)$ clearly contain redundancies; for example, if (y, z) is an element of $\Sigma_{0,0}(\pi)$, then (y, y, z, z) is in $\Sigma_{1,1}(\pi)$. These redundancies are detected by the action of the permutation groups in a way which will be made precise below.

The situation is analogous to that found in simplicial homology between the ordered and alternating complexes [14, 34]. The complexes we have at this point are analogous to the ordered complex. In finding an analogue of the alternating, there is a real problem here in that we have no obvious notion of a simplex or of orientation. However, by considering the action of the permutation groups on Krieger's invariant, we are able to construct a version of the alternating complex. In fact, there are three double complexes which take into account the actions of the groups S_{L+1}, S_{M+1} and $S_{L+1} \times S_{M+1}$, respectively.

Definition 5.1.5. *Let π be an s/u-bijective pair for the Smale space (X, φ). Let $L, M \geq 0$.*

1. *We define $D^s_{\mathcal{B},}(\Sigma_{L,M}(\pi))$ to be the subgroup of $D^s(\Sigma_{L,M}(\pi))$ which is generated by*
   - *(a) all elements b such that $b = b \cdot (\alpha, 1)$, for some transposition α in S_{L+1}, and*
   - *(b) all elements of the form $a - sgn(\alpha) a \cdot (\alpha, 1)$, where a is in $D^s(\Sigma_{L,M}(\pi))$ and α is in S_{L+1}.*
2. *We define $D^s_{\mathcal{Q},}(\Sigma_{L,M}(\pi))$ to be the quotient of $D^s(\Sigma_{L,M}(\pi))$ by the subgroup $D^s_{\mathcal{B},}(\Sigma_{L,M}(\pi))$ and we let Q denote the quotient map.*
3. *We define $D^s_{,\mathcal{A}}(\Sigma_{L,M}(\pi))$ to be the subgroup of all elements a in $D^s(\Sigma_{L,M}(\pi))$ satisfying $a = sgn(\beta) a \cdot (1, \beta)$, for all β in S_{M+1} and we let J denote the inclusion map.*
4. *We define $D^s_{\mathcal{Q},\mathcal{A}}(\Sigma_{L,M}(\pi))$ to be the image in $D^s_{\mathcal{Q},}(\Sigma_{L,M}(\pi))$ of $D^s_{,\mathcal{A}}(\Sigma_{L,M}(\pi))$ under Q. We let $Q_{\mathcal{A}}$ denote the restriction of Q to $D^s_{,\mathcal{A}}(\Sigma_{L,M}(\pi))$ and $J_{\mathcal{Q}}$ denote the inclusion of $D^s_{\mathcal{Q},\mathcal{A}}(\Sigma_{L,M}(\pi))$ in $D^s_{\mathcal{Q},}(\Sigma_{L,M}(\pi))$.*

There are analogous definitions of $D^u_{\mathcal{A},}(\Sigma_{L,M}(\pi)), D^u_{,\mathcal{Q}}(\Sigma_{L,M}(\pi))$ and $D^u_{\mathcal{A},\mathcal{Q}}(\Sigma_{L,M}(\pi))$.

We have eight complexes, four based on Krieger's D^s and four based on D^u. We will now concentrate our attention on the first four. Analogous results hold for the others.

Definition 5.1.7. *Let π be an s/u-bijective pair for the Smale space (X, φ). For $L, M \geq 0$, we define*

$$\begin{aligned} C^s_{\mathcal{Q},}(\pi)_{L,M} &= D^s_{\mathcal{Q},}(\Sigma_{L,M}(\pi)), \\ C^s_{,\mathcal{A}}(\pi)_{L,M} &= D^s_{,\mathcal{A}}(\Sigma_{L,M}(\pi)), \\ C^s_{\mathcal{Q},\mathcal{A}}(\pi)_{L,M} &= D^s_{\mathcal{Q},\mathcal{A}}(\Sigma_{L,M}(\pi)). \end{aligned}$$

We also let

1. *$d^s_{\mathcal{Q},}(\pi)_{L,M}$ be the map induced by $d^s(\pi)_{L,M}$ on the quotient $D^s_{\mathcal{Q},}(\Sigma_{L,M}(\pi))$,*
2. *$d^s_{,\mathcal{A}}(\pi)_{L,M}$ to be the restriction of $d^s(\pi)_{L,M}$ to $D^s_{,\mathcal{A}}(\Sigma_{L,M}(\pi))$, and*
3. *$d^s_{\mathcal{Q},\mathcal{A}}(\pi)_{L,M}$ to be the restriction of $d^s_{\mathcal{Q},}(\pi)_{L,M}$ to $D^s_{\mathcal{Q},\mathcal{A}}(\Sigma_{L,M}(\pi))$.*

We summarize the situation as follows.

Theorem 5.1.8. *Let π be an s/u-bijective pair for the Smale space (X, φ). We have a commutative diagram of chain complexes and chain maps as shown:*

$$\begin{array}{ccc} (C^s_{,\mathcal{A}}, d^s_{,\mathcal{A}}) & \xrightarrow{J} & (C^s, d^s) \\ \downarrow{\scriptstyle Q_{\mathcal{A}}} & & \downarrow{\scriptstyle Q} \\ (C^s_{\mathcal{Q},\mathcal{A}}, d^s_{\mathcal{Q},\mathcal{A}}) & \xrightarrow{J_{\mathcal{Q}}} & (C^s_{\mathcal{Q},}, d^s_{\mathcal{Q},}) \end{array}$$

The first important general result about our four complexes is the following.

Theorem 5.1.10. *Let π be an s/u-bijective pair for the Smale space (X, φ). Let L_0 and M_0 be such that $\#\pi_s^{-1}\{x\} \leq L_0$ and $\#\pi_u^{-1}\{x\} \leq M_0$, for all x in X (which exist by Theorem 2.5.3).*

1. *If $L \geq L_0$, then $C^s_{\mathcal{Q},}(\pi)_{L,M} = 0$.*
2. *If $M \geq M_0$, then $C^s_{,\mathcal{A}}(\pi)_{L,M} = 0$.*
3. *If either $L \geq L_0$ or $M \geq M_0$, then $C^s_{\mathcal{Q},\mathcal{A}}(\pi)_{L,M} = 0$.*

Thus, the complex $C^s_{\mathcal{Q},\mathcal{A}}(\pi)$ is the simplest since it has only finitely many non-zero entries. This is our analogue of the alternating complex in Cech cohomology. Here is our main definition.

Definition 5.1.11. *Let π be an s/u-bijective pair for the Smale space (X, φ).*

1. *We define $H^s_*(\pi)$ to be the homology of the double complex $(C^s_{\mathcal{Q},\mathcal{A}}(\pi), d^s_{\mathcal{Q},\mathcal{A}}(\pi))$. That is, for each integer N, we have*

$$H^s_N(\pi) = Ker(\oplus_{L-M=N} d^s_{\mathcal{Q},\mathcal{A}}(\pi)_{L,M}) / Im(\oplus_{L-M=N+1} d^s_{\mathcal{Q},\mathcal{A}}(\pi)_{L,M}).$$

2. *We define $H^u_*(\pi)$ to be the homology of the double complex $(C^u_{\mathcal{A},\mathcal{Q}}(\pi), d^u_{\mathcal{A},\mathcal{Q}}(\pi))$. That is, for each integer N, we have*

$$H^u_N(\pi) = Ker(\oplus_{L-M=N} d^u_{\mathcal{A},\mathcal{Q}}(\pi)_{L,M}) / Im(\oplus_{L-M=N-1} d^u_{\mathcal{A},\mathcal{Q}}(\pi)_{L,M}).$$

The first nice property of this homology (and one requested as part of Bowen's original conjecture) are the two finiteness properties in the following theorem. Recall that an abelian group H is *finite rank* if $H \otimes_{\mathbb{Z}} \mathbb{Q}$ is a finite-dimensional rational vector space.

Theorem 5.1.12. *Let π be an s/u-bijective pair for the Smale space (X, φ) and let L_0 and M_0 be as in Theorem 5.1.10.*

1. *The homology groups $H^s_N(\pi)$ are finite rank, for each integer N.*

2. *The homology groups $H^s_N(\pi)$ are zero for $N \leq -M_0$ and for $N \geq L_0$.*

The principal aim of Section 5.3 is the following two results which show that three of our four complexes all yield exactly the same homology.

Theorem 5.3.1. *Let π be an s/u-bijective pair for the Smale space (X, φ). The chain map*

$$Q_{\mathcal{A}} : (C^s_{,\mathcal{A}}(\pi), d^s_{,\mathcal{A}}(\pi)) \to (C^s_{\mathcal{Q},\mathcal{A}}(\pi), d^s_{\mathcal{Q},\mathcal{A}}(\pi))$$

induces an isomorphism on homology.

Theorem 5.3.2. *Let π be an s/u-bijective pair for the Smale space (X, φ). The chain map*

$$J_{\mathcal{Q}} : (C^s_{\mathcal{Q},\mathcal{A}}(\pi), d^s_{\mathcal{Q},\mathcal{A}}(\pi)) \to (C^s_{\mathcal{Q},}(\pi), d^s_{\mathcal{Q},}(\pi))$$

induces an isomorphism on homology.

In section 5.4, we discuss the functorial properties of of our homology $H^s(\pi)$. Not surprisingly, they are analogous to those of Krieger's invariant.

In section 5.5, we establish perhaps the most important fact that the homology $H^s(\pi)$ does not depend on the choice of the s/u-bijective pair π, but only on the underlying Smale space (X, φ), as follows.

Theorem 5.5.1. *Let (X, φ) be a Smale space and suppose that $\pi = (Y, \psi, \pi_s, Z, \zeta, \pi_u)$ and $\pi' = (Y', \psi', \pi'_s, Z', \zeta', \pi'_u)$ are two s/u-bijective pairs for (X, φ). Then there are canonical isomorphisms between $H^s_N(\pi)$ and $H^s_N(\pi')$ and between $H^u_N(\pi)$ and $H^u_N(\pi')$, for all integers N.*

With this, we adopt the notation $H^s(X, \varphi)$. We stress that the definition *does* depend on the existence of an s/u-bijective pair, which we know to be the case for all non-wandering Smale spaces and the basic sets for Smale's Axiom A systems.

Definition 5.5.2. *Let (X, φ) be a Smale space which has an s/u-bijective pair π. We define $H^s_N(X, \varphi) = H^s_N(\pi)$ and $H^u_N(X, \varphi) = H^u_N(\pi)$, for all integers N.*

With this notation we also establish the desired functorial properties.

Theorem 5.5.3.

1. *The functor which associates the sequence of abelian groups $H^s_*(X, \varphi)$ to a (non-wandering) Smale space (X, φ) is covariant for s-bijective factor maps and contravariant for u-bijective factor maps.*

2. *The functor which associates the sequence of abelian groups $H^u_*(X, \varphi)$ to a (non-wandering) Smale space (X, φ) is contravariant for s-bijective factor maps and covariant for u-bijective factor maps.*

If $\rho : (Y, \psi) \to (X, \varphi)$ is an s-bijective factor map between non-wandering Smale spaces, we use ρ^s_N to denote the natural map from $H^s_N(Y, \psi)$ to $H^s_N(X, \varphi)$ and ρ^{u*}_N to denote the natural map from $H^u_N(X, \varphi)$ to $H^u_N(Y, \psi)$. There is analogous notation for u-bijective factor maps.

Chapter 6 is devoted to the Lefschetz formula. The key ingredient in the proof is Manning's argument for the rationality of the zeta function [24], which was the original impetus for Bowen's question.

We begin by observing that, for any Smale space (X, φ), φ and φ^{-1} are each s-bijective and u-bijective factor maps from (X, φ) to itself.

Theorem 6.1.1. *Let (X, φ) be a Smale space which has an s/u-bijective pair. Then, for every $n \geq 1$, we have*

$$\begin{aligned}
\#\{x \in X \mid \varphi^n(x) = x\} &= \sum_{N \in \mathbb{Z}} (-1)^N Tr(((\varphi^{-1})^s_N \otimes 1_{\mathbb{Q}})^n) \\
&= \sum_{N \in \mathbb{Z}} (-1)^N Tr((\varphi^u_N \otimes 1_{\mathbb{Q}})^n) \\
&= \sum_{N \in \mathbb{Z}} (-1)^N Tr((\varphi^{s*}_N \otimes 1_{\mathbb{Q}})^n) \\
&= \sum_{N \in \mathbb{Z}} (-1)^N Tr(((\varphi^{-1})^{u*}_N \otimes 1_{\mathbb{Q}})^n).
\end{aligned}$$

In addition, we derive a Corollary which shows that the zeta function for an irreducible Smale space is not just rational, but actually has a *canonical* decomposition as a product of polynomials and their inverses which is provided in a natural way by the homology.

Corollary 6.1.2. *Let (X, φ) be an irreducible Smale space. For each integer N and real number t, define $p_N(t)$ to be the determinant of the linear transformation*

$$I - t(\varphi^{-1})^s_N \otimes 1_{\mathbb{R}} : H^s_N(X, \varphi) \otimes \mathbb{R} \to H^s_N(X, \varphi) \otimes \mathbb{R}.$$

Each of the vector spaces is finite dimensional and all but finitely many are trivial so that $p_N(t)$ is well-defined and all but finitely many are identically one. Then we have

$$\begin{aligned}
\zeta_\varphi(t) &= \Pi_{N \in \mathbb{Z}} p_N(t)^{(-1)^{N+1}} \\
&= \frac{\Pi_{N \ odd} \, p_N(t)}{\Pi_{N \ even} \, p_N(t)}.
\end{aligned}$$

Chapter 7 provides computations of some simple examples: shifts of finite type (where we recover Krieger's original invariant), solenoids and a hyperbolic toral automorphism. In particular, we prove the following.

Theorem 7.1.1. *Let (Σ, σ) be a shift of finite type.*

1. *We have*

$$H^s_N(\Sigma, \sigma) = \begin{cases} D^s(\Sigma, \sigma) & N = 0, \\ 0 & N \neq 0. \end{cases}$$

2. We have

$$H_N^u(\Sigma, \sigma) = \begin{cases} D^u(\Sigma, \sigma) & N = 0, \\ 0 & N \neq 0. \end{cases}$$

In Chapter 8, we formulate several questions for further investigation, and in some cases, provide a few thoughts on their possible solution.

Chapter 2

Dynamics

This chapter contains an introduction to most of the basic dynamical ideas which will be used. Two very useful standard references are Katok and Hasselblatt [20] and Robinson [30]. In addition, there is an interesting book by Aoki and Hiraide [1] where the setting is completely topological and hence rather closer to ours. Unfortunately, this seems now to be out of print.

We begin with some basics including the definition of a Smale space, due to David Ruelle. These are systems which possess canonical local coordinates (called the local stable sets and the local unstable sets) of contracting and expanding directions. The second section concentrates on shifts of finite type. These are described as arising from graphs, but they are precisely the Smale spaces which are totally disconnected.

The third section discusses some basic results on factor maps between Smale spaces. In the fourth section, we discuss the notion of a fibred product of two factor maps and a subsequent special case where the fibred product is taken from a single map with itself, and then this is iterated. We refer to the sequence of such systems as self-products. Particular attention is paid to the cases of Smale spaces and shifts of finite type.

In the fifth section, we discuss factor maps between Smale space possessing various special properties. These are called s-resolving, u-resolving, s-bijective and u-bijective maps. In further developments, these play a crucial role.

In the sixth section, we discuss the notion of an s/u-bijective pair for a Smale space. Essentially, this is a refinement of Bowen's seminal result that a Smale space may be covered by a shift of finite type. The weakness in Bowen's Theorem is that the factor map constructed does not possess any of the resolving properties of the previous section. For irreducible Smale spaces, we have a version of this result which does supply factor maps which may be decomposed as compositions of maps with the desired special properties.

Finally, in the seventh section, we assemble a number of technical results which play an important role in the later development.

2.1 Smale spaces

In the entire paper, our dynamical systems will always consist of a compact space, together with a homeomorphism. Although this is a little more restrictive than is usual, we make the following definition.

Definition 2.1.1. *A* dynamical system *is a pair* (X, φ) *where* X *is a topological space and* φ *is a homeomorphism of* X*.*

For the most part, we will restrict our attention to the case that X is actually a compact metric space. If (X, d) is a metric space, we let $X(x, \epsilon)$ denote the ball at x of radius ϵ for any x in X and $\epsilon > 0$.

We recall the definition of an expansive dynamical system.

Definition 2.1.2. *If* X *is a metric space, we say that a dynamical system* (X, φ) *is* expansive *if there is a constant* $\epsilon > 0$ *such that, for any* $x \neq y$ *in* X*, there exists an integer* n *such that* $d(\varphi^n(x), \varphi^n(y)) \geq \epsilon$*.*

A key property will be the notion of a non-wandering point.

Definition 2.1.3. *For a dynamical system* (X, φ)*, a point* x *in* X *is called* non-wandering *if for every open set* U *containing* x*, there is a positive integer* N *such that* $\varphi^N(U) \cap U$ *is non-empty. We say that* (X, φ) *is non-wandering if every point of* X *is non-wandering.*

Another important idea will be the notion of irreducibility. There are several equivalent formulations, but we will use the following.

Definition 2.1.4. *A dynamical system* (X, φ) *is said to be* irreducible *if, for every (ordered) pair of non-empty open sets* U, V*, there is a positive integer* N *such that* $\varphi^N(U) \cap V$ *is non-empty.*

It is plain that every irreducible system is also non-wandering.

Finally, we recall the notion of topological mixing.

Definition 2.1.5. *A dynamical system* (X, φ) *is said to be* mixing *if, for every (ordered) pair of non-empty open sets* U, V*, there is a positive integer* N *such that* $\varphi^n(U) \cap V$ *is non-empty for all* $n \geq N$*.*

It is clear that any mixing system is also irreducible.

We give our most important definition, a Smale space. Two words of warning are in order before we embark on the definition. First, it is long. Second, it is opaque. After we complete the correct technical definition and note one important result, we will take a moment to give a more intuitive description.

We begin with a compact metric space, (X, d), a homeomorphism, φ, of X, constants $\epsilon_X > 0$ and $0 < \lambda < 1$ and a continuous map from

$$\Delta_{\epsilon_X} = \{(x, y) \in X \times X \mid d(x, y) \leq \epsilon_X\}$$

to X. The image of a pair (x, y) under this map is denoted $[x, y]$. These satisfy a number of axioms (which will take a little time to describe).

First, we assume that

$$\begin{array}{ll} B1 & [x, x] = x, \\ B2 & [x, [y, z]] = [x, z], \\ B3 & [[x, y], z] = [x, z], \\ B4 & [\varphi(x), \varphi(y)] = \varphi[x, y], \end{array}$$

for all x, y, z in X whenever both sides of an equation are defined.

We pause to note that if $[x, y] = x$, then we also have $[y, x] = [y, [x, y]] = [y, y] = y$, using the second and first conditions above. Similarly, if $[x, y] = y$ then $[y, x] = x$.

Second, we assume the following two conditions. We have

$$C1 \qquad d(\varphi(x), \varphi(y)) \leq \lambda d(x, y), \text{ whenever } [x, y] = y,$$

and

$$C2 \qquad d(\varphi^{-1}(x), \varphi^{-1}(y)) \leq \lambda d(x, y), \text{ whenever } [x, y] = x.$$

Definition 2.1.6. *A* Smale space *is a dynamical system* (X, φ) *together with a metric* d *on* X*, constants* ϵ_X, λ *and map* $[,]$ *as above which satisfy the Axioms B1, B2, B3, B4, C1 and C2.*

For a given x in X, the sets where the estimates C1 and C2 hold are called the local stable and unstable sets, respectively. In fact, it is useful to include a parameter to control their size, as follows.

Definition 2.1.7. *Let (X, φ) be a Smale space. For any x in X and $0 < \epsilon \leq \epsilon_X$, we define*

$$\begin{aligned} X^s(x, \epsilon) &= \{y \in X \mid d(x, y) < \epsilon, [x, y] = y\}, \\ X^u(x, \epsilon) &= \{y \in X \mid d(x, y) < \epsilon, [x, y] = x\} \end{aligned}$$

These sets are called local stable *and* local unstable *sets.*

Let us just observe that that if x and y are close so that $d(x, y)$ is defined and within distance $\epsilon \leq \epsilon_X$ of x, then $[x, [x, y]] = [x, y]$ by axiom B2 and so $[x, y]$ is in $X^s(x, \epsilon)$. Similarly, if it is within distance $\epsilon \leq \epsilon_X$ of y, then it is in $X^u(y, \epsilon)$.

The main feature of a Smale space is that it is locally the product of its local stable and local unstable sets, which we summarize as follows.

Proposition 2.1.8. *Let x be in X and $0 < \epsilon \leq \epsilon_X/2$ satisfy*

$$[X(x, \epsilon), X(x, \epsilon)] \subset X(x, \epsilon_X/2).$$

The map $[,]$ defines a homeomorphism between $X^u(x, \epsilon) \times X^s(x, \epsilon)$ and a neighbourhood of x. The inverse map sends a point y to $([y, x], [x, y])$. As ϵ varies, these sets form a neighbourhood base at x.

Now is a convenient moment to give a more intuitive description of a Smale space. Usually, if one is given an example, one can see first the "local product structure" with the contracting dynamics on the local stable set and the expanding (meaning "inverse contracting") dynamics on the local unstable sets as in Proposition 2.1.8 above and axioms C1 and C2. As an example, consider Arnold's cat map: the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ gives an automorphism of the 2-torus which we consider as $\mathbb{R}^2/\mathbb{Z}^2$. This matrix has two independent eigenvectors v_s and v_u whose respective eigenvalues λ_s and λ_u satisfy $0 < \lambda_s < 1$ and $\lambda_u > 1$. Thus the local stable and unstable sets are given by short line segments through a point in the respective eigendirections.

Having the local stable and unstable sets, the bracket of two points x and y, $[x, y]$, is then defined to be the unique point in the intersection of the local stable set of x with the local unstable set of y. To summarize, it is intuitively easier to see the bracket as coming from the local stable and unstable sets, but for theory, it is easier to view the local stable and unstable sets as arising from the bracket.

As we have described above, Arnold's cat map is an example of a Smale space. It can easily be generalized to hyperbolic toral automorphisms: let A be any $N \times N$ integer matrix with integer entries, determinant either 1 or -1 and no eigenvalues on the unit circle in the complex plane. Then this matrix gives a homeomorphism of the N-torus, $\mathbb{R}^N/\mathbb{Z}^N$. Other examples of Smale spaces are various solenoids, substitution tiling spaces and, most importantly, the basic sets for Smale's Axiom A systems. Another key class of examples for our development are the shifts of finite type, which we describe in detail in the next section.

We next note the following result - see 6.4.10 of [20] or Corollary 3.3 of [30].

Proposition 2.1.9. *A Smale space is expansive.*

The Smale space then has two basic positive constants: one which controls the domain of the bracket and the other which is the expansiveness constant. As we may freely replace either a smaller constant, by replacing them by their minimum, we may assume that ϵ_X is also the expansiveness constant for X.

In addition to the notion of local stable and unstable sets, there is also the notion of global stable and unstable sets.

Definition 2.1.10. *Let (X, φ) be a Smale space. We say that two points x and y in X are* stably equivalent *if*

$$\lim_{n \to +\infty} d(\varphi^n(x), \varphi^n(y)) = 0.$$

It is clear that this is an equivalence relation and we let $X^s(x)$ denote the equivalence class of x. Similarly, we say that two points x and y in X are unstably equivalent *if*

$$\lim_{n \to -\infty} d(\varphi^n(x), \varphi^n(y)) = 0.$$

It is clear that this is also an equivalence relation and we let $X^u(x)$ denote the equivalence class of x.

Obviously, the notation suggests that there is a close relation between the local stable sets of a point x and $X^s(x)$, which we call its global stable set. It follows fairly easily from the axioms that $X^s(x, \epsilon) \subset X^s(x)$, for any positive ϵ. Much more precisely, we have the following (6.4.9 of [20]).

Proposition 2.1.11. *Let (X, φ) be a Smale space. For any x in X and $0 < \epsilon \leq \epsilon_X$, we have*

$$X^s(x) = \cup_{n=0}^{+\infty} \varphi^{-n}(X^s(\varphi^n(x), \epsilon)),$$

and

$$X^u(x) = \cup_{n=0}^{+\infty} \varphi^{n}(X^u(\varphi^{-n}(x), \epsilon)).$$

This result is very useful, but slightly deceptive. While the local stable and unstable sets are usually quite easy to describe, the global ones are very complicated. This can probably be most easily seen in the geometric example of Arnold's cat map; the local stable and unstable sets are given by the eigendirections of the matrix while the global ones are leaves of a Kronecker foliation and are dense.

As suggested by the last comment above, the global stable and unstable sets, with their relative topologies from X, are rather badly behaved (i.e. not locally compact). They do possess very natural finer topologies which are much more suitable.

Proposition 2.1.12. *Let (X, φ) be a Smale space and let x_0 be in X.*

1. *The collection of sets $\{X^s(x, \epsilon) \mid x \in X^s(x_0), 0 < \epsilon \leq \epsilon_X\}$ forms a neighbourhood base for a topology on $X^s(x_0)$ which is second countable and locally compact.*

2. *The function*

$$d^s(x, y) = \begin{cases} d(x, y) & \text{if } y \in X^s(x, \epsilon_X) \\ \epsilon_X & \text{otherwise} \end{cases}$$

   defines a metric on $X^s(x_0)$ for this topology.

3. *A sequence $\{x_n \mid n \geq 1\}$ converges to x in $X^s(x_0)$ if and only if it converges to x in X and, for all n sufficiently large, $[x_n, x] = x$.*

Proof. If x, x' are in $X^s(x_0)$, ϵ, ϵ' are positive and x'' is in $X^s(x, \epsilon) \cap X^s(x', \epsilon')$, then choose

$$0 < \delta < \min\{\epsilon - d(x, x''), \epsilon' - d(x', x'')\} \leq \epsilon_X.$$

If $d(y, x'') < \delta$, then $d(y, x') < \epsilon$ and $d(y, x'') < \epsilon'$ as usual. If in addition, $[x'', y] = y$, then it follows that $[x, y] = [[x'', x], y] = [x'', y] = y$. In a similar way, we have $[x', y] = y$ and so

$$x'' \in X^s(x'', \delta) \subset X^s(x, \epsilon) \cap X^s(x', \epsilon').$$

For the second part, we leave the proof that this function is a metric to the reader. If $\epsilon \geq \epsilon_X$, then any ball in d^s of radius ϵ is just $X^s(x_0)$. If $\epsilon < \epsilon_X$, then it is easy to see that the d^s ball of radius ϵ is just $X^s(x, \epsilon)$. The conclusion of the second part follows.

For the third part, first suppose the sequence $x_n, n \geq 1$ is converging to x in d^s. For sufficiently large n, $d^s(x_n, x) < \epsilon_X$. It follows that x_n is in $X^s(x, \epsilon_X)$ and hence $[x_n, x] = x_n$, or, equivalently, $[x, x_n] = x$. For such an x_n, we have $d^s(x_n, x) = d(x_n, x)$. From this we conclude that x_n converges to x. Conversely, if $[x, x_n] = x$, then x_n is in $X^s(x, \epsilon_X)$ and again we have $d^s(x_n, x) = d(x_n, x)$: if x_n is also converging to x, then it is doing so in the metric d^s as well. □

We complete this summary of Smale spaces with Smale's spectral decomposition, which summarizes the relations between non-wandering, irreducible and mixing in a simple fashion [7, 31].

Theorem 2.1.13. *Let (X, φ) be a Smale space.*

1. *If (X, φ) is non-wandering, then there exists a partition of X into a finite number of closed and open, pairwise disjoint subsets, each of which is invariant under φ and so that the restriction of φ to each is irreducible. Moreover, this decomposition is unique.*

2. *If (X, φ) is irreducible, then then there exists a partition of X into a finite number of closed and open, pairwise disjoint subsets which are cyclicly permuted by φ. If the number of these sets is N, then φ^N (which leaves each invariant) is mixing on each element of the partition.*

2.2 Shifts of finite type

We give a description of the fundamental class of Smale spaces called shifts of finite type. The data needed is a finite, directed graph. All of our graphs will be finite and directed and we will simply use the term "graph".

We develop the basic ideas and show that these systems are Smale space, giving explicit descriptions of the bracket and the local stable and unstable sets. The main result here is that such systems are precisely the totally disconnected Smale space, up to topological conjugacy.

Definition 2.2.1. *A* graph *G consists of finite sets G^0 and G^1 and maps $i, t : G^1 \to G^0$. The elements of G^0 are called vertices and the elements of G^1 are called edges. The notation for the maps is meant to suggest* initial *and* terminal *and the graph is drawn by depicting each vertex as a dot and each edge, e, as an arrow from $i(e)$ to $t(e)$.*

We follow with a standard definition of a path in a graph.

Definition 2.2.2.
1. *Let G be a graph and let $K \geq 2$. A* path of length K *in G is a sequence $(e^1, e^2, \ldots, e^K)$ where e^k is in G^1, for each $1 \leq k \leq K$ and $t(e^k) = i(e^{k+1})$, for $1 \leq k < K$.*
2. *We also let G^K denote the graph whose vertex set is G^{K-1} and whose edge set is G^K with initial and terminal maps*

$$\begin{aligned} i(e^1, e^2, \ldots, e^K) &= (e^1, e^2, \ldots, e^{K-1}), \\ t(e^1, e^2, \ldots, e^K) &= (e^2, e^3, \ldots, e^K) \end{aligned}$$

We just remark that for every $1 \leq L \leq K$, we have a maps $i^L, t^L : G^K \to G^{K-L}$ which are simply the L-fold iterations of i and t, respectively. That is, i^K simply deletes the last K entries, while t^K deletes the first K entries.

Finally, we note the obvious definition of a graph homomorphism.

Definition 2.2.3. *Let G and H be graphs. A* graph homomorphism, *$\theta : H \to G$, from H to G consists of of a pair of maps $\theta : H^0 \to G^0$ and $\theta : H^1 \to G^1$ (this notation will cause no confusion) such that $\theta(i(e)) = i(\theta(e)), t(\theta(e)) = \theta(t(e))$, for all e in H^1.*

Observe that a graph homomorphism $\theta : H \to G$ induces maps $\theta : H^K \to G^K$, for all $K \geq 2$. Each of these is also a graph homomorphism.

With these preliminary notions established, we are ready to describe the shift of finite type associated with a graph.

Definition 2.2.4. *Let G be a graph. Its associated shift space, denoted Σ_G, consists of all doubly-infinite paths in G. That is, the elements are sequences $(e^k)_{k \in \mathbb{Z}}$ such that e^k is in G^1 and $t(e^k) = i(e^{k+1})$, for all k in $\mathbb{Z}$.*

The map $\sigma : \Sigma_G \to \Sigma_G$ is the left shift: $\sigma(e)^k = e^{k+1}$, for all e in Σ_G.

We need to define a metric on the space Σ_G and toward that end, we introduce the following notation that will be used frequently in other contexts. If e is in Σ_G and $K \leq L$ are integers, we let $e^{[K,L]} = (e^K, e^{K+1}, \ldots, e^L)$, which is in G^{L-K+1}. It will also be convenient to define $e^{[K+1,K]} = t(e^K) = i(e^{K+1})$, which is in G^0.

Proposition 2.2.5. *Let G be a graph. The function*

$$d(e,f) = \inf\{1, 2^{-K-1} \mid K \geq 0, e^{[1-K,K]} = f^{[1-K,K]}\}$$

defines a metric on Σ_G. With this metric, (Σ_G, σ) becomes a Smale space with constant $\epsilon_{\Sigma_G} = 1$, $\lambda = \frac{1}{2}$ and

$$[e,f]^k = \begin{cases} f^k & k \leq 0, \\ e^k & k \geq 1 \end{cases}$$

The proof is routine and we omit it. We remark that the topology for this metric coincides with the relative topology viewing Σ_G as a closed subset of the infinite product space $\Pi_{n\in\mathbb{Z}} G^1$. Of course, the main feature of the metric is that, for any e, f in Σ_G and $K \geq 1$, $d(e,f) < 2^{-K}$ if and only if $e^{[1-K,K]} = f^{[1-K,K]}$. The product structure may be understood fairly easily as follows. To any vertex v in G^0, we may consider the set Σ_v to be those elements of Σ_G with $t(e^0) = i(e^1) = v$. These sets form a partition of Σ_G into pairwise disjoint clopen sets. In fact, the reader may check that each set is the open ball of radius 1 at any of its points. For v fixed, Σ_v is a product of the semi-infinite sequences starting at v with the semi-infinite sequences terminating at v. This is a rather informal way of putting things. More concretely, we have the following.

Lemma 2.2.6. *Let G be a graph. For any e in Σ_G and $K \geq 1$, we have*

$$\begin{aligned} \Sigma_G^s(e, 2^{-K}) &= \{f \in \sigma_G \mid f^k = e^k, k \geq 1-K\}, \\ \Sigma_G^u(e, 2^{-K}) &= \{f \in \sigma_G \mid f^k = e^k, k \leq K\} \end{aligned}$$

Proof. We prove the first part only. Let f be in $\Sigma_G^s(e, 2^{-K})$. First, this means that $d(e,f) < 2^{-K}$ and so $e^{[1-K,K]} = f^{[1-K,K]}$. Secondly, we know that $[e,f] = f$. For any $k \geq 1$, we have $e^k = [e,f]^k = f^k$ and we are done.

Conversely, if $f^k = e^k$, for all $k \geq 1-K$, then it is clear that $e^{[1-K,K]} = f^{[1-K,K]}$ and so $d(e,f) \leq 2^{-K-1} < 2^{-K}$. It also means that $[e,f]$ is defined. For $k \leq 0$, we have $[e,f]^k = f^k$, from the definition of $[,]$ while for $k \geq 1$, $[e,f]^k = e^k = f^k$ as desired. □

We are now ready to give our definition of a shift of finite type.

Definition 2.2.7. *A* shift of finite type *is any system* (Σ, σ) *which is topologically conjugate to* (Σ_G, σ) *for some graph* G*. In this case, we say that* G *is a presentation of* (Σ, σ)*.*

This is not the usual definition which supposes, first, that Σ is a closed subsystem of a full shift [23] and is described by a finite set of forbidden words. For our purposes, this definition will be more convenient.

The main result of this section is the following.

Theorem 2.2.8. *A Smale space* (X, φ) *is a shift of finite type if and only if* X *is totally disconnected.*

We refer the reader to Proposition 18.7.8 of [20] for a proof. It should be noted though that the hypothesis that the set is a locally maximal hyperbolic set of a diffeomorphism is not needed as only the bracket and notions of local stable and unstable sets are used in the proof. This also means that no irreducibility hypothesis is necessary.

Notice that if $\theta : H \to G$ is a graph homomorphism, then it induces an obvious map, also denoted by θ, from Σ_H to Σ_G. We note the following result whose proof is elementary.

Proposition 2.2.9. *Let* G *be a graph and let* $L > K \geq 1$*. The graph homomorphism* i^{L-K} *induces a homeomorphism* $i^{L-K} : \Sigma_{G^L} \to \Sigma_{G^K}$ *and satisfies* $\sigma \circ i^{L-K} = i^{L-K} \circ \sigma$*. A similar statement holds for* t^{L-K}*.*

The graphs G^K are usually referred to as higher block presentations for the dynamical system (Σ_G, σ).

2.3 Maps

In this section, we give some basic definitions of maps and factor maps between dynamical systems and establish basic properties of them in the case that both domain and range are Smale spaces. The first is basically that a map is necessarily compatible with the two bracket operations. We then consider the case when the domain is a shift of finite type and introduce a concept of regularity for a factor map.

Definition 2.3.1. *Let* (X, φ) *and* (Y, ψ) *be dynamical systems. A* map $\pi : (Y, \psi) \to (X, \varphi)$, *is a continuous function* $\pi : Y \to X$ *such that* $\pi \circ \psi = \varphi \circ \pi$. *A* factor map *from* (Y, ψ) *to* (X, φ) *is a map for which* $\pi : Y \to X$ *is also surjective.*

Theorem 2.3.2. *Let* (Y, ψ) *and* (X, φ) *be Smale spaces and let*

$$\pi : (Y, \psi) \to (X, \varphi)$$

be a map. There exists $\epsilon_\pi > 0$ *such that, for all* y_1, y_2 *in* Y *with* $d(y_1, y_2) \leq \epsilon_\pi$, *then both* $[y_1, y_2], [\pi(y_1), \pi(y_2)]$ *are defined and*

$$\pi([y_1, y_2]) = [\pi(y_1), \pi(y_2)].$$

Proof. Let ϵ_X, ϵ_Y be the Smale space constants for X and Y, respectively. As Y is compact and π is continuous, we may find a constant $\epsilon > 0$ such that, for all y_1, y_2 is in Y with $d(y_1, y_2) < \epsilon$, we have $d(\pi(y_1), \pi(y_2)) < \epsilon_X/2$. From the continuity of the bracket map, we may choose ϵ_π such that $0 < \epsilon_\pi < \epsilon_Y$ and for all y_1, y_2 in X with $d(y_1, y_2) \leq \epsilon_\pi$, we have

$$d(y_1, [y_1, y_2]), d(y_2, [y_1, y_2]) < \epsilon.$$

Now assume y_1, y_2 are in Y with $d(y_1, y_2) \leq \epsilon_\pi$. It follows that, $[y_1, y_2]$ is defined and we have the estimates above. Then, inductively for all $n \geq 0$, we have

$$d(\psi^n(y_1), \psi^n[y_1, y_2]) \leq \lambda^n d(y_1, [y_1, y_2]) \leq \epsilon$$

and also

$$d(\psi^{-n}(y_2), \psi^{-n}[y_1, y_2]) \leq \lambda^n d(y_2, [y_1, y_2]) \leq \epsilon$$

It follows from the choice of ϵ that, for all $n \geq 0$, we have

$$\begin{aligned} d(\pi(\psi^n(y_1)), \pi(\psi^n[y_1, y_2])) &\leq \epsilon_X/2 \\ d(\varphi^n(\pi(y_1)), \varphi^n(\pi[y_1, y_2])) &\leq \epsilon_X/2 \end{aligned}$$

and similarly

$$d(\varphi^{-n}(\pi(y_2)), \varphi^{-n}(\pi[y_1, y_2])) \leq \epsilon_X/2.$$

On the other hand, these two estimates are also satisfied replacing $\varphi^n(\pi[y_1, y_2])$ by $\varphi^n[\pi(y_1), \pi(y_2)]$ and so, by expansiveness of (X, φ), we have the desired conclusion. □

In the case where the domain is a shift of finite type with a specific presentation, we introduce the notion of a map being regular. It is really just an analogue of the conclusion of the last Theorem appropriate for shifts of finite type.

Definition 2.3.3. *Let G be a graph, (Σ_G, σ) be the associated shift of finite type and (X, φ) be a Smale space. We say that a map $\pi : (\Sigma_G, \sigma) \to (X, \varphi)$ is* regular *if, for all e, f in Σ_G with $t(e^0) = t(f^0)$, we have $d(\pi(e), \pi(f)) \leq \epsilon_X$ and*

$$\pi[e, f] = [\pi(e), \pi(f)].$$

The first and most obvious case of a regular map is the following; the proof is trivial and we omit it.

Proposition 2.3.4. *Let G and H be finite directed graphs and let $\theta : H \to G$ be a graph homomorphism. The associated map $\theta : (\Sigma_H, \sigma) \to (\Sigma_G, \sigma)$ is regular.*

In fact, after replacing the domain by a higher block presentation, we may assume any map from a shift of finite type is regular. The following is an easy consequence of Proposition 2.2.9 and Theorem 2.3.2 and we omit the proof.

Theorem 2.3.5. *Let (Σ, σ) be a shift of finite type, (X, φ) be a Smale space and $\pi : (\Sigma, \sigma) \to (X, \varphi)$ be a map. Then there exists a graph G and a conjugacy $h : (\Sigma_G, \sigma) \to (\Sigma, \sigma)$ such that $\pi \circ h$ is regular.*

When G satisfies the conclusion of this Theorem, we will say (rather imprecisely) that G is a presentation of π.

2.4 Self-products of a map

We begin with the standard definition of a fibred product or a pull-back, which will be used frequently. One can apply this definition to a map and itself, and then continue iteratively. We refer to the sequence of resulting systems as "self-products" of the map. We go on to consider the special case when this is applied to Smale spaces and show the resulting systems are also Smale spaces. Finally, we specialize to the case where the domain is a shift of finite type. If it is presented by a graph,and the map is regular in our earlier sense, then we give specific graphs which present the fibred products.

Definition 2.4.1. *Suppose that* $(X, \varphi), (Y_1, \psi_1)$ *and* (Y_2, ψ_2) *are dynamical systems and that* $\pi_1 : (Y_1, \psi_1) \to (X, \varphi)$ *and* $\pi_2 : (Y_2, \psi_2) \to (X, \varphi)$ *are maps. Their* fibred product *is the space*

$$Z = \{(y_1, y_2) \mid y_1 \in Y_1, y_2 \in Y_2, \pi_1(y_1) = \pi_2(y_2)\}$$

equipped with the relative topology from $Y_1 \times Y_2$ *together with the map* $\zeta = \psi_1 \times \psi_2$*, which is clearly a homeomorphism of this space. There are canonical maps from this system to* (Y_1, ψ_1) *and* (Y_2, π_2) *defined by* $\rho_1(y_1, y_2) = y_1$ *and* $\rho_2(y_1, y_2) = y_2$ *which satisfy* $\pi_1 \circ \rho_1 = \pi_2 \circ \rho_2$.

We observe the following. Its proof is routine and we omit it.

Theorem 2.4.2. *Suppose that* $(X, \varphi), (Y_1, \psi_1)$ *and* (Y_2, ψ_2) *are Smale spaces and that* $\pi_1 : (Y_1, \psi_1) \to (X, \varphi)$ *and* $\pi_2 : (Y_2, \psi_2) \to (X, \varphi)$ *are maps. The fibred product is also a Smale space with the metric*

$$d((y_1, y_2), (y_1', y_2')) = \max\{d(y_1, y_1'), d(y_2, y_2')\},$$

constant $\epsilon_Z = \min\{\epsilon_{\pi_1}, \epsilon_{\pi_2}\}$ *and bracket*

$$[(y_1, y_2), (y_1', y_2')] = ([y_1, y_1'], [y_2, y_2']),$$

provided $d(y_1, y_1'), d(y_2, y_2') \leq \epsilon_Z$.

We will make extensive use of a variation of this idea. We consider a map between two Smale spaces

$$\pi : (Y, \psi) \to (X, \varphi)$$

First, we may consider the fibred product of this map with itself. In addition, we may iterate this process, taking fibred product with the same (Y, ψ) each time. The result is a sequence of Smale spaces which records where the map is many-to-one.

Definition 2.4.3. *Let*

$$\pi : (Y, \psi) \to (X, \varphi)$$

be a map. For each $N \geq 0$*, we define*

$$\begin{aligned} Y_N(\pi) \;&=\; \{(y_0, y_1, y_2, \ldots, y_N) \in Y^{N+1} \mid \pi(y_i) = \pi(y_j), \\ &\qquad \textit{for all } 0 \leq i, j \leq N\}. \end{aligned}$$

We also define

$$\psi(y_0, y_1, y_2, \ldots, y_N) = (\psi(y_0), \psi(y_1), \psi(y_2), \ldots, \psi(y_N))$$

for all $(y_0, y_1, y_2, \ldots, y_N)$ *in* $Y_N(\pi)$.

It will often be the case that no confusion will be caused by omitting the π in the notation, writing just Y_N.

Finally, we note the obvious fact that the permutation group on $\{0, 1, 2, \ldots, N\}$, S_{N+1}, acts on $Y_N(\pi)$, commuting with ψ. We use the following notation; if y is in $Y_N(\pi)$ and α is in S_{N+1}, for some $N \geq 0$, we let $y \cdot \alpha = (y_{\alpha(0)}, y_{\alpha(1)}, \ldots, y_{\alpha(N)})$.

It is worth noting that $Y_N(\pi)$ is rarely irreducible for $N \geq 1$, even if Y and X are. We will also use the notation π for the map from $Y_N(\pi)$ to X defined by $\pi(y_0, y_1, \ldots, y_N) = \pi(y_0)$. It is easy to see that, for all $N \geq 1$, $(Y_N(\pi), \psi)$ is the fibred product of (Y, ψ) and $(Y_{N-1}(\pi), \psi)$. This, and Theorem 2.4.2, immediately yields the following.

Proposition 2.4.4. *If* $\pi : (Y, \psi) \to (X, \varphi)$ *is a map between Smale spaces then, for each* $N \geq 0$, $(Y_N(\pi), \psi)$ *is also a Smale space with* $\epsilon_{Y_N(\pi)} = \epsilon_\pi$ *and bracket*

$$[(y_0, \ldots, y_N), (y'_0, \ldots, y'_N)] = ([y_0, y'_0], \ldots, [y_N, y'_N]),$$

for all $(y_0, \ldots, y_N), (y'_0, \ldots, y'_N)$ *in* $Y_N(\pi)$ *within distance* ϵ_π.

Before proceeding further, let us introduce a standard piece of notation which we will use frequently throughout the paper. If A is any set and $N \geq 1$, we define, for each $0 \leq n \leq N$, $\delta_n : A^{N+1} \to A^N$ by

$$\delta_n(a_0, \ldots, a_N) = (a_0, \ldots, a_{n-1}, a_{n+1}, \ldots, a_N) = (a_0, \ldots, \widehat{a_n}, \ldots, a_N).$$

We also note that if we regard $(Y_N(\pi), \psi)$ as the fibred product of (Y, ψ) and $(Y_{N-1}(\pi), \psi)$ then δ_0 is simply the map ρ_2 of Definition 2.4.1.

We now specialize further by considering the situation of a regular map

$$\pi : (\Sigma_G, \sigma) \to (X, \varphi),$$

where G is a graph and (X, φ) is a Smale space. In view of Theorem 2.3.5, we will lose no generality in considering this case.

For each $N \geq 0$, we define graphs $G_N(\pi)$ by:

$$\begin{aligned} G_N(\pi)^0 &= \{t(e^0) = (t(e_0^0), t(e_1^0), \ldots, t(e_N^0)) \mid \\ &\quad e = (e_0, e_1, \ldots, e_N) \in \Sigma_N(\pi)\} \\ G_N(\pi)^1 &= \{e^0 = (e_0^0, e_1^0, \ldots, e_N^0) \mid \\ &\quad e = (e_0, e_1, \ldots, e_N) \in \Sigma_N(\pi)\} \end{aligned}$$

If no confusion will arise, we will drop the π from the notation. Also, the graph $G_N(\pi)$ has an action of S_{N+1}. We begin by establishing a few simple properties of the graphs.

We note that $\Sigma_N(\pi)$ consists of $N+1$-tuples of bi-infinite sequences in G^1. On the other hand, $\Sigma_{G_N(\pi)}$ consists of bi-infinite sequences of $N+1$-tuples from G^1. If we regard an $N+1$-tuples of bi-infinite sequences as also being a bi-infinite sequences of $N+1$-tuples, we have the following.

Theorem 2.4.5. *Let G be a graph, (X, φ) be a Smale space and*

$$\pi : (\Sigma_G, \sigma) \to (X, \varphi),$$

be a regular map. For each $N \geq 0$, we have

$$(\Sigma_{G_N(\pi)}, \sigma) = (\Sigma_N(\pi), \sigma).$$

Proof. It is clear that we have containment $\supset$. It remains to prove the other direction. We will show by induction on $K \geq 1$, that if p is any path of length K in $G_N(\pi)$, then there exists an e in $\Sigma_N(\pi)$ such that $e^{[1,K]} = p$. The conclusion follows from an easy compactness argument. The case $K = 1$ is clear. Now suppose that the statement is true for $K \geq 1$ and we prove it for $K+1$. Let p be a path in $G_N(\pi)$ of length $K+1$. By induction hypothesis, we may find e in Σ_N such that $e^{[1,K]} = p^{[2,K+1]}$ and also a f in Σ_N with $f^0 = p^1$. Since p is a path, $t(f^0) = i(e^1)$ and so we may form $\sigma^{-1}[e, f]$ which satisfies the desired conclusion. □

2.5 s/u-resolving and s/u-bijective maps

In this section, we discuss special classes of maps called s-resolving, u-resolving, s-bijective and u-bijective maps. These maps possess many nice properties but the most important is that our invariants will behave in a functorial way with respect to them.

It is an easy consequence of the definitions that if (Y, ψ) and (X, φ) are Smale spaces and

$$\pi : (Y, \psi) \to (X, \varphi)$$

is a map, then $\pi(Y^s(y)) \subset X^s(\pi(y))$ and $\pi(Y^u(y)) \subset X^u(\pi(y))$. We recall the following definition due to David Fried [18].

Definition 2.5.1. *Let (X, φ) and (Y, ψ) be Smale spaces and let*

$$\pi : (Y, \psi) \to (X, \varphi)$$

*be a map. We say that π is s-*resolving *(or u-*resolving*) if, for any y in Y, its restriction to $Y^s(y)$ (or $Y^u(y)$, respectively) is injective.*

The following is a useful technical preliminary result.

Proposition 2.5.2. *Let (X, φ) and (Y, ψ) be Smale spaces and let*

$$\pi : (Y, \psi) \to (X, \varphi)$$

be an s-resolving (or u-resolving) map. With ϵ_π as in Theorem 2.3.2, if y_1, y_2 are in Y with $\pi(y_1)$ in $X^u(\pi(y_2), \epsilon_X)$ (or $\pi(y_1)$ in $X^u(\pi(y_2), \epsilon_X)$, respectively) and $d(y_1, y_2) \leq \epsilon_\pi$, then $y_2 \in Y^u(y_1, \epsilon_\pi)$ ($y_2 \in Y^s(y_1, \epsilon_\pi)$, respectively).

Proof. It follows at once the from hypotheses that

$$\pi[y_1, y_2] = [\pi(y_1), \pi(y_2)] = \pi(y_1).$$

On the other hand $[y_1, y_2]$ is stably equivalent to y_1 and, since π is s-resolving, $[y_1, y_2] = y_1$. □

Resolving maps have many nice properties, the first being that they are finite-to-one. We establish this, and a slight variant of it, as follows.

Theorem 2.5.3. *Let (X, φ) and (Y, ψ) be Smale spaces and let*

$$\pi : (Y, \psi) \to (X, \varphi)$$

be an s-resolving map. There is a constant $M \geq 1$ such that

1. *for any x in X, there exist $y_1, \ldots, y_K$ in Y with $K \leq M$ such that*

$$\pi^{-1}(X^u(x)) = \cup_{k=1}^{K} Y^u(y_k),$$

and

2. for any x in X, we have $\#\pi^{-1}\{x\} \leq M$, .

Proof. Cover Y with balls of radius $\epsilon_\pi/2$, then extract a finite subcover, whose elements we list as $B_m, 1 \leq m \leq M$. We claim this M satisfies the desired conclusions.

For the first statement, given x in X and y in $\pi^{-1}(X^u(x))$, it is clear that $Y^u(y) \subset X^u(x)$. We must show that there exist at most M unstable equivalence classes in $\pi^{-1}(X^u(x))$. For this, it suffices to show that if $y_i, 1 \leq i \leq M+1$, are in Y with $\pi(y_i)$ and $\pi(y_j)$ unstably equivalent, for all i, j, then y_i and y_j are unstably equivalent for some $i \neq j$. Choose $n \leq 0$ such that $\varphi^n(\pi(y_i))$ is in $X^u(\varphi^n(\pi(y_j)), \epsilon_X)$, for all $1 \leq i, j \leq M+1$. From the pigeon hole principle, there exists distinct i and j such that $\psi^n(y_i)$ and $\psi^n(y_j)$ lie in the same B_m, for some $1 \leq m \leq M$. These points satisfy the hypotheses of 2.5.2 and it follows that they are unstably equivalent. Then y_i are y_j are also unstably equivalent.

For the second statement, suppose $\pi^{-1}\{x\}$ contains distinct points $y_1, \ldots, y_{M+1}$. Let δ denote the minimum distance, $d(y_i, y_j)$, over all $i \neq j$. Choose $n \geq 1$ such that $\lambda^n \epsilon_\pi < \delta$. Consider the points $\psi^n(y_i), 1 \leq i \leq M+1$. By the pigeon-hole principle, there exists $i \neq j$ with $\psi^n(y_i)$ and $\psi^n(y_j)$ in the same set B_m. We have

$$\pi(\psi^n(y_i)) = \varphi^n(\pi(y_i)) = \varphi^n(x) = \varphi^n(\pi(y_j)) = \pi(\psi^n(y_j)).$$

From Proposition 2.5.2, $\psi^n(y_i)$ is in $Y^u(\psi^n(y_j), \epsilon_\pi)$. This implies that y_i is in $Y^u(y_j, \lambda^n\epsilon_\pi)$. As $\lambda^n\epsilon_\pi < \delta$, this is a contradiction. □

Although the definition of s-resolving is given purely at the level of the stable sets as sets, various nice continuity properties follow.

Theorem 2.5.4. *Let (X, φ) and (Y, ψ) be Smale spaces and let*

$$\pi : (Y, \psi) \to (X, \varphi)$$

be either an s-resolving or a u-resolving map. For each y in Y, the maps

$$\pi : Y^s(y) \to X^s(\pi(y)), \pi : Y^u(y) \to X^u(\pi(y))$$

are continuous and proper, where the sets above are given the topologies of Proposition 2.1.12.

Proof. From the symmetry of the statement, it suffices to consider the case that π is s-resolving.

We use the characterization of limits in $Y^s(y)$ and $X^s(\pi(y))$ given in Proposition 2.1.12. From this, and Theorem 2.3.2, it is easy to see that π is continuous on $Y^s(y)$. The same argument covers the case of π on $Y^u(y)$.

To see the map π on $Y^s(y)$ is proper, it suffices to consider a sequence y_n in $Y^s(y)$ such that $\pi(y_n)$ is convergent in the topology of $X^s(\pi(y))$, say with limit x, and show that it has a convergent subsequence. As Y is compact in its usual topology, we may find y' which is a limit point of a convergent subsequence $y_{n_k}, k \geq 1$. It follows that

$$\pi(y') = \pi(\lim_k y_{n_k}) = \lim_k \pi(y_{n_k}) = \lim_n \pi(y_n) = x.$$

We also have, for k sufficiently large,

$$\pi[y_{n_k}, y'] = [\pi(y_{n_k}), \pi(y')] = [\pi(y_{n_k}), x] = x,$$

since $\pi(y_{n_k})$ is converging to x in the topology on $X^s(\pi(y))$ and using Proposition 2.1.12. We know that $\pi^{-1}\{x\}$ is finite and contains y' and $[y_{n_k}, y']$, for all k sufficiently large. Moreover, y' is the limit of the sequence $[y_{n_k}, y']$. It follows that there is K such that $[y_{n_k}, y'] = y'$, for all $k \geq K$. From this, we see that y' is in $Y^s(y_{n_k}) = Y^s(y)$ and that the subsequence y_{n_k} converges to y' in $Y^s(y)$.

To see the map π on $Y^u(y)$ is proper, we begin in the same way with a sequence y_n such that $\pi(y_n)$ has limit x in the topology of $X^u(\pi(y))$. Again we obtain a subsequence y_{n_k} with limit y' in Y. Then we have, for k sufficiently large,

$$\pi[y_{n_k}, y'] = [\pi(y_{n_k}), \pi(y')] = [\pi(y_{n_k}), x] = \pi(y_{n_k}),$$

since $\pi(y_{n_k})$ is converging to x in the topology on $X^u(\pi(y))$ and using Proposition 2.1.12. On the other hand, $[y_{n_k}, y']$ and y_{n_k} are stably equivalent and since π is s-resolving, this implies they are equal. It follows that y' is in $Y^u(y_{n_K}) = Y^u(y)$ and y_{n_k} is converging to y' in the topology of $Y^u(y)$. □

There has been extensive interest in s/u-resolving maps. We will need a slightly stronger condition, which we refer to as s/u-bijective maps.

Definition 2.5.5. *Let (X, φ) and (Y, ψ) be Smale spaces and let*

$$\pi : (Y, \psi) \to (X, \varphi)$$

be a map. We say that π is s-bijective *(or* u-bijective*) if, for any y in Y, its restriction to $Y^s(y)$ (or $Y^u(y)$, respectively) is a bijection to $X^s(\pi(y))$ (or $X^u(\pi(y))$, respectively).*

It is relatively easy to find an example of a map which is s-resolving, but not s-bijective and we will give one in a moment. However, one important distinction between the two cases should be pointed out at once. The image of a Smale space under an s-resolving map is not necessarily a Smale space. The most prominent case is where the domain and range are both shifts of finite type and the image is a sofic shift, which is a much broader class of systems. (See [23].) This is not the case for s-bijective maps (or u-bijective maps).

Theorem 2.5.6. *Let (Y, ψ) and (X, ϕ) be Smale spaces and let*

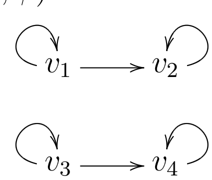
$$\pi : (Y, \psi) \to (X, \varphi)$$

be either an s-bijective map or a u-bijective map. Then $(\pi(Y), \varphi|_{\pi(Y)})$ is a Smale space.

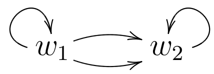
Proof. The only property which is not clear is the existence of the bracket: if y_1 and y_2 are in Y and $d(\pi(y_1), \pi(y_2)) < \epsilon_X$, then it is clear that $[\pi(y_1), \pi(y_2)]$ is defined, but we must see that it is in $\pi(Y)$. If π is s-bijective, then $[\pi(y_1), \pi(y_2)]$ is stably equivalent to $\pi(y_1)$ and hence in the set $\pi(Y^s(y_1))$ and hence in $\pi(Y)$. A similar argument deals with the case π is u-bijective. □

If $\pi : (Y, \psi) \to (X, \varphi)$ is a factor map and every point in the system (Y, ψ) is non-wandering (including the case that (Y, ψ) is irreducible), then it follows that the same is true of (X, φ) and in this case, any s-resolving factor map is also s-bijective, as we will show. The distinction is important for us; although we are mainly interested in our homology theory for irreducible systems, the various self-products we constructed earlier, and which will be used in the later definitions, will almost never be irreducible.

Example 2.5.7. *Consider (Y, ψ) to be the shift of finite type associated with the following graph:*



and (X, φ) to be the shift of finite type associated with the following graph:



It is clear that there is a factor map from (Y, ψ) to (X, φ) obtained by mapping the loops in the first graph to those in the second 2-to-1, while mapping the other two edges injectively. The resulting factor map is s-resolving and u-resolving but not s-bijective or u-bijective.

Theorem 2.5.8. *Let (X, φ) and (Y, ψ) be Smale spaces and let*

$$\pi : (Y, \psi) \to (X, \varphi)$$

be an s-resolving factor map. Suppose that each point of (Y, ψ) is non-wandering. Then π is s-bijective.

The proof will be done in a series of Lemmas, beginning with the following quite easy one.

Lemma 2.5.9. *Let $\pi : Y \to X$ be a continuous map and let x_0 be in X with $\pi^{-1}\{x_0\} = \{y_1, y_2, \ldots, y_N\}$ finite. For any $\epsilon > 0$, there exists $\delta > 0$ such that $\pi^{-1}(X(x_0, \delta)) \subset \cup_{n=1}^{N} Y(y_n, \epsilon)$.*

Proof. If there is no such δ, we may construct a sequence $x^k, k \geq 1$ in X converging to x_0 and a sequence $y^k, k \geq 1$ with $\pi(y^k) = x^k$ and y^k not in $\cup_{n=1}^{N} Y(y_n, \epsilon)$. Passing to a convergent subsequence of the y^k, let y be the limit point. Then y is not in $\cup_{n=1}^{N} Y(y_n, \epsilon)$, since that set is open, while $\pi(y) = \lim_k \pi(y^k) = \lim_k x^k = x_0$. This is a contradiction to $\pi^{-1}\{x_0\} = \{y_1, y_2, \ldots, y_N\}$. □

Lemma 2.5.10. *Let $\pi : (Y, \psi) \to (X, \varphi)$ be a finite-to-one factor map between Smale spaces and suppose that (X, φ) is non-wandering. There exists a periodic point x in X such that*

$$\#\pi^{-1}\{x\} = \min\{\#\pi^{-1}\{x'\} \mid x' \in X\}.$$

Proof. Choose x_0 in X which minimizes $\#\pi^{-1}\{x_0\}$ and let $\pi^{-1}\{x_0\} = \{y_1, \ldots, y_N\}$. Choose $\epsilon_Y/2 > \epsilon > 0$ so that the sets $Y(y_n, \epsilon), 1 \leq n \leq N$ are pairwise disjoint. Apply the last Lemma to find δ satisfying the conclusion there for this ϵ.

As (X, φ) is non-wandering, the periodic points are dense, so choose a periodic point x in $X(x, \delta)$. We claim that, for any $1 \leq n \leq N$, the set $\pi^{-1}\{x\} \cap Y(y_n, \epsilon)$ contains at most one point. Note that from this, it follows at once that $\#\pi^{-1}\{x\} \leq N = \#\pi^{-1}\{x_0\}$. The reverse inequality is trivial from the choice of x_0 and the proof will be complete. Suppose that both y and y' are in $\pi^{-1}\{x\} \cap Y(y_n, \epsilon)$. As they both map to x under π, they are both periodic. As $\epsilon < \epsilon_Y/2$, we may bracket y and y'. We have

$$\pi[y, y'] = [\pi(y), \pi(y')] = [x, x] = x.$$

This implies that $[y, y']$ is also periodic. The periodic points y and $[y, y']$ are stably equivalent and hence must be equal. Similarly, the periodic points y' and $[y, y']$ are unstably equivalent and hence must be equal. We conclude that $y = y'$ as desired. □

We now prove a version of Lemma 2.5.9 for local stable sets.

Lemma 2.5.11. *Let $\pi : (Y, \psi) \to (X, \varphi)$ be a factor map between Smale spaces and suppose x_0 in X is periodic and $\pi^{-1}\{x_0\} = \{y_1, y_2, \ldots, y_N\}$. Given $\epsilon_0 > 0$, there exist $\epsilon_0 > \epsilon > 0$ and $\delta > 0$ such that*

$$\pi^{-1}(X^s(x_0, \delta)) \subset \cup_{n=1}^{N} Y^s(y_n, \epsilon).$$

Proof. First, since x_0 is periodic, so is each y_n. Choose $p \geq 1$ such that $\psi^p(y_n) = y_n$, for all $1 \leq n \leq N$, and hence $\varphi^p(x_0) = x_0$. The system (Y, ψ^p) is also a Smale space. Choose $\epsilon_0 > \epsilon > 0$ to be less than the Smale space constant for this. Also, choose ϵ sufficiently small so that the sets $Y(y_n, \epsilon), 1 \leq n \leq N$ are pairwise disjoint and so that $\psi^p(Y(y_n, \epsilon)) \cap B(y_m, \epsilon) = \emptyset$, for $m \neq n$. Use the Lemma 2.5.9 to find δ such that $\pi^{-1}\{x\} \subset \cup_{n=1}^{N} Y(y_n, \epsilon)$.

Now suppose that x is in $X^s(x_0, \delta)$ and $\pi(y) = x$. It follows that y is in $Y(y_m, \epsilon)$, for some m. Now consider $k \geq 1$. We have $\pi(\psi^{kp}(y)) = \varphi^{kp}(x)$ which is in $X^s(x_0, \lambda^{kp}\delta) \subset X(x, \delta)$. It follows that $\psi^{kp}(y)$ is in $\cup_{n=1}^{N} Y^s(y_n, \epsilon)$ for all $k \geq 1$. It then follows from the choice of ϵ and induction that $\psi^{kp}(y)$ is in $Y^s(y_m, \epsilon)$ for all $k \geq 1$. This means that y is in $Y^s(y_m, \epsilon)$. □

We are now prepared to give a proof of Theorem 2.5.8.

Proof. In view of the structure Theorem 2.1.13, it suffices for us to consider the case that (X, φ) is irreducible. First, choose a periodic point x_0 satisfying the conclusion of Lemma 2.5.10. Let $\pi^{-1}\{x_0\} = \{y_1, \ldots, y_N\}$. We will first show that, for each $1 \leq n \leq N$, $\pi : Y^s(y_n) \to X^s(x_0)$ is open and onto. We choose $\epsilon_0 > 0$ so that the sets $Y(y_n, \epsilon_0), 1 \leq n \leq N$, are pairwise disjoint. We then choose $\epsilon_0 > \epsilon > 0$ and $\delta > 0$ as in Lemma 2.5.11. Let x be any point in $B(x_0, \delta)$. We know that $\pi^{-1}(\{x\})$ is contained in $\cup_{n=1}^N Y^s(y_n, \epsilon_0)$. As the map π is s-resolving, it is injective when restricted to each of the sets $Y^s(y_n, \epsilon)$. This means that $\pi^{-1}\{x\}$ contains at most one point in each of these sets. On the other hand, it follows from our choice of x_0 that $\pi^{-1}\{x\}$ contains at least N points. We conclude that, for each n, $\pi^{-1}\{x\} \cap Y^s(y_n, \epsilon)$ contains exactly one point. Let $W_n = \pi^{-1}(X(x, \delta)) \cap Y^s(y_n, \epsilon)$. The argument above shows that π is a bijection from W_n to $X^s(x_0, \delta)$, for each n. It is clearly continuous and we claim that is actually a homeomorphism. To see this, it suffices to show that, for any sequence y^k in W_n such that $\pi(y^k)$ converges to some x in $X^s(x_0, \delta)$, it follows that y^k converges to some y in W_n. As $\epsilon < \epsilon_0$, the closure of W_n is a compact subset of $Y^s(y_n, \epsilon_0)$. So the sequence y^k has limit points; let y be one of them. By continuity, $\pi(y) = x$. On the other hand, there is a unique point y' in W_n such that $\pi(y') = x$. Thus, y and y' are both in $Y^s(y_n, \epsilon_0)$ and have image x under π. As π is s-resolving, $y = y'$ and so y is in W_n. So the only limit point of the sequence y^k is y' and this completes the proof that π is a homeomorphism.

Since W_n is an open subset of $Y^s(y_n, \epsilon)$, we know that $Y^s(y_n) = \cup_{l \geq 0} \psi^{-l}(W_n)$ and the topology is the inductive limit topology. Similarly, $X^s(x_0) = \cup_{l \geq 0} \varphi^{-l}(X^s(x_0, \delta))$ and the topology is the inductive limit topology. It follows at once that π is a homeomorphism from the former to the latter.

Now we turn to arbitrary point y in Y and $x = \pi(y)$ in X and show that $\pi : Y^s(y) \to X^s(x)$ is onto. We choose x_0 and $\{y_1, \ldots, y_N\}$ to be periodic points as above so that $\pi : Y^s(y_n) \to X^s(x_0)$ are homeomorphisms. By replacing x_0 by another point in its orbit (which will satisfy the same condition), we may assume that x is in the closure of $X^s(x_0)$. Then, we may choose y_n such that y is in the closure of $Y^s(y_n)$. There exists a point y' is $Y^s(y_n)$ in $Y^u(y, \epsilon_Y/2)$ and so that $x' = \pi(y')$ is in $X^u(x, \epsilon_X/2)$. The map π may be written as the composition of three maps. The first from $Y^s(y, \epsilon_Y/2)$ to $Y^s(y', \epsilon_Y)$ sends z to $[y', z]$. The second from $Y^s(y', \epsilon_Y)$ to $X^s(x')$ is simply π. The third is the map from $X^s(x', \epsilon_X/2)$ to $X^s(x, \epsilon_X)$ sends z to $[x, z]$. Each is defined on an open set containing y, y' and x', respectively and is an open map. The conclusion is that there exists some $\epsilon' > 0$ such that $\pi(Y^s(y, \epsilon')) = U$ is an open set in $X^s(x)$ containing x. It then follows that

$$X^s(x) = \cup_{l \geq 0} \varphi^{-l}(U) = \cup_{l \geq 0} \pi(\psi^{-l}(Y^s(y, \epsilon))) \subset \pi(Y^s(y)).$$

□

Now we want to observe that although the property of a map being s-bijective is defined purely at the level of stable sets, continuity properties follow as a consequence.

Theorem 2.5.12. *Let (X, φ) and (Y, ψ) be Smale spaces and let*

$$\pi : (Y, \psi) \to (X, \varphi)$$

be an s-bijective (or u-bijective) map. Then for each y in Y, the map $\pi : Y^s(y) \to X^s(\pi(y))$ (or $\pi : Y^u(y) \to X^u(\pi(y))$, respectively) is a homeomorphism.

Proof. The proof is the general fact that if A, B are locally compact Hausdorff spaces and $f : A \to B$ is a continuous, proper bijection, then f is a homeomorphism. This can be seen as as follows. Let A^+ and B^+ denote the one-point compactifications of A and B, respectively. That the obvious extension of f to a map between these spaces is continuous follows from the fact that f is proper. Since this extension is a continuous bijection between compact Hausdorff spaces, it is a homeomorphism. The result follows from this argument and Theorem 2.5.4. □

We have established a number of properties of s/u-bijective maps. We now want to consider the constructions from the last section of fibred products and self-products as they pertain to s/u-resolving maps and s/u-bijective maps. The first basic result is the following.

Theorem 2.5.13. *Let $(Y_1, \psi_1), (Y_2, \psi_2)$ and (X, φ) be Smale spaces and suppose that*

$$\pi_1 : (Y_1, \psi_1) \to (X, \varphi), \quad \pi_2 : (Y_2, \psi_2) \to (X, \varphi)$$

are maps. Let (Z, ζ) denote the fibred product:

$$Z = \{(y_1, y_2) \in Y_1 \times Y_2 \mid \pi_1(y_1) = \pi_2(y_2)\}, \zeta = \psi_1 \times \psi_2.$$

We have maps $\rho_1 : (Z, \zeta) \to (Y_1, \psi_1)$, $\rho_2 : (Z, \zeta) \to (Y_2, \psi_2)$ defined by $\rho_1(y_1, y_2) = y_1$, $\rho_2(y_1, y_2) = y_2$.

1. *If π_1 is a factor map, then so is ρ_2.*

2. *If π_1 is s-bijective (or u-bijective), then ρ_2 is s-bijective (or u-bijective, respectively).*

Proof. We will not prove the first statement since it is well-known. A proof can be found in Proposition 8.3.3 of [23], but is also very similar to what we demonstrate below for the other part.

We assume that π_1 is s-resolving and show that ρ_2 is injective on stable equivalence classes. Let (y_1, y_2) and (y_1', y_2') be stably equivalent points in Z and satisfy $\rho_2(y_1, y_2) = \rho_2(y_1', y_2')$. The first statement implies that y_1 and y_1' are stably equivalent. The second implies that $y_2 = y_2'$. Since these points are in Z, we have $\pi_1(y_1) = \pi_2(y_2) = \pi_2(y_2') = \pi_1(y_1')$. As π_1 is s-resolving, we have $y_1 = y_1'$ and hence $(y_1, y_2) = (y_1', y_2')$ as desired.

Next, we show that ρ_2 maps stable equivalence classes surjectively. Let (y_1, y_2) be in Z and y_2' be stably equivalent to $\rho_2(y_1, y_2) = y_2$. As (y_1, y_2) is in Z, we have $\pi_1(y_1) = \pi_2(y_2)$, which is stably equivalent to $\pi_2(y_2')$. As π_1 is s-bijective, we may find y_1' in Y_1 such that y_1' is stably equivalent to y_1 and $\pi_1(y_1') = \pi_2(y_2')$. This means that (y_1', y_2') is in Z and is stably equivalent to (y_1, y_2). Finally, we note $\rho_2(y_1', y_2') = y_2'$. □

Theorem 2.5.14. *Let (Y, ψ) and (X, φ) be Smale spaces and let $\pi : (Y, \psi) \to (X, \varphi)$ be a map.*

1. *If π is a factor map, then so is*

$$\delta_n : Y_N(\pi) \to Y_{N-1}(\pi)$$

then for all $N \geq 1$ and $0 \leq n \leq N$.

2. *If π is an s-bijective (or u-bijective) map, then so is*

$$\delta_n : Y_N(\pi) \to Y_{N-1}(\pi)$$

for all $N \geq 1$ and $0 \leq n \leq N$.

Proof. We first consider the case $N = 1$. The result follows at once from Theorem 2.5.13, since $Y_1(\pi)$ is the fibred product of Y with itself. If we have established the result for some $N \geq 1$, it follows for $N + 1$ by again applying Theorem 2.5.13 and observing that $Y_{N+1}(\pi)$ is the fibred product of Y with $Y_N(\pi)$. □

The final part of this section deals with the special case of maps between two shifts of finite type. We begin with the following easy result.

Lemma 2.5.15. *Let $\pi : (\Sigma_G, \sigma) \to (X, \varphi)$ be a regular map. Suppose $N, k \geq 1$ and p is in $G_N^k(\pi)$.*

1. *If π is s-resolving and $t^k(p_i) = t^k(p_j)$ for some i, j, then $p_i = p_j$.*
2. *If π is u-resolving and $i^k(p_i) = i^k(p_j)$ for some i, j, then $p_i = p_j$.*

Proof. We prove the second part only. Let e be any point in $\Sigma_N(\pi)$ such that $e^{[1,k]} = p$. It follows that $i(e_i^1) = i^k(p_i) = i^k(p_j) = i(e_j^1)$, and so we may form $[e_i, e_j]$. As π is regular, we have $\pi([e_i, e_j]) = [\pi(e_i), \pi(e_j)] = \pi(e_i)$. As e_i and $[e_i, e_j]$ are stably equivalent and π is s-resolving, $e_i = [e_i, e_j]$. The same argument also shows $e_j = [e_i, e_j]$. We conclude that $p_i = p_j$ as desired. □

As we are discussing s/u-resolving maps between shifts of finite type, we describe a simple condition on the underlying graphs which is related.

Definition 2.5.16. *Let G and H be graphs. A graph homomorphism $\theta : H \to G$ is* left-covering *if it is surjective and, for every v in H^0, the map $\theta : t^{-1}\{v\} \to t^{-1}\{\theta(v)\}$ is a bijection. Similarly, π is* right-covering *if it is surjective and, for every v in H^0, the map $\theta : i^{-1}\{v\} \to i^{-1}\{\theta(v)\}$ is a bijection.*

The following result is obvious and we omit the proof.

Theorem 2.5.17. *If G and H are graphs and $\theta : H \to G$ is a left-covering (or right-covering) graph homomorphism, then the associated map $\theta : (\Sigma_H, \sigma) \to (\Sigma_G, \sigma)$ is an s-bijective (or u-bijective, respectively) factor map.*

The following is a result due to Kitchens - see Proposition 1 of [11] and the discussion just preceeding it.

Theorem 2.5.18. *Let $\pi : (\Sigma, \sigma) \to (\Sigma_G, \sigma)$ be an s-bijective (or u-bijective) factor map. There exists a graph H, a left-covering graph homomorphism (or right-covering graph homomorphism, respectively) $\theta : H \to G$ and a conjugacy $h : (\Sigma_H, \sigma) \to (\Sigma, \sigma)$ such that $\pi \circ h = \theta$.*

Remark 2.5.19. *At first glance, this would seem to indicate that, as long as we are concerned with s/u-bijective factor maps between shifts of finite type, we may effectively restrict our attention to ones coming from left-covering graph homomorphisms. We must warn readers that this is not correct. The result has some very serious limitations for our applications. Specifically, if* ρ *an s-bijective factor map defined on a shift of finite type* (Σ, σ)*, we will consider the maps* $\delta_n : (\Sigma_N(\rho), \sigma) \to (\Sigma_{N-1}(\rho), \sigma)$*. The last theorem states that we may recode* $\Sigma_N(\rho)$ *so that* δ_n *is left-covering. Unfortunately, to do so would destroy the presentation of* $\Sigma_N(\rho)$ *as* $N+1$*-tuples from* Σ*. If* G *is a graph which presents* Σ_G *such that* ρ *is regular, then in view of Theorem 2.4.5,* δ_n *is plainly given by a graph homomorphism (in our notation, it is also* δ_n*), but we cannot assume is left-covering.*

2.6 s/u-bijective pairs

The main motivation for this section is the seminal result of Bowen, which we state now.

Theorem 2.6.1. *Let* (X, φ) *be a non-wandering Smale space. There exists a shift of finite type,* (Σ, σ) *and a factor map*

$$\pi : (\Sigma, \sigma) \to (X, \varphi).$$

Moroever, these may be chosen such that π *is finite-to-one and one-to-one on a dense* G_δ *subset of* Σ*.*

The weakness of this result for us is that the map π does not necessarily possess nice properties such as the ones we studied in the last section. The most naive question, whether π might be chosen to be s-resolving can be seen to have a negative general answer, as follows. If (X, φ) is, say, a hyperbolic toral automorphism, then its local stable and unstable sets are Euclidean spaces, while those of Σ are totally disconnected. In this case, π cannot be s-bijective or u-bijective, by Theorem 2.5.14.

The next possibility is that π might be factored as a composition of s-bijective and u-bijective maps. This is indeed the case - see Corollary 2.6.7 - but we begin in a slightly different way. One can view Bowen's Theorem above as saying that the Smale space (X, φ) may be "disconnected" in some controlled way. Our improvement to this is to ask that we disconnect only the stable sets, while leaving the unstable sets unchanged. Of course, we can also reverse the roles of the local stable and unstable sets. This is the content of the following definition.

Definition 2.6.2. *Let (X, φ) be a Smale space. We say that $\pi = (Y, \psi, \pi_s, Z, \zeta, \pi_u)$ is an s/u-*bijective pair *for (X, φ) if*

1. *(Y, ψ) and (Z, ψ) are Smale spaces,*
2. *$\pi_s : (Y, \psi) \to (X, \varphi)$ is an s-bijective factor map,*
3. *$Y^u(y)$ is totally disconnected, for every y in Y,*
4. *$\pi_u : (Z, \zeta) \to (X, \varphi)$ is a u-bijective factor map,*
5. *$Z^s(z)$ is totally disconnected, for every z in Z.*

The main result of this section is the following. As we will explain shortly, it may be seen as an improved version of Bowen's Theorem.

Theorem 2.6.3. *If (X, φ) is a non-wandering Smale space, then there exists an s/u-bijective pair for (X, φ).*

Proof. By Theorem 2.1.13, it suffices to consider the case that (X, φ) is irreducible. We apply Corollary 1.4 of [29] or Corollary 1.5 of [15] to find an irreducible shift of finite type, (Σ, σ), an irreducible Smale space, (Y, ψ), and factor maps

$$\pi_1 : (\Sigma, \sigma) \to (Y, \psi), \quad \pi_2 : (Y, \psi) \to (X, \varphi)$$

where π_1 is u-resolving and π_2 is s-resolving. Since the systems are irreducible, π_1, π_2 are s-bijective and u-bijective by Theorem 2.5.8. Let $\pi_s = \pi_2$. As π_1 is u-bijective, the unstable classes in Y are homeomorphic to those in Σ, by Theorem 2.5.12, and hence totally disconnected.

A similar argument shows the existence of (Z, ζ, π_u). □

Definition 2.6.4. *Let $\pi = (Y, \psi, \pi_s, Z, \zeta, \pi_u)$ be an s/u-bijective pair for the Smale space (X, φ). For each $L, M \geq 0$, we define*

$$\begin{aligned}\Sigma_{L,M}(\pi) &= \{(y_0, \ldots, y_L, z_0, \ldots, z_M) \mid y_l \in Y, z_m \in Z, \\ &\quad \pi_s(y_l) = \pi_u(z_m), 0 \leq l \leq L < 0 \leq m \leq M\}\end{aligned}$$

For convenience, we also let $\Sigma(\pi) = \Sigma_{0,0}(\pi)$*, which is simply the fibred product of the spaces* Y *and* Z*. We let* $\rho_u(y,z) = y$ *and* $\rho_s(y,z) = z$ *denote the usual maps from* $\Sigma(\pi)$ *to* Y *and* Z *respectively.*

For all, $L, M \geq 0$*, we also define*

$$\sigma(y_0, \ldots, y_L, z_0, \ldots, z_M) = (\psi(y_0), \ldots, \psi(y_L), \zeta(z_0), \ldots, \zeta(z_M)),$$

for all $(y_0, \ldots, y_L, z_0, \ldots, z_M)$ *in* $\Sigma_{L,M}(\pi)$*.*

Finally, for $L \geq 1$ *and* $0 \leq l \leq L$*, we let* $\delta_{l,} : \Sigma_{L,M}(\pi) \to \Sigma_{L-1,M}(\pi)$ *be the map which deletes entry* y_l*. Similarly, for* $M \geq 1$ *and* $0 \leq m \leq M$*, we let* $\delta_{,m} : \Sigma_{L,M}(\pi) \to \Sigma_{L,M-1}(\pi)$ *be the map which deletes entry* z_m*.*

We note the following whose proof (like 2.4.4) is straightforward and omitted.

Proposition 2.6.5. *If* π *is an s/u-bijective pair for* (X, φ)*, then for all* $L, M \geq 0$*,* $(\Sigma_{L,M}(\pi), \sigma)$ *is a Smale space.*

In fact, we can say more.

Theorem 2.6.6. *If* π *is an s/u-bijective pair for* (X, φ)*, then for all* $L, M \geq 0$*,* $(\Sigma_{L,M}(\pi), \sigma)$ *is a shift of finite type.*

Proof. We begin with the case $L = M = 0$. The map $\rho_u : (\Sigma(\pi), \sigma) \to (Y, \psi)$ is u-bijective. Hence it is a homeomorphism on unstable sets 2.5.12. On the other hand, by definition, the unstable sets of (Y, ψ) are totally disconnected. A similar argument using Z in stead of Y shows that the stable sets of $\Sigma(\pi)$ are totally disconnected. It follows that $\Sigma(\pi)$ is totally disconnected and so, by Theorem 2.2.8, it is a shift of finite type.

For $L, M \geq$, the map sending $(y_0, \ldots, y_L, z_0, \ldots, z_M)$ to $((y_l, z_m))_{l,m}$ is an injection from $\Sigma_{L,M}(\pi)$ to $\Sigma(\pi)^{(L+1)(M+1)}$. It follows that $\Sigma_{L,M}(\pi)$ is also totally disconnected and hence a shift of finite type. □

Let us restate our main result (Theorem 2.6.3), so that it more closely resembles Bowen's result. It requires the additional hypothesis of non-wandering, simply because 2.6.3 does.

Corollary 2.6.7. *Let* (X, φ) *be a non-wandering Smale space. Then there exists a shift of finite type,* (Σ, σ)*, and a factor map*

$$\pi : (\Sigma, \sigma) \to (X, \varphi).$$

such that π *may be written as the composition of an s-bijective factor map with a u-bijective factor map and vice verse.*

The fact that Bowen's result (with the stronger hypothesis of non-wandering) is a consequence is a little bit misleading: the proof of Corollary 1.4 of [29] which is used in the proof of 2.6.3 uses Bowen's Theorem.

We may find a graph G such that $(\Sigma(\pi), \sigma)$ is conjugate to (Σ_G, σ). For a point (y, z) in $(\Sigma(\pi), \sigma)$, we denote by $e(y, z)$ the corresponding point in (Σ_G, σ). For a given integer k, we let $e^k(y, z)$ denote the kth entry of $e(y, z)$ which is an edge of G. We may find G satisfying the following definition.

Definition 2.6.8. *Let π be an s/u-bijective pair for (X, φ). We say that a graph G together with a conjugacy*

$$e : \Sigma(\pi) \to \Sigma_G$$

is a presentation *of π if, for any $(y, z), (y', z')$ in $\Sigma(\pi)$ with $t(e^0(y, z)) = t(e^0(y', z'))$, it follows that $[y, y'], [z, z']$ and $[\pi(y), \pi(y')]$ are all defined and*

$$[e(y, z), e(y', z')] = e([y, y'], [z, z']),$$

for all $(y, z), (y', z')$ in $\Sigma(\pi)$.

For $L, M \geq 0$, we define the graph $G_{L,M}$ as follows. The vertices are those $(L+1) \times (M+1)$ arrays of entries of G^0 obtained as $t(e^0(y_l, z_m)), 0 \leq l \leq L, 0 \leq m \leq M$, where $(y_0, \ldots, y_L, z_0, \ldots, z_M)$ is in $\Sigma_{L,M}(\pi)$. Similarly, the edge set $G^1_{L,M}$ is the set of those $(L+1) \times (M+1)$ arrays of entries of G^1 obtained as $e^0(y_l, z_m), 0 \leq l \leq L, 0 \leq m \leq M$, where $(y_0, \ldots, y_L, z_0, \ldots, z_M)$ is in $\Sigma_{L,M}(\pi)$. The definition of the maps i and t are obvious. The proof of the following Theorem is much the same as that of 2.4.5, and we omit it.

Theorem 2.6.9. *Let π be an s/u-bijective pair for (X, φ) and suppose that G is a presentation of π. Then for every $L, M \geq 0$,*

$$(\Sigma_{L,M}(\pi), \sigma) \cong (\Sigma_{G_{L,M}}, \sigma).$$

Although it is not needed now, it will be convenient for us to have other descriptions of these systems. Toward that end, we make the following additional definition.

Definition 2.6.10. *1. For each $L \geq 0$, let $\rho_{L,} : \Sigma_{L,0}(\pi) \to Y_L(\pi_s)$ be the map defined by*

$$\rho_{L,}(y_0, \ldots, y_L, z_0) = (y_0, \ldots, y_L).$$

2. *For each $M \geq 0$, let $\rho_{,M} : \Sigma_{0,M}(\pi) \to Z_M(\pi_u)$ be the map defined by*

$$\rho_{,M}(y_0, z_0, \ldots, z_M) = (z_0, \ldots, z_M).$$

Theorem 2.6.11. *Let π be an s/u-bijective pair for (X, φ).*

1. *For all $L \geq 0$, $\rho_{L,}$ is a u-bijective factor map.*
2. *For all $M \geq 0$, $\rho_{,M}$ is an s-bijective factor map.*

Proof. We prove the first statement only. It is clear that $\rho_{L,}$ is continuous and intertwines the dynamics. We check that it is onto. Let $(y_0, \ldots, y_L)$ be in $Y_L(\pi_s)$. As π_u is onto, we may find z in Z such that $\pi_u(z) = \pi_s(y_0)$. Then $(y_0, \ldots, y_L, z)$ is in $\Sigma_{L,0}(\pi)$ and its image under $\rho_{L,}$ is $(y_0, \ldots, y_L)$.

Next, we check that $\rho_{L,}$ is u-resolving. Suppose that $(y_0, \ldots, y_L, z_0)$ and $(y_0', \ldots, y_L', z_0')$ are unstably equivalent and have the same image under $\rho_{L,}$. The first fact implies, in particular, that z_0 and z_0' are unstably equivalent. The second fact just means that $(y_0, \ldots, y_L) = (y_0', \ldots, y_L')$. Since the points are in $\Sigma_{L,0}$, we also have $\pi_u(z_0) = \pi_s(y_0) = \pi_s(y_0') = \pi_u(z_0')$. As π_u is u-resolving, we conclude that $z_0 = z_0'$.

Finally, we check that $\rho_{L,}$ is u-bijective. Suppose that $(y_0, \ldots, y_L, z_0)$ is in $\Sigma_{L,0}$ and $(y_0', \ldots, y_L')$ in $Y_L(\pi_s)$ is unstably equivalent to $\rho_{L,}(y_0, \ldots, y_L, z_0)$. It follows that y_0 is unstably equivalent to y_0' and hence $\pi_u(z_0) = \pi_s(y_0)$ is unstably equivalent to $\pi_s(y_0')$. As π_u is u-bijective, we may find z_0' in Z unstably equivalent to z_0 and with $\pi_u(z_0') = \pi_s(y_0')$. It follows that $(y_0', \ldots, y_L', z_0')$ is in $\Sigma_{L,0}$, is unstably equivalent to $(y_0, \ldots, y_L, z_0)$ and has image $(y_0', \ldots, y_L')$ under $\rho_{L,}$. □

Consider now the factor map $\rho_{L,} : \Sigma_{L,0}(\pi) \to Y_L(\pi_s)$. We are free to take self-products of this map with itself as in Definition 2.4.3. That is, let $M \geq 0$ and consider $M + 1$ points in $\Sigma_{L,0}(\pi)$ which all have the same image under $\rho_{L,}$. Using our earlier notation, this set is written as $(\Sigma_{L,0}(\pi))_M(\rho_{L,})$. Each element of such an $M + 1$-tuple has the form $(y_0, \ldots, y_L, z_0)$ and the condition that they have the same image under $\rho_{L,}$ simply means that the $y_0, \ldots, y_L$ entries of each one are all the same. Thus, we could list them as $(y_0, \ldots, y_L, z_0), (y_0, \ldots, y_L, z_1), \ldots, (y_0, \ldots, y_L, z_M)$. It is then easy to see that $(y_0, \ldots, y_L, z_0, \ldots, z_M)$ is actually in $\Sigma_{L,M}$.

What we have described is a bijection between $(\Sigma_{L,0}(\pi))_M(\rho_{L,})$ and $\Sigma_{L,M}$. It is an easy matter to see it is invariant for the actions and a homeomorphism. Its biggest complication would be in writing it explicitly, which we avoid. The same kind of analysis applies to $(\Sigma_{0,M}(\pi))_L(\rho_{,M})$. We have proved the following.

Theorem 2.6.12. *For any $L, M \geq 0$, we have*

$$(\Sigma_{L,0}(\pi))_M(\rho_{L,}) = \Sigma_{L,M}(\pi) = (\Sigma_{0,M}(\pi))_L(\rho_{,M}).$$

This result has the following easy consequence (although it could also be proved directly earlier). It will be crucial in our development of the homology theory in Chapter 5.

Theorem 2.6.13. *Let π be an s/u-bijective pair for (X, φ).*

1. *For all $L \geq 1$, $M \geq 0$ and $0 \leq l \leq L$, the map $\delta_{l,} : \Sigma_{L,M}(\pi) \to \Sigma_{L-1,M}(\pi)$ which deletes y_l from $(y_0, \ldots, y_L, z_0, \ldots, z_M)$ is an s-bijective factor map.*

2. *For all $L \geq 0$, $M \geq 1$ and $0 \leq m \leq M$, the map $\delta_{,m} : \Sigma_{L,M}(\pi) \to \Sigma_{L,M-1}(\pi)$ which deletes z_m from $(y_0, \ldots, y_L, z_0, \ldots, z_M)$ is a u-bijective factor map.*

Proof. The first statement follows immediately from the second equality of Theorem 2.6.12, the fact that $\rho_{,M}$ is s-bijective in Theorem 2.6.11 and Theorem 2.5.14. The second statement uses the first equality of Theorem 2.6.12 and the same two other results. □

2.7 Technical results

In this section, we assemble a number of technical results which will be crucial in our arguments later on. These fall into two groups. Let us take a moment to motivate the first.

Recall that Theorem 2.5.12 states that an s-bijective factor map is actually a homeomorphism on stable sets. We wish to establish analogues of this result for factor maps between shifts of finite type. Of course, the statements look rather different - one sees integer parameters instead of ϵ's and δ's, but this is typical in symbolic dynamics. In doing so, we obtain slightly stronger versions of 2.5.12; the continuity is uniform in some sense.

We begin considering a pair of graphs and a graph homomorphism between them and the associated factor map between shifts of finite type. If the graph homomorphism is left-covering, then our estimates are both simpler to state and easier to prove. But we remind the reader that the case of the a homomorphism $\delta_n : G_N(\pi) \to G_{N-1}(\pi)$ cannot be assumed to have this property.

Lemma 2.7.1. *Let G, H be graphs and $\theta : H \to G$ be a graph homomorphism.*

1. *If the associated map on the shift spaces is an s-resolving factor map, then there is a constant $K_\theta \geq 0$ such that, if e, f are in Σ_H and are stably equivalent and k_0 is an integer such that $\theta(e)^k = \theta(f)^k$, for all $k \geq k_0$, then $e^k = f^k$, for all $k \geq k_0 + K_\theta$.*
2. *If θ is left-covering, then $K_\theta = 0$ will satisfy the condition of part 1.*
3. *If the associated map on the shift spaces is a u-resolving factor map, then there is a constant $K_\theta \geq 0$ such that, if (e, f) are in Σ_H and are unstably equivalent and k_0 is an integer such that $\theta(e)^k = \theta(f)^k$, for all $k \leq k_0$, then $e^k = f^k$, for all $k \leq k - K_\theta$.*
4. *If θ is right-covering, then $K_\theta = 0$ will satisfy the condition of part 3.*

Proof. It obviously suffices to prove the first and second statements only. We consider the first. If it is false, then for every K we may find a stably equivalent pair of points satisfying the hypothesis, but not the conclusion. This means that they are distinct, but stably equivalent so we may shift them so that they are equal in all positive entries and different at zero. We let e_K, f_K denote these points after applying a suitable power of σ so that $e_K^0 \neq f_K^0$, while $e_K^k = f_K^k$, for all $k \geq 1$. The fact that the points satisfied the hypothesis means that $\theta(e)^k = \theta(f)^k$, for all $k \geq -K$. After passing to a subsequence, we may assume these have limit points e, f, respectively. It follows immediately that $e^0 \neq f^0$, $e^k = f^k$ for all $k \geq 1$ and $\theta(e)^k = \theta(f)^k$, for all integers k. This contradicts θ being s-resolving.

The proof of the second statement is clear from the fact that θ is injective on $t^{-1}\{v\}$, for every vertex v in H^0. □

The next result is, in some way, a more general form of the last one in that there is no requirement that the range be a shift of finite type.

Lemma 2.7.2. *Let G be a graph, (X, φ) be a Smale space and $\rho : (\Sigma_G, \sigma) \to (X, \varphi)$ be a regular, s-resolving factor map. There exists a constant $K_\rho \geq 0$*

such that, for e_0, e_1, f_0, f_1 in Σ_G and k_0 such that $\rho(e_0) = \rho(e_1)$, $\rho(f_0) = \rho(f_1)$, $e_0^k = f_0^k$, for all $k \geq k_0$ and e_1, f_1 are stably equivalent, we have $e_1^k = f_1^k$, for all $k \geq k_0 + K_\rho$.

In the special case that (X, φ) is a shift of finite type associated with some graph and ρ is induced by a graph homomorphism θ, then the constant K_θ of Lemma 2.7.1 satisfies the conclusion.

Proof. As before, we let $\Sigma_1(\rho)$ denote the fibred product of the map ρ with itself. It is a shift of finite type. Let $\delta_0(e, f) = e$ be the usual factor map from $\Sigma_1(\rho)$ onto Σ_G. It is s-resolving by Theorem 2.5.13. The result follows at once from considering $e = (e_0, e_1)$ and $f = (f_0, f_1)$ in Σ and using $K_\rho = K_{\delta_0}$. The second statement is trivial and we omit the details. □

As we see from the proof above, there is a close relation between the constant K_ρ of a factor map $\rho : (\Sigma_G, \sigma) \to (X, \varphi)$ and that of the factor map $\delta_0 : (\Sigma_1(\rho), \sigma) \to (\Sigma_G, \sigma)$. In fact, the same relation exists for all higher self-products of (Σ_G, σ), as follows. The proof is straightforward and we omit it.

Lemma 2.7.3. *Let G be a graph, (X, φ) be a Smale space and $\rho : (\Sigma_G, \sigma) \to (X, \varphi)$ be a regular, s-resolving factor map. If K satisfies the condition of Lemma 2.7.2 for ρ, then for all $N \geq 1$ and $0 \leq n \leq N$, it also satisfies the conclusion of Lemma 2.7.1 for the map $\delta_n : (\Sigma_N(\rho), \sigma) \to (\Sigma_{N-1}(\rho), \sigma)$.*

Lemma 2.7.4. *Let G and H be graphs and (X, φ) be a Smale space. Suppose that $\theta : H \to G$ is a graph homomorphism such that the induced map $\theta : (\Sigma_H, \sigma) \to (\Sigma_G, \sigma)$ is an s-resolving factor map. Also suppose that $\rho : (\Sigma_G, \sigma) \to (X, \varphi)$ is an s-resolving factor map. If K_θ and K_ρ are constants satisfying the conclusion of Lemma 2.7.2 for θ and ρ, respectively, then $K_\theta + K_\rho$ satisfies 2.7.2 for the map $\rho \circ \theta$.*

Proof. Let e_0, e_1, f_0, f_1 be in Σ_H and satisfy $\rho \circ \theta(e_0) = \rho \circ \theta(e_1)$, $\rho \circ \theta(f_0) = \rho \circ \theta(f_1)$, $e_0^k = f_0^k, k \geq k_0$ and $e_1^k = f_1^k, k \geq k_0 + K_\theta + K_\rho$. We apply the hypothesis on the map ρ to the four points $\theta(e_0), \theta(e_1), \theta(f_0), \theta(f_1)$. The second part of 2.7.2 implies that $\theta(e_1)^k = \theta(f_1)^k, k \geq k_0 + K_\theta$. One more application of the second part of 2.7.2, this time to θ, yields the result. □

The next technical result will be used frequently. Again, we consider a graph homomorphism and its associated maps between shifts of finite type. Under the hypothesis that this map is an s-bijective factor map, we know from Theorem 2.5.12 that it is a homeomorphism on stable sets. We need to have a local version of the surjectivity and injectivity; i.e. a result that applies to the maps between finite paths in the graphs. It is provided below. Notice that the first and third statements are existence results (surjectivity), while the second and fourth are uniqueness statements (injectivity).

Lemma 2.7.5. *Let G, H be graphs and $\theta : H \to G$ be a graph homomorphism.*

1. *Suppose the factor map $\theta : (\Sigma_H, \sigma) \to (\Sigma_G, \sigma)$ is s-bijective and suppose K satisfies the condition of Lemma 2.7.1. For any $k_0 \geq K$, $k_1 \geq 0$, p in $G^{k_0+k_1}$ and q in H^{k_0} satisfying $\theta(q) = t^{k_1}(p)$, there exists q' in $H^{k_0+k_1}$ such that $t^{k_1+K}(q') = t^K(q)$ and $\theta(q') = p$.*

2. *Suppose the factor map $\theta : (\Sigma_H, \sigma) \to (\Sigma_G, \sigma)$ is s-bijective and suppose K satisfies the condition of Lemma 2.7.1. For any $k_0 \geq K$, if q, q' are in H^k and satisfy $\pi(q) = \pi(q')$ and $t^{k_0}(q) = t^{k_0}(q')$, then $t^K(q) = t^K(q')$.*

3. *Suppose the factor map $\theta : (\Sigma_H, \sigma) \to (\Sigma_G, \sigma)$ is u-bijective and suppose K satisfies the condition of Lemma 2.7.1. For any $k_0 \geq K$, $k_1 \geq 0$, p in $G^{k_0+k_1}$ and q in H^{k_0} satisfying $\theta(q) = i^{k_1}(p)$, there exists q' in $H^{k_0+k_1}$ such that $i^{k_1+K}(q') = i^K(q)$ and $\theta(q') = p$.*

4. *Suppose the factor map $\theta : (\Sigma_H, \sigma) \to (\Sigma_G, \sigma)$ is u-bijective and suppose K satisfies the condition of Lemma 2.7.1. For any $k_0 \geq K$, if q, q' are in H^k and satisfy $\pi(q) = \pi(q')$ and $i^{k_0}(q) = i^{k_0}(q')$, then $i^K(q) = i^K(q')$.*

Proof. We prove the first and second statements only. For the first, choose e in Σ_G such that $e^{[1-k_1,k_0]} = p$ and f in Σ_H such that $f^{[1,k_0]} = q$. It follows that $e' = [\theta(f), e]$ is defined and is stably equivalent to $\theta(f)$. As θ is s-bijective and e' is stably equivalent to $\theta(f)$, we may choose f' in Σ_H stably equivalent to f such that $\theta(f') = e'$. For all $k \geq 1$, we have $\theta(f')^k = \theta(f)^k$. It follows from Lemma 2.7.1 that $(f')^k = f^k$, for $k \geq 1 + K$. Let $q' = (f')^{[1-k_1,k_0]}$. and so $t^{k_1+K}(q') = t^K(q)$. For $1 - k_1 \leq k \leq 0$, we have $\theta(f')^k = (e')^k = e^k$, while for $1 \leq k \leq k_0$, we have $\theta(f')^k = \theta(f)^k$. It follows that $\theta(q') = p$ as desired.

For the second statement, choose f and f' from Σ_H so that $f^{[1-k_0,0]} = q$ and $f'^{[1-k_0,0]} = q'$. Since $t^{k_0}(q) = t^{k_0}(q')$, we know that $\bar{f} = [f, f']$ is defined and $\bar{f}^k = f^k$, for $k \geq 1$. Now consider $1 - k_0 \leq k \leq 0$. We have

$$\theta(\bar{f}^k) = \theta(f'^k) = \theta(q'^{k+k_0}) = \theta(q^{k+k_0}) = \theta(f^k).$$

Thus, $\theta(\bar{f})^k = \theta(f)^k$, for $k \geq 1 - k_0$ and it follows from Lemma 2.7.1 that $\bar{f}^k = f^k$, for $1 \geq 1 - k_0 + K$. This implies the desired conclusion. □

This brings us to the second set of technical results. These are concerned with the action of $S_{L+1} \times S_{M+1}$ of the system $\Sigma_{L,M}(\pi)$ and the graph $G_{L,M}(\pi)$, associated with an s/u-bijective pair.

It is fairly easy to see that, given an action of $S_{L+1} \times S_{M+1}$ on a set, the subgroups $S_{L+1} \times 1$ and $1 \times S_{M+1}$ may have trivial isotropy at a point, without the whole action doing so. (The reader may consider the case of $L = M = 1$ acting on the set of 2×2 matrices and the identity matrix.) However, it is clear that this does not occur for the actions on $\Sigma_{L,M}(\pi)$. Roughly speaking, we wish to see this special property is present for the actions of the graphs $G^K_{L,M}$, provided K is sufficiently large.

To begin, we want to establish analogue of the fairly simple result for s/u-resolving maps in Lemma 2.5.15 for s/u-bijective pairs.

Lemma 2.7.6. *Let $\pi = (Y, \psi, \pi_s, Z, \zeta, \pi_u)$ be an s/u-bijective pair for the Smale space (X, φ). Let G be a graph which presents π. There exists $K_0 \geq 0$ such that if $(y_0, \ldots, y_L, z_0, \ldots, z_M)$ is in $\Sigma_{L,M}(\pi)$ and satisfies $e^k(y_{l_1}, z_{m_1}) = e^k(y_{l_2}, z_{m_2})$ for some $0 \leq l_1, l_2 \leq L$, some $0 \leq m_1, m_2 \leq M$ and all $1 - K_0 \leq k \leq K_0$, then*

1. *$e^k(y_{l_1}, z_m) = e^k(y_{l_2}, z_m)$, for all $k \leq 0$ and $0 \leq m \leq M$, and*
2. *$e^k(y_l, z_{m_1}) = e^k(y_l, z_{m_2})$, for all $k \geq 0$ and $0 \leq l \leq L$.*

Proof. We use the continuity of the map $t \circ e^0 : \Sigma(\pi) \to G^0$ to assert that there is a constant $0 < \epsilon < \epsilon_Y, \epsilon_Z$ such that $d(y, y'), d(z, z') < \epsilon$ implies $t(e^0(y, z)) = t(e^0(y', z'))$. Next, we choose $K_0 \geq 1$ such that $d(e(y, z), e(y', z')) \leq 2^{-K_0}$ implies $d(y, y'), d(z, z') < \epsilon$. Then with $(y_0, \ldots, y_L, z_0, \ldots, z_M)$ as above, we prove the first statement only. Let $0 \leq m \leq M$ be fixed. It follows from the hypothesis and our choice of K_0 that $[y_{l_1}, y_{l_2}]$ is defined and $\pi_s([y_{l_1}, y_{l_2}]) = [\pi_s(y_{l_1}), \pi_s(y_{l_2})] = \pi_s(y_{l_1})$. As π_s is s-bijective, it follows that $[y_{l_1}, y_{l_2}] = y_{l_1}$. It also follows from our choice of ϵ that $t(e^0(y_{l_1}, z_m)) = t(e^0(y_{l_2}, z_m))$ and so we may compute, for $k \leq 0$,

$$e^k(y_{l_2}, z_m) = [e(y_{l_1}, z_m), e(y_{l_2}, z_m)]^k = e^k([y_{l_1}, y_{l_2}], [z_m, z_m]) = e^k(y_{l_1}, z_m).$$

□

Proposition 2.7.7. *Let $\pi(Y, \psi, \pi_s, Z, \zeta, \pi_u)$ be an s/u-resolving pair for the Smale space (X, φ) and let G be a graph which presents π. Suppose that $K \geq 2K_0$, as in Lemma 2.7.6 and $L, M \geq 0$.*

1. *If p is in $G^K_{L,M}$ satisfies $p_{l_1,m_1} = p_{l_2,m_2}$ for some $0 \leq l_1, l_2 \leq L$ and $0 \leq m_1, m_2 \leq M$, then $i^{K_0}(p)\cdot((l_1\, l_2), 1) = i^{K_0}(p)$ and $t^{K_0}(p)\cdot(1, (m_1\, m_2)) = t^{K_0}(p)$.*

2. *If p is in $G^K_{L,M}$ and β is in S_{M+1} satisfy $p_{l_1,m} = p_{l_2,\beta(m)}$, for some l_1, l_2 and all $0 \leq m \leq M$, then $p_{l_1,m} = p_{l_2,m}$, for all $0 \leq m \leq M$.*

3. *If p is in $G^K_{L,M}$ and α is in S_{L+1} satisfy $p_{l,m_1} = p_{\alpha(l),m_2}$, for some m_1, m_2 and all $0 \leq l \leq L$, then $p_{l,m_1} = p_{l,m_2}$, for all $0 \leq l \leq L$.*

4. *If p is in $G^K_{L,M}$, α is in S_{L+1} and β is in S_{M+1} satisfy $p \cdot (\alpha, \beta) = p$, then*
$$p \cdot (\alpha, 1) = p \cdot (1, \beta) = p.$$

5. *For p in $G^{K+1}_{L,M}$, if $t(p)\cdot(\alpha, 1) = t(p)$, for some α in S_{L+1}, then $p\cdot(\alpha, 1) = p$.*

6. *For p in $G^{K+1}_{L,M}$, if $i(p)\cdot(1, \beta) = i(p)$, for some β in S_{M+1}, then $p\cdot(1, \beta) = p$.*

Proof. For the first four parts, we begin by choosing $(y_0, \ldots, y_L, z_0, \ldots, z_M)$ in $G^K_{L,M}$ such that
$e^{[1-K_0,K-K_0]}(y_l, z_m) = p_{l,m}$, for all l, m. The first statement follows at once from the last Lemma.

For the second part, it follows from Lemma 2.7.6 that $e^k(y_{l_1}, z_m) = e^k(y_{l_2}, z_m)$ for all $k \leq$ and all m. For $k \geq 1$, the same result states that, for any m, $e^k(y_{l_2}, z_m) = e^k(y_{l_2}, z_{\beta(m)})$. In addition, for $1 - K_0 \leq k \leq K_0$, we also know $e^k(y_{l_2}, z_{\beta(m)}) = e^k(y_{l_1}, z_m)$. Putting these last two together, we have $e^k(y_{l_2}, z_m) = e^k(y_{l_1}, z_m)$, for $1 \leq k \leq K_0$ and all m. Together with the first step, we have the desired conclusion

The proof of the third part is analogous to that of the second. The fourth part follows from the second using $l_1 = l$ and $l_2 = \alpha(l)$ and the third using $m_1 = m$ and $m_2 = \beta(m)$.

For the fifth part, we begin by choosing $(y_0, \ldots, y_L, z_0, \ldots, z_M)$ in $G^K_{L,M}$ such that
$e^{[-K_0,K-K_0]}(y_l, z_m) = p_{l,m}$, for all l, m. For any l, we apply Lemma 2.7.6

with $l_1 = l, l_2 = \alpha(l)$ and any $m_1 = m_2 = m$. It follows that $e^k(y_l, z_m) = e^k(y_{\alpha(l)}, z_m)$ with $k \leq 0$. Applying $k = -K$ yields the result. The proof of the last statement is analogous to the fifth and we omit it. $\square$

Chapter 3

Dimension groups

In this chapter, we present background material on Krieger's theory of dimension group invariants for shifts of finite type.

The first section presents some very simple observations on free abelian groups. The second discusses two abelian groups, $D^s(G)$ and $D^u(G)$, associated with a graph G. The third section outlines Krieger's invariants, $D^s(\Sigma, \sigma)$ and $D^u(\Sigma, \sigma)$ associated with a shift of finite type. The main result here is Krieger's, that if G is a graph, then the invariant $D^s(\Sigma_G, \sigma)$ is isomorphic to that of the graph, $D^s(G)$. Of course, we establish explicit isomorphisms between the two.

The reason we have two different presentations for the invariant is simple: described as $D^s(\Sigma, \sigma)$, it is clearly dependent only on the dynamics, while the description as $D^s(G)$ gives an effective tool for its computation. This becomes particularly important in the next sections where we consider the properties of these invariants as functors. That is, when $\pi : (\Sigma', \sigma) \to (\Sigma, \sigma)$ is a map between shifts of finite type, does it induce group homomorphisms between the invariants? The answer is subtle: for s-bijective maps, D^s is covariant while D^u is contravariant. The fourth section considers the covariant situation, while the fifth section deals with the contravariant case. The other important results here are to establish concrete formulae for the group homomorphisms in terms of $D^s(G)$ and $D^u(G)$.

3.1 Free abelian groups

In this section, we establish some very simple ideas about free abelian groups and homomorphisms between them. Simply put, we are transforming combinatorial objects to algebraic ones.

Let A be any (finite) set. The free abelian group on A, denoted by $\mathbb{Z}A$ is the set of all formal integral combinations of elements of A. It is isomorphic to $\mathbb{Z}^A$, but it will be most convenient for us to regard A as being a subset of the group (which is not so convenient in $\mathbb{Z}^A$). Its main feature is that any function $\alpha : A \to G$, where G is an abelian group, has a unique extension to a group homomorphism $\alpha : \mathbb{Z}A \to G$.

We will also make use of the following notation: for a subset C of A, we let

$$Sum(C) = \sum_{a \in C} a.$$

Some care must be taken. For example, if $\alpha : A \to B$ is any function between finite sets and C is a subset of A, $\alpha(Sum(C))$ and $Sum(\alpha(C))$ are not equal in general if α is not one-to-one.

For a finite set A, we introduce an integer-valued bilinear form $<,>_A$ on the group $\mathbb{Z}A$ by setting

$$< a, b >= \begin{cases} 1 & \text{if } a = b, \\ 0 & \text{if } a \neq b \end{cases}$$

for any a, b in A and extend to pairs in $\mathbb{Z}A$ so that it is additive in each variable. This is non-degenerate in the sense that, for any a in $\mathbb{Z}A$, if $< a, b >= 0$, for all b in $\mathbb{Z}A$, then $a = 0$.

If A and B are two finite sets and $h : \mathbb{Z}A \to \mathbb{Z}B$ is any homomorphism, there is a unique homomorphism $h^* : \mathbb{Z}B \to \mathbb{Z}A$ such that

$$< h(a), b >_B = < a, h^*(b) >_A,$$

for all a in $\mathbb{Z}A$ and b in $\mathbb{Z}B$. If h_1 and h_2 are two homomorphisms (with appropriate domains), then $(h_1 \circ h_2)^* = h_2^* \circ h_1^*$.

The following result is a trivial consequence of the definitions and we state it without proof.

Lemma 3.1.1. *Let $\alpha : A \to B$ be any function. Also denote by α its unique extension to a group homomorphism from $\mathbb{Z}A$ to $\mathbb{Z}B$. Then the function*

$\alpha^* : \mathbb{Z}B \to \mathbb{Z}A$ satisfies

$$\alpha^*(b) = \sum_{a \in \alpha^{-1}\{b\}} a = Sum(\alpha^{-1}\{b\})$$

for all b in B.

3.2 The dimension group of a graph

Expanding on the ideas of the last section, we construct two abelian groups, denoted $D^s(G)$ and $D^u(G)$, from a graph G. These are called the dimension groups associated with G. After presenting the definition, we will observe one result: that the dimension groups associated with the graphs $G^K, K \geq 1$ from Definition 2.2.2 are all naturally isomorphic to those of G.

Let G be a graph. First, consider $\mathbb{Z}G^0$, the free abelian group on the generating set G^0 and we define a homomorphism

$$\gamma_G^s : \mathbb{Z}G^0 \to \mathbb{Z}G^0$$

by

$$\gamma_G^s(v) = \sum_{t(e)=v} i(e) = i \circ t^*(v),$$

for all v in G^0. If no confusion will arise, we suppress the subscript G. We then let $D^s(G)$ be the inductive limit of the system

$$\mathbb{Z}G^0 \xrightarrow{\gamma^s} \mathbb{Z}G^0 \xrightarrow{\gamma^s} \cdots$$

The usual definition is in terms of a universal property, but we will provide a more concrete definition since it will be more useful.

On the set $\mathbb{Z}G^0 \times \mathbb{N}$, we define $(a, m) \sim_s (b, n)$ if there exists $l \geq 0$ such that $(\gamma_G^s)^{n+l}(a) = (\gamma_G^s)^{m+l}(b)$. It is easy to see this is an equivalence relation. It is also easy to see that this is the equivalence relation generated by $(a, m) \sim (\gamma_G^s(a), m + 1)$, for (a, m) in $\mathbb{Z}G^0 \times \mathbb{N}$. For (a, m) in $\mathbb{Z}G^0 \times \mathbb{N}$, we denote its equivalence class by $[a, m]_s$.

The second dimension group associated with G is obtained by replacing the map γ_G^s by

$$\gamma_G^u(v) = \sum_{i(e)=v} t(e) = t \circ i^*(v)$$

and taking inductive limits. We denote the result by $D^u(G)$. It is immediate that this coincides with the dimension group of the opposite graph $G^{op} = (G^0, G^1, t, i)$.

The equivalence relation $\sim_u$ is defined replacing γ^s_G by γ^u_G and the equivalence class of (a, m) is denoted by $[a, m]_u$. If no confusion will arise, we drop the subscripts.

Definition 3.2.1. *Let G be a graph. The set of equivalence class in $\mathbb{Z}G^0 \times \mathbb{N}$ in the relation $\sim_s$ is denoted $D^s(G)$. It is an abelian group with the operation $[a, m]_s + [b, m]_s = [a + b, m]_s$, for all a, b in $\mathbb{Z}G^0$ and m in $\mathbb{N}$. The group $D^u(G)$ is obtained in analogous way, using the relation $\sim_u$.*

We remark that an important aspect in Krieger's theory is that the group $\mathbb{Z}G^0$ has a natural order structure: the map γ^s is positive and the inductive limit as actually taken in the category of ordered abelian groups. (For more on this, see [23].) For our purposes here, we will ignore the order structure, but it should presumably feature in further developments.

Of course, the same construction may be applied to any of the graphs $G^K, K \geq 1$, described in Definition 2.2.2. We recall that the vertex set of G^K is G^{K-1} and so $D^s(G^K)$ is the inductive limit of the system

$$\mathbb{Z}G^{K-1} \stackrel{\gamma^s}{\to} \mathbb{Z}G^{K-1} \stackrel{\gamma^s}{\to} \mathbb{Z}G^{K-1} \stackrel{\gamma^s}{\to} \cdots$$

where the map $\gamma^s = \gamma^s_{G^K} = i \circ t^*$.

It is worth noting the following easy result.

Lemma 3.2.2. *For $K \geq 1$, as maps defined on $\mathbb{Z}G^K$, we have*

$$\begin{aligned} i \circ t^* &= t^* \circ i, \\ t \circ i^* &= i^* \circ t. \end{aligned}$$

For $K > j$, as maps defined on $\mathbb{Z}G^K$, we have

$$\begin{aligned} (\gamma^s)^j &= i^j \circ t^{j*} = t^{j*} \circ i^j, \\ (\gamma^u)^j &= t^j \circ i^{j*} = i^{j*} \circ t^j. \end{aligned}$$

Proof. Let p be in G^K. By definition $i \circ t^* = \sum_{t(q)=p} i(q)$, while $t^* \circ i(p) = \sum_{t(q)=i(p)} q$. We claim that $i : \{q \mid t(q) = p\} \to \{q' \mid t(q') = i(p)\}$ is a bijection. Since we suppose $K \geq 1$, if q is such that $t(q) = p$, then $t(i(q)) = i(t(q)) = i(p)$. Moreover, the map sending $q^1 \cdots q^k$ to $q^1 \cdots q^K p^K$

is the inverse of i and this establishes the claim. The conclusion follows at once from this.

The second part is proved in the same way and the last two statements are easy applications of the first two. □

For a fixed graph G, its higher block presentations, $G^K, K \geq 1$, all have the same D^s and D^u invariants , stated precisely as follows. The proof is left to the reader.

Theorem 3.2.3. *Let G be a graph and let $K, K' \geq 0$.*

1. *The maps sending $[a, j]$ in $D^s(G^K)$ to $[t^{K'*}(a), j]$ in $D^s(G^{K+K'})$ and $[b, j]$ in $D^s(G^{K+K'})$ to $[i^{K'}(b), j + K']$ in $D^s(G^K)$ are inverse to each other and hence implement inverse isomorphisms between the groups $D^s(G^K)$ and $D^s(G^{K+K'})$.*

2. *The maps sending $[a, j]$ in $D^u(G^K)$ to $[i^{K'*}(a), j]$ in $D^u(G^{K+K'})$ and $[b, j]$ in $D^u(G^{K+K'})$ to $[t^{K'}(b), j + j]$ in $D^u(G^K)$ are inverse to each other and hence implement inverse isomorphisms between the groups $D^u(G^K)$ and $D^u(G^{K+K'})$.*

3.3 The dimension group of a shift of finite type

Let (Σ, σ) be a shift of finite type. We will assign to (Σ, σ) two abelian groups, denoted $D^s(\Sigma, \sigma)$ and $D^u(\Sigma, \sigma)$. This construction is to due to Krieger [22] but also see [23] or [10] for other treatments. Once again, these are actually ordered abelian groups, but we will not concern ourselves with the order structure here.

Let $CO^s(\Sigma, \sigma)$ be the collection of all non-empty, compact, open subsets of $\Sigma^s(e)$, over all e in Σ. Notice that if E is in $CO^s(\Sigma, \sigma)$, then any non-empty subset which is compact and open relative to E is also in $CO^s(\Sigma, \sigma)$. Let $\sim$ be the smallest equivalence relation on $CO^s(\Sigma, \sigma)$ such that $E \sim F$ if $[E, F] = E$ and $[F, E] = F$ (meaning that both sets are defined) and such that $E \sim F$ if and only if $\sigma(E) \sim \sigma(F)$. We let $[E]$ denote the equivalence class of E.

Suppose that G is a graph. For e in Σ_G and $K \geq 0$, recall from Lemma 2.2.6 that

$$\Sigma^s_G(e, 2^{-K}) = \{f \in \Sigma_G \mid e^k = f^k, \text{ for all } k \geq 1 - K\}.$$

It is easy to see that such sets are in $CO^s(\Sigma_G, \sigma)$ and that if e, e' are in Σ_G and satisfy $i(e^{1-K}) = i(e'^{1-K})$ then $\Sigma^s_G(e, 2^{-K}) \sim \Sigma^s_G(e', 2^{-K})$. Moreover, every set in $CO^s(\Sigma_G, \sigma)$ may be expressed as a finite union of such sets (allowing both e and K to vary).

In a similar way, we let $CO^u(\Sigma, \sigma)$ be the collection of all non-empty, compact, open subsets of $\Sigma^u(e)$. Let $\sim$ be the smallest equivalence relation on $CO^u(\Sigma, \sigma)$ such that $E \sim F$ if $[E, F] = F$ and $[F, E] = E$ (meaning that both sets are defined) and such that $E \sim F$ if and only if $\sigma(E) \sim \sigma(F)$. We let $[E]$ denote the equivalence class of E.

If G is a graph, for e in Σ_G and $K \geq 0$, recall that

$$\Sigma^u_G(e, 2^{-K}) = \{f \in \Sigma_G \mid e^k = f^k, \text{ for all } k \leq K\}.$$

It is easy to see that such sets are in $CO^u(\Sigma_G, \sigma)$ and that if e, e' are in Σ_G and satisfy $t(e^K) = t(e'^K)$ then $\Sigma^u_G(e, 2^{-K}) \sim \Sigma^u_G(e', 2^{-K})$.

Remark 3.3.1. *Krieger's original definition appears slightly differently. There, the clopen subsets of a stable set are considered equivalent if there exists an element of an ample group which moves one to another. For the case of the past and future dimension groups considered in section 4 of [22], this is exactly the same as the equivalence relation we introduced above. A second small variation is the following. Krieger defines addition of clopen sets by means of taking unions. This requires some type of aperiodicity assumption which we avoid by taking free abelian groups in the following.*

Definition 3.3.2. *Let (Σ, σ) be a shift of finite type. The group $D^s(\Sigma, \sigma)$ is defined to be the free abelian group on the $\sim$-equivalence classes of $CO^s(\Sigma, \sigma)$, modulo the subgroup generated by $[E \cup F] - [E] - [F]$, where E, F and $E \cup F$ are in $CO^s(\Sigma, \sigma)$ and E and F are disjoint.*

The group $D^u(\Sigma, \sigma)$ is defined to be the free abelian group on the $\sim$-equivalence classes of $CO^u(\Sigma, \sigma)$, modulo the subgroup generated by $[E \cup F] - [E] - [F]$, where E, F and $E \cup F$ are in $CO^u(\Sigma, \sigma)$ and E and F are disjoint.

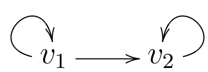
The following theorem is due to Krieger (4.1 of [22]). It asserts that the dimension group associated to a shift of finite type presented by a graph G is the same as that of the underlying graph. This provides a concrete method of computing the invariant. It will be useful for us to describe this isomorphism in terms of the invariant $D^s(G^k)$, for all $k \geq 1$. Recall in the formulas below, that for any e in Σ_G and integer k, $e^{[k+1,k]} = i(e^{k+1}) = t(e^k)$, by convention.

Theorem 3.3.3. *Let G be a graph, (Σ_G, σ) be the associated shift of finite type and $k \geq 1$.*

1. *The map sending $[\Sigma^s_G(e, 2^{-j})], e \in \Sigma_G, j \geq k$ to $[e^{[1-j,k-j-1]}, j-k+1]$ extends to an isomorphism from $D^s(\Sigma_G, \sigma)$ to $D^s(G^k)$.*
2. *The map sending $[\Sigma^u_G(e, 2^{-j})], e \in \Sigma_G, j \geq k$ to $[e^{[j-k+2,j]}, j-k+1]$ extends to an order isomorphism from $D^u(\Sigma_G, \sigma)$ to $D^u(G^k)$.*

Moreover, these isomorphisms, for different values of K are compatible with the natural isomorphisms of Theorem 3.2.3.

Example 3.3.4. *Let (Σ, σ) be the shift of finite type associated with the following graph*



It is fairly easy to see that Σ may be identified with the two-point compactification of the integers, $\{-\infty, \ldots, -2, -1, 0, 1, 2, \ldots, +\infty\}$. The points $-\infty$ and $+\infty$ correspond to the paths in the graph which only use the left loop and the right loop respectively. The integer n corresponds to the unique path which has the non-loop at entry n. Under this identification, σ sends n to $n-1$, for n finite and each infinite point to itself. Each singleton $\{+\infty\}$ and $\{n\}, n \in \mathbb{Z}$ is in $CO^s(\Sigma, \sigma)$. Now, we have some notational difficulties because our space carries an obvious order structure and we would like to look at intervals, such as $[-\infty, n] = \{a \mid -\infty \leq a \leq n\}$. Unfortunately, as we are in a Smale space, the bracket has another meaning. We use $[,]$ in the order sense only. Moreover, each interval $[-\infty, n]$, $n \in \mathbb{Z}$ is also in $CO^s(\Sigma, \sigma)$. Notice that, for $n \geq 1$, the Smale bracket of n with $+\infty$ is $+\infty$ and so $\{n\} \sim \{+\infty\}$. It follows that in the group $D^s(\Sigma, \sigma)$, $< [-\infty, n] >=< [-\infty, m] >$, for every $m, n \in \mathbb{Z}$ and $< \{n\} >=< \{+\infty\} >$, for every n in $\mathbb{Z}$. Moreover, $D^s(\Sigma, \sigma)$ is isomorphic to $\mathbb{Z}^2$, with these elements as generators. The order is lexicographic.

3.4 The dimension group as a covariant functor

The construction of the dimension group has various subtle, but interesting functorial properties. This is given in Theorem 3.2 of [10], in the implication 1 implies 2 only in the case $m = 1$. From our point of view, the result of [10] is not completely satisfactory. The method there uses a recoding of domain of the map which we would prefer to avoid (Remark 2.5.19). Secondly, the key property of a functor, namely that the map induced by a composition is the composition of the induced maps, is not clear with this approach. So we give a complete treatment here, starting from Definition 3.3.2.

We begin by showing that the invariant $D^s(\Sigma, \sigma)$ is covariant for s-bijective maps. Having established this, we next assume that our shifts of finite type are presented by graphs H and G and that the s-bijective map arises from a graph homomorphism, π. We have seen in the last section that $D^s(\Sigma_H, \sigma)$ and $D^s(\Sigma_G, \sigma)$ may be computed in terms of H and G. We need to have specific formulae for the maps induced by π. Essentially, the situation of [10] is to assume that π is actually left-covering. As was the case in the last chapter, the results in that case are quite simple. But we need to work in greater generality. The main result is summarized in Theorem 3.4.4, but we also derive some other useful facts in Theorem 3.4.5 and Theorem 3.4.6. Finally, we deal with an important special case. In general, any dynamical system (X, φ) always comes with a natural automorphism, namely φ. In the context of the this section, if (Σ, σ) is a shift of finite type, then both σ and σ^{-1} may be regarded as maps from (Σ, σ) to itself. We obtained formulae for their induced maps in the dimension group invariants in Theorem 3.4.7.

We begin with the following result which shows that the construction of $D^s(\Sigma, \sigma)$ is covariant for s-bijective maps. Of course, there is an analogous result which we do not state that $D^u(\Sigma, \sigma)$ is covariant for u-bijective factor maps. The idea is simple enough: suppose

$$\pi : (\Sigma, \sigma) \to (\Sigma', \sigma)$$

is an s-bijective map, then for E in $CO^s(\Sigma, \sigma)$, we will show that $\pi(E)$ is in $CO^s(\Sigma', \sigma)$. Moreover, this map extends in a well-defined fashion to a map from $D^s(\Sigma, \sigma)$ to $D^s(\Sigma', \sigma)$.

Theorem 3.4.1. *Let (Σ, σ) and (Σ', σ) be shifts of finite type and let*

$$\pi : (\Sigma, \sigma) \to (\Sigma', \sigma)$$

be an s-bijective map.

1. *If E is in $CO^s(\Sigma, \sigma)$, then $\pi(E)$ is in $CO^s(\Sigma', \sigma)$.*

2. *If E and F are in $CO^s(\Sigma, \sigma)$ and $E \sim F$, then $\pi(E) \sim \pi(F)$.*

3. *If E, F and $E \cup F$ are all in $CO^s(\Sigma, \sigma)$ with E and F disjoint, then $\pi(E)$ and $\pi(F)$ are also disjoint.*

In particular, the map defined by $\pi^s[E] = [\pi(E)]$ induces a well-defined group homomorphism $\pi^s : D^s(\Sigma, \sigma) \to D^s(\Sigma', \sigma)$. Finally, if $\pi' : (\Sigma', \sigma) \to (\Sigma'', \sigma)$ is another s-bijective map, then $(\pi' \circ \pi)^s = \pi'^s \circ \pi^s$.

Proof. The first and third statements follow easily from the fact that $\pi : \Sigma^s(y) \to \Sigma^s(\pi(y))$ is a homeomorphism (Theorem 2.5.12). The second statement is an easy consequence of the definitions and Proposition 2.5.2. The last statements follow from the first three. □

Example 3.4.2. *Consider the graphs H and G of Example 2.5.7. Arguing as in Example 3.3.4, it is easy to show that Σ_H may be identified with $\{a, b\} \times \{-\infty, \ldots, -2, -1, 0, 1, 2, \ldots, +\infty\}$, while Σ_G may be identified with $\{-\infty, +\infty\} \cup \{a, b\} \times \{\ldots, -2, -1, 0, 1, 2, \ldots\}$. The factor map sends $(a, \pm\infty)$ and $(b, \pm\infty)$ to $\pm\infty$ and is the identity on the other points. Fix n in $\mathbb{Z}$ and notice that $E = \{(a, x) \mid -\infty \leq x \leq n\}$ is in $CO^s(\Sigma_H, \sigma)$, while its image under the factor map is* not *in $CO^s(\Sigma_G, \sigma)$. Hence, the first part of the result above fails with the hypothesis of s-bijective replaced by s-resolving. Although we have not computed the dimension groups explicitly, it is not difficult if one begins by looking at Example 3.3.4. Let D denote the group $\mathbb{Z}^2$ with lexicographic order. We have $D^s(\Sigma_H, \sigma) \cong D \oplus D$, while $D^s(\Sigma_H, \sigma) \cong D$. There does indeed exist a well-defined homomorphism from the former to the latter, but it is* not *induced dynamically.*

The next objective is to consider the case that the two shifts (Σ, σ) and (Σ', σ) are presented by graphs, H and G, and the map is induced by a graph homomorphism, π. Identifying the invariants $D^s(\Sigma, \sigma), D^s(\Sigma', \sigma)$ with the dimension groups of their graphs as in 3.3.3, we want to have an explicit formula for the map π^s. Toward that end, we begin by defining the symbolic presentations for the induced map. Unlike 3.4.1, we give a complete statement for both D^s and D^u invariants.

Definition 3.4.3. *Let G and H be graphs and let $\pi : H \to G$ be a graph homomorphism.*

1. *If $\pi : (\Sigma_H, \sigma) \to (\Sigma_G, \sigma)$ is an s-bijective map, then for any $K, k \geq 0$, define $\pi^{s,K} : \mathbb{Z}H^k \to \mathbb{Z}G^{k+K}$ by*

$$\pi^{s,K}(q) = Sum\{\pi(q') \mid q' \in H^{k+K}, t^K(q') = q\},$$

for q in H^k.

2. *If $\pi : (\Sigma_H, \sigma) \to (\Sigma_G, \sigma)$ is a u-bijective map, then for any $K, k \geq 0$, define $\pi^{u,K} : \mathbb{Z}H^k \to \mathbb{Z}G^{k+K}$ by*

$$\pi^{u,K}(q) = Sum\{\pi(q') \mid q' \in H^{k+K}, i^K(q') = q\},$$

for q in H^k.

We remark, concerning the set $\{\pi(q') \mid q' \in H^{k+K}, t^K(q') = q\}$, that the map π on the set of all q' with $t^K(q') = q$ may not be injective. This means our sum above is *not* the same as the sum of $\pi(q')$ over all q' with $t^K(q') = q$. The latter may contain some repetitions while the former does not.

The relevance of the definition above is summarized by the following result which states that the maps $\pi^{s,K}$ give presentations for π^s on the invariants for the graphs.

Theorem 3.4.4. *Let G and H be graphs and let $\pi : H \to G$ be a graph homomorphism. Let $k \geq 1$.*

1. *Suppose the associated map $\pi : (\Sigma_H, \sigma) \to (\Sigma_G, \sigma)$ is an s-bijective map and K satisfies the conclusion of Lemma 2.7.1. If we implicitly identify $D^s(\Sigma_H, \sigma)$ with $D^s(H^k)$ and $D^s(\Sigma_G, \sigma)$ with $D^s(G^{k+K})$ using the isomorphism of Theorem 3.3.3, then we have*

$$\pi^s[a, j] = [\pi^{s,K}(a), j],$$

for all $j \geq 1$ and a in $\mathbb{Z}H^{k-1}$.

2. *Suppose the associated map $\pi : (\Sigma_H, \sigma) \to (\Sigma_G, \sigma)$ is a u-bijective map and K satisfies the conclusion of Lemma 2.7.1. If we implicitly identify $D^u(\Sigma_H, \sigma)$ with $D^u(H^k)$ and $D^u(\Sigma_G, \sigma)$ with $D^u(G^{k+K})$ using the isomorphism of Theorem 3.3.3, then we have*

$$\pi^u[a, j] = [\pi^{u,K}(a), j],$$

for all $j \geq 1$ and a in $\mathbb{Z}H^{k-1}$.

Proof. We prove the first part only. It suffices to prove the result for $a = q$, where q is in H^{k-1}. Select f in Σ_H with $f^{[-j-k+2,-j]} = q$. From Theorem 3.3.3, the set $\Sigma^s_H(f, 2^{-j-k+1})$ in $CO^s(\Sigma_H, \sigma)$ is identified with $[q, j]$ in $D^s(H^k)$. Let $A = \{\pi(q') \mid q' \in H^{K+k-1}, t^K(q') = q\}$. It follows from Definition 3.4.3 that $\pi^{s,K}(q) = SumA$.

For each p in A, choose e_p an element of Σ_G with $e_p^{[-j-k-K+2,-j]} = p$. By replacing e_p with $\sigma^{-j}[\sigma^j(\pi(f)), \sigma^j(e_p)]$, we may also assume that $e^l_p = \pi(f)^l$, for all $l > -j$. The sets $\Sigma^s_G(e_p, 2^{-j-K-k+1})$ are in $CO^s(\Sigma_G, \sigma)$ and are pairwise disjoint for different values of p. Moreover, for each p, the isomorphism of Theorem 3.3.3 identifies $\Sigma^s_G(e_p, 2^{-j-k-K+1})$ with $[p, j]$ in $D^s(G^{k+K})$. So it suffices for us to prove that $\pi(\Sigma^s_H(f, 2^{-j-k+1}))$ is the union of the sets $\Sigma^s_G(e_p, 2^{-j-k-K+1}), p \in A$.

Suppose that f_1 is in $\Sigma^s_H(f, 2^{-j-k+1})$. Letting $p = \pi(f_1)^{[-j-k-K+2,-j]}$, it is immediate that $\pi(f_1)$ is in $\Sigma^s_G(e_p, 2^{-j-K-k+1})$. In addition, $p = \pi(f_1^{[-j-k-K+2,-j]})$ with

$$t^K(f_1^{[-j-k-K+2,-j]}) = f_1^{[-j-k+2,-j]} = f^{[-j-k+2,-j]} = q$$

and so p is in A. Hence, we have

$$\pi(\Sigma^s_H(f, 2^{-j-k+1})) \subset \cup_{p \in A} \Sigma^s_G(e_p, 2^{-j-k-K+1}).$$

Conversely, suppose that p is in A and e is in $\Sigma^s_G(e_p, 2^{-j-k-K+1})$. From the definition of A, there exists q' in H^{K+k-1} such that $\pi(q') = p$ and $t^K(q') = q$. Let f_1 be any element of Σ_H such that $f_1^{[-j-k-K+2,-j]} = q'$. It follows that $f_2 = \sigma^{-j}[\sigma^j(f), \sigma^j(f_1)]$ is defined and $f^l_2 = f^l$ for $l > -j$, while $f^l_2 = f^l_1$ for $l \leq -j$. In particular, we have $f_2^{[-j-k-K+2,-j]} = q'$. For $l > -j$, we have

$$\pi(f^l_2) = \pi(f^l) = e^l_p = e^l,$$

while

$$\pi(f_2^{[-j-k-K+2,-j]}) = \pi(q') = p = e_p^{[-j-k-K+2,-j]} = e^{[-j-k-K+2,-j]}.$$

Since $\pi(f_2)$ and e are stably equivalent, there is f_3 in Σ_H with $\pi(f_3) = e$ and f_3 stably equivalent to f_2. By Lemma 2.7.1, since $\pi(f^l_2) = e^l$, for $l > -j - k - K + 2$, we have $f^l_2 = f^l_3$, for $l > -j - k + 2$. We have seen already that $f^l_2 = f^l$, for $l > -j - k + 2$ and so f_3 is in $\Sigma^s_H(f, 2^{-j-k+1})$. □

The following result will be used frequently.

Theorem 3.4.5. *Let G and H be graphs.*

1. *Suppose that $\pi : H \to G$ is a graph homomorphism, the associated map $\pi : (\Sigma_H, \sigma) \to (\Sigma_G, \sigma)$ is s-bijective and suppose K satisfies the conclusion of Lemma 2.7.1 for this map. We have*

$$\begin{aligned} \pi^{s,K} \circ t^* &= \pi^{s,K+1} = t^* \circ \pi^{s,K} \\ \pi^{s,K} \circ i &= i \circ \pi^{s,K}, \\ \pi^{s,K} \circ \gamma^s_H &= \gamma^s_G \circ \pi^{s,K}. \end{aligned}$$

2. *Suppose that $\pi : H \to G$ is a graph homomorphism, the associated map $\pi : (\Sigma_H, \sigma) \to (\Sigma_G, \sigma)$ is u-bijective and K satisfies the conclusion of Lemma 2.7.1 for this map . We have*

$$\begin{aligned} \pi^{u,K} \circ i^* &= \pi^{u,K+1} = i^* \circ \pi^{u,K} \\ \pi^{u,K} \circ t &= t \circ \pi^{u,K}, \\ \pi^{u,K} \circ \gamma^u_H &= \gamma^u_G \circ \pi^{u,K}. \end{aligned}$$

Proof. We prove the first part only. For the first two equalities, it suffices to consider the case that the domain is $\mathbb{Z}H^0$ and the range is $\mathbb{Z}G^{K+1}$. Let v be in H^0. It follows directly from the definitions that $\pi^{s,K} \circ t^*(v) = \sum_{t(e)=v} SumA_e$, where $A_e = \{\pi(q) \mid q \in H^{K+1}, t^K(q) = e\}$. We claim that the sets A_e are disjoint for distinct e with $t(e) = v$. Suppose that $t(e) = t(e') = v$ and there exist q, q' with $t^K(q) = e$ and $t^K(q') = e'$ and $\pi(q) = \pi(q')$. It follows that

$$t^{K+1}(q) = t(t^K(q)) = t(e) = v = t(e') = t^{K+1}(q')$$

so by part 4 of Lemma 2.7.5, we conclude $e = t^K(q) = t^K(q') = e'$. This means that $\pi^{s,K} \circ t^*(v) = SumA$, where $A = \cup_{t(e)=v} A_e$.

Next, from the definition, we have $\pi^{s,K+1}(v) = SumB$, where

$$B = \{\pi(q) \mid q \in H^{K+1}, t^{K+1}(q) = v\}.$$

If q is in H^{K+1} and $t^{K+1}(q) = v$, then letting $e = t^K(q)$, we see that $t(e) = v$ and $t^K(q) = e$, so $\pi(q)$ is in A_e. Conversely, if $t(e) = v$ and a is in A_e so that $a = \pi(q)$, where $t^K(q) = e$, then $t^{K+1}(q) = t(t^K(q)) = t(e) = v$ and so $\pi(q)$ is in B. We have shown $A = B$ and this proves the first desired equality.

For the second equality, again using the definitions, we have

$$t^* \circ \pi^{s,K}(v) = \sum_{\{\pi(q) \mid q \in H^K, t^K(q)=v\}} \sum_{t(q')=\pi(q)} q'.$$

Let C denote the collection of q' in G^{K+1} such that $t(q') = \pi(q)$, for some q in H^K with $t^K(q) = v$ so that $t^* \circ \pi^{s,K}(v) = SumC$. First, we claim that $B \subset C$, for if q is in H^{K+1} with $t^{K+1}(q) = v$, then $t(q)$ is in H^K, $t(\pi(q)) = \pi(t(q))$ and $t^K(i(q)) = v$, so $\pi(q)$ is in C.

Conversely, let us show $C \subset B$. Suppose that q' is in C. Then there exists q in H^K with $t(q') = \pi(q)$ and $t^K(q) = v$. We may find f in Σ_H such that $f^{[1-K,0]} = q$ and e in Σ_G such that $e^{[-K,0]} = q'$. It follows that we may form $[\pi(f), e]$ which is stably equivalent to $\pi(f)$. As π is s-bijective, we may find f_1, stably equivalent to f such that $\pi(f_1) = [\pi(f), e]$. It follows that for $k \geq 1$, $\pi(f_1)^k = \pi(f)^k$, while $\pi(f_1)^{[-K,0]} = e^{[-K,0]} = q'$. In particular, $\pi(f_1)^{[1-K,0]} = e^{[1-K,0]} = t(q') = \pi(q) = \pi(f)^{[1-K,0]}$. It follows from Lemma 2.7.1 that $f_1^k = f^k$, for $k \geq 1$. Consider $f_1^{[-K,0]}$, which is in H^{K+1}. We have

$$i(f_1^1) = i(F^1) = t^K(f^{[1-K,0]}) = t^K(q) = v$$

and $\pi(f^{[-K,0]}) = q'$. Thus q' is B. This completes the proof of the second equality.

We now consider the second line. Let v be in H^k, with $k \geq 1$. It follows immediately from the definitions that $\pi^{s,K} \circ i(v) = SumA$, where

$$A = \{\pi(q) \mid q \in H^{k+K-1}, t^K(q) = i(v)\}$$

and $i \circ \pi^{s,K}(v) = SumB$, where

$$B = \{i(\pi(q)) = \pi(i(q)) \mid q \in H^{k+K}, t^K(q) = v\}.$$

We claim that $A = B$, and the result follows. Suppose a is in A so that $a = \pi(q)$, q in H^{k+K-1} with $t^K(q) = i(v)$. Let $q' = qv^k$ which is in H^{k+K} and $t^K(q') = t^K(qv^k) = t^K(q)v^k = i(v)v^k = v$. This means that $\pi(i(q'))$ is in B and $\pi(i(q')) = \pi(q) = a$. Conversely, suppose that b is in B so that $b = \pi(i(q))$, for some q in H^{k+K} with $t^K(q) = i(v)$. Then $i(q)$ is in H^{k+K-1} and $t^K(i(q)) = i(t^K(q)) = i(v)$. It follows that $\pi(i(q))$ is in A and $\pi(i(q)) = b$.

The third equation follows immediately from the earlier ones as $\gamma^s = i \circ t^*$. □

Theorem 3.4.6. *Let I, H and G be graphs, $\eta : I \to H$ and $\pi : H \to G$ graph homomorphisms such that their induced maps on the shift spaces are s-bijective. Let K_η and K_π satisfy the conclusion of Lemma 2.7.1 for these maps. Then we have*

$$(\pi \circ \eta)^{s,K_\pi+K_\eta} = \pi^{s,K_\pi} \circ \eta^{s,K_\eta}.$$

Proof. Let $k \geq 0$ and q be in I^k. It follows directly from the definitions that $\eta^{s,K_\eta}(q) = Sum(A_0)$, where

$$A_0 = \{\eta(q') \mid q' \in I^{k+K_\eta}, t^{K_\eta}(q') = q\}.$$

For each q' in I^{k+K_η} with $t^{K_\eta}(q') = q$, let

$$A(q') = \{\pi(q'') \mid q'' \in H^{k+K_\eta+K_\pi}, t^{K_\pi}(q'') = \eta(q')\}.$$

It is clear that if $\eta(q') = \eta(\bar{q}')$, then $A(q') = A(\bar{q}')$. We claim that if $A(q')$ and $A(\bar{q}')$ have non-empty intersection, then $\eta(q') = \eta(\bar{q}')$. Suppose there is a q'' with $t^{K_\pi}(q'') = \eta(q'), t^{K_\pi}(\bar{q}'') = \eta(\bar{q}')$ and $\pi(q'') = \pi(\bar{q}'')$. It follows that

$$t^{K_\pi+K_\eta}(q'') = t^{K_\eta}(\eta(q')) = \eta(t^{K_\eta}(q')) = \eta(q) = \eta(t^{K_\eta}(\bar{q}')) = t^{K_\pi+K_\eta}(\bar{q}'').$$

It follows from the uniqueness statement of 2.7.5 that $t^{K_\pi}(q'') = t^{K_\pi}(\bar{q}'')$; but the former is $\eta(q')$ and the latter $\eta(\bar{q}')$ and the conclusion follows.

From the claim above and the definition, we see that

$$\pi^{s,K_\pi} \circ \eta^{s,K_\eta}(q) = \sum_{q' \in R} Sum(A(q')),$$

where R is a subset of the possible q''s with the value $\eta(q')$ represented exactly once by an element of R. This means that $\pi^{s,K_\pi} \circ \eta^{s,K_\eta}(q) = Sum(A)$, where

$$A = \{\pi(q'') \mid q'' \in H^{k+K_\eta+K_\pi}, \exists q' \in I^{k+K_\eta}, t^{K_\pi}(q'') = \eta(q'), t^{K_\eta}(q') = q\}.$$

On the other hand, it follows from the definitions that $(\pi \circ \eta)^{s,K_\pi+K_\eta}(q) = Sum(B)$, where

$$B = \{\pi(\eta(\bar{q})) \mid \bar{q} \in I^{k+K_\pi+K_\eta}, t^{K_\pi+K_\eta}(\bar{q}) = q\}.$$

We now claim that $A = B$. For the containment $A \supset B$, if $\pi(\eta(\bar{q}))$ is in B, with $\bar{q}$ as described above, then let $q'' = \eta(\bar{q})$ and $q' = t^{K_\eta}(\bar{q})$. It is immediate that $\pi(q'') = \pi(\eta(\bar{q}))$ is in A.

Now suppose that q'' and q' are as given if the definition of A, so that $\pi(q'')$ is in A. We apply Lemma 2.7.5 to the graph homomorphism η, the paths q'' in $H^{k+K_\eta+K_\pi}$ and q' in I^{k+K_η} to find $\bar{q}$ in $I^{k+K_\eta+K_\pi}$ with $\eta(\bar{q}) = q''$ and $t^{K_\eta+K_\pi}(\bar{q}) = t^{K_\eta}(q')$. The second condition immediately implies $t^{K_\eta+K_\pi}(\bar{q}) = q$ while the former means that $\pi(\eta(\bar{q})) = \pi(q'')$ is in B. □

We conclude by examining the special case where we consider σ and σ^{-1} as maps from (Σ, σ) to itself. We work with a fixed specific presention by a graph G.

Theorem 3.4.7. *Let G be a graph and (Σ_G, σ) be the associated shift of finite type.*

1. *If we consider $\sigma^{-1} : (\Sigma_G, \sigma) \to (\Sigma_G, \sigma)$, then it is an s-bijective factor map. For any $K \geq 1$, if we identity $D^s(\Sigma_G, \sigma)$ with $D^s(G^K)$ as in Theorem 3.3.3, then the map $(\sigma^{-1})^s$ is given by*

$$(\sigma^{-1})^s[p, k] = [\gamma^s(p), k],$$

for any p in G^{K-1} and $k \geq 1$.

2. *If we consider $\sigma : (\Sigma_G, \sigma) \to (\Sigma_G, \sigma)$, then it is an u-bijective factor map. For any $K \geq 1$, if we identity $D^u(\Sigma_G, \sigma)$ with $D^u(G^K)$ as in Theorem 3.3.3, then the map σ^u is given by*

$$\sigma^u[p, k] = [\gamma^u(p), k],$$

for any p in G^{K-1} and $k \geq 1$.

Proof. We begin with the first statement. The properties claimed for σ^{-1} are obvious. Let $J = k + K - 1$ and e in Σ_G such that $e^{[1-J, J-K+1]} = p$ so that, under the isomorphism of 3.3.3, $[p, k] = [\Sigma^s_G(e, 2^{-J})]$. It is immediate from the definitions that

$$\sigma^{-1}(\Sigma^s_G(e, 2^{-J})) = \Sigma^s_G(\sigma^{-1}(e), 2^{-J+1}).$$

Going back to $D^s(G)$, $\Sigma^s_G(\sigma^{-1}(e), 2^{-J+1})$ corresponds to

$$\begin{aligned} [\sigma^{-1}(e)^{[1-J+1, K-J]}, J - 1 + K - 1] &= [e^{[1-J, J-K+1]}, K - J] \\ &= [p, K - J] \\ &= [\gamma^s(p), K - J + 1] \\ &= [\gamma^s(p), k]. \end{aligned}$$

For the second statement, again let $k = J - K + 1$ and choose e such that $e^{[J-2+K, J]} = p$ so that $\Sigma^u_G(e, 2^{-J})$ is associated with $[p, k]$. Again it follows from the definitions that

$$\sigma(\Sigma^u_G(e, 2^{-J})) = \Sigma^u_G(\sigma(e), 2^{-J+1})$$

and under 3.3.3, this is associated with

$$\begin{aligned}
[\sigma(e)^{[J-K+1,J-1]}, J-K] &= [e^{[J-K+2,J]}, J-K] \\
&= [p, J-K] \\
&= [\gamma^u(p), J-K+1] \\
&= [\gamma^u(p), k].
\end{aligned}$$

□

3.5 The dimension group as a contravariant functor

In contrast to the situation of the last section, the invariant $D^s(\Sigma, \sigma)$ is contravariant for u-bijective maps, while $D^u(\Sigma, \sigma)$ is contravariant for s-bijective maps. This section is devoted to establishing this fact and others which are direct analogies with the results from the last section. At the end, there is one new result, Theorem 3.5.11, which deals with the case where we have a commuting diagram consisting of four shifts of finite type and four maps, two of which are s-bijective and two which are u-bijective. The conclusion is that, under certain hypotheses, the associated maps on our invariant are also commuting in an appropriate way.

The precise statement of D^s as a contravariant functor for u-bijective maps follows. In its simplest form, the idea is that, if $\pi : (\Sigma, \sigma) \to (\Sigma', \sigma)$ is u-bijective and E' is in $CO^s(\Sigma', \sigma)$, then $\pi^{-1}(E')$ is in $CO^s(\Sigma, \sigma)$. As stated, this is false, simply because $\pi^{-1}(E')$ is not contained in a single stable equivalence class. However, we show that $\pi^{-1}(E')$ may be written (uniquely, in a certain sense) as a finite disjoint union of elements of $CO^u(\Sigma, \sigma)$. Just as in Theorem 3.4.1, we state the following result for the D^s-invariant only.

Theorem 3.5.1. *Let (Σ, σ) and (Σ', σ) be shifts of finite type and let*

$$\pi : (\Sigma, \sigma) \to (\Sigma', \sigma)$$

be a u-bijective map.

If E' is in $CO^s(\Sigma', \sigma)$, then there exist a finite collection $E_1, \ldots, E_L$ in $CO^s(\Sigma, \sigma)$ such that,

1. *for $1 \leq i \neq j \leq L$, no point of E_i is stably equivalent to any point of E_j, and*

2. $\pi^{-1}(E') = \cup_{l=1}^{L} E_l$.

The collection satisfying these two conditions, is unique, up to re-ordering.

If E' and F' are in $CO^s(\Sigma', \sigma)$ and $E' \sim F'$ and $\pi^{-1}(E') = \cup_{l=1}^{L} E_l$ and $\pi^{-1}(F') = \cup_{m=1}^{M} F_m$ are as above, then

$$[E_1] + [E_2] + \cdots + [E_L] = [F_1] + [F_2] + \cdots + [F_M]$$

in $D^s(\Sigma, \sigma)$.

In particular, the map defined by $\pi^{s*}[E'] = \sum_{l=1}^{L}[E_l]$ induces a well-defined group homomorphism $\pi^{s*} : D^s(\Sigma', \sigma) \to D^s(\Sigma, \sigma)$. Finally, if $\pi' : (\Sigma'', \sigma) \to (\Sigma', \sigma)$ is another u-bijective factor map, then $(\pi \circ \pi')^{s*} = \pi'^{s*} \circ \pi^{s*}$.

Proof. Let e' be any point in E', so that $E' \subset \Sigma'^s(e')$. From Theorem 2.5.3, $\pi^{-1}(\Sigma'^s(e'))$ can be written as the union of a finite number of distinct stable equivalence classes, $\Sigma^s(e_1), \ldots, \Sigma^s(e_L)$. Let $E_l = \pi^{-1}(E') \cap \Sigma^s(e_l)$, for $1 \leq l \leq L$. The fact that each E_l is compact and open follows from the fact that $\pi : \Sigma^s(e_l) \to \Sigma'^s(e')$ is proper and continuous (Theorem 2.5.4). The desired properties of this collection and its uniqueness are immediate.

For the second statement, we suppose that $\Sigma = \Sigma_H$ and $\Sigma' = \Sigma_G$, for some graphs G and H, and π is induced by a graph homomorphism. Suppose that E', F' are in $CO^s(\Sigma', \sigma)$ satisfy $[E', F'] = E', [F', E'] = F'$. Let e'_0 be in E' and f'_0 be in F'. It follows from the fact that $[E', F']$ is defined that every element of E' and F' passes through the same vertex, denoted v, between edges zero and one. So E' is contained in $\Sigma'^s(e'_0, 1)$ and F' is contained in $\Sigma'^s(f'_0, 1)$ and $t(e'^0_0) = t(f'^0_0) = v$. Let K be as in Lemma 2.7.1 and P be the collection of all paths of length K terminating at v. For each p in P, define

$$E'_p = \{e' \in E' \mid e'^{[1-K,0]} = p\}, F'_p = \{f' \in F' \mid f'^{[1-K,0]} = p\}.$$

It is easy to see that each E'_p is in $CO^s(\Sigma', \sigma)$, they are pairwise disjoint and their union is E'. Analogous statements hold for the F'_p. We claim that $[E'_p, F'_p] = E'_p, [F'_p, E'_p] = F'_p$. It suffices to show the first condition. If e' is in E'_p and f' is in F'_p, then $[e', f']$ is in $[E', F'] = E'$ and $[e', f']^{[1-K,0]} = e'^{[1-K,0]} = p$ and hence we have $[E'_p, F'_p] \subset E'_p$. For the reverse inclusion, let f' be in F'_p. Then f' is in F' and hence equals $[f'', e']$, for some e' in E' and f'' in F'. But since F' is contained in $\Sigma'^s(f'_0, 1)$, we have $f'' = [f', f'']$ and hence

$$f' = [f'', e'] = [[f', f''], e'] = [f', e'].$$

Also, $e'^{[1-K,0]} = [f', e']^{[1-K,0]} = f'^{[1-K,0]} = p$ and is therefore in E'_p. This completes the proof of the claim.

So it suffices to prove the desired conclusion under the added hypothesis that E' is contained in $\Sigma'^s(e'_0, 2^{-K})$, F' is contained in $\Sigma'^s(f'_0, 2^{-K})$, for some e'_0 in E' and f'_0 in F' with $e'^k_0 = f'^k_0$, for all $k < 0$.

As e varies over $\pi^{-1}(E')$, the sets $\Sigma'^s(e, 2^{-K})$ form an open cover. Also, any two such sets are either equal or disjoint, so we may find a finite subcover and write $\pi^{-1}(E')$ as the disjoint union of $E_i = \pi^{-1}(E') \cap \Sigma'^s(e_i, 2^{-K})$, $1 \leq i \leq I$, each being in $CO^s(\Sigma, \sigma)$. We claim that, if $t(e_i^{-K}) = t(e_j^{-K})$, for some i, j, then $E_i = E_j$. Consider the points e_i and $\bar{e} = \sigma^K[\sigma^{-K}(e_j), \sigma^{-K}(e_i)]$, which is defined since $t(e_i^{-K}) = t(e_j^{-K})$. It is clear that $e_i^k = \bar{E}^k$, for all $k \leq K$. On the other hand, for $k > K$, we have $\pi(e_i^k) = e'^k$ and $\pi(\bar{e}^k) = \pi(e_j^k) = e'^k$. It follows that e_i and $\bar{e}$ are unstably equivalent and have the same image under π. Since π is u-resolving, it follows that $\bar{e} = e_i$ and it follows that $\Sigma'^s(e_i, 2^{-K}) = \Sigma'^s(\bar{e}, 2^{-K}) = \Sigma'^s(e_j, 2^{-K})$. Henceforth, we assume these sets are all distinct and so $t(e_i^{-K}) \neq t(e_j^{-K})$, for $i \neq j$.

For each i, the point $[f'_0, \pi(e_i)]$ is in F'. It is also unstably equivalent to $\pi(e_i)$ and since π is u-bijective, we may find a unique point f_i in $\pi^{-1}(F')$, unstably equivalent to e_i and such that $\pi(f_i) = [f'_0, \pi(e_i)]$. It follows from Lemma 2.7.1 that $e_i^k = f_i^k$, for all $k < -K$. For each i, we define $F_i = \sigma^{-K}[\sigma^K(f_i), \sigma^K(E_i)]$ which is a compact open subset of $F \cap \Sigma^s(f_i, 2^{-K})$ and hence in $CO^s(\Sigma, \sigma)$. Since $t(f_i^{-K}) = t(e_i^{-K}) \neq t(e_j^{-K}) = t(f_j^{-K})$, for $i \neq j$, these sets are pairwise disjoint. Moreover, we have $E_i \sim F_i$, for all i.

We claim that the union of the F_i is all of F. Let f be in F so that $\pi(f)$ is in F'. Let $e' = [e'_0, \pi(f)]$ which is in E'. As π is u-bijective, there exists e, stably equivalent to f, such that $\pi(e) = e'$. Hence, e is in E and is in E_i, for some i. As $\pi(f)^k = \pi(e)^k$, for all $k \leq 0$, we have $f^k = e^k$, for all $k \leq -K$, by Lemma 2.7.1. As $t(f_i^{-K}) = t(e_i^{-K}) = t(e^{-K}) = t(f^{-K})$ an argument like the one above showing that the sets E_i are disjoint or equal then implies that $f_i^k = f^k$, for all $k > -K$. It follows that $f = \sigma^K[\sigma^{-K}(f_i), \sigma^{-K}(e)]$ and hence is in F_i as desired. □

Remark 3.5.2. *A word of warning is in order regarding our notation. If $\pi : (\Sigma, \sigma) \to (\Sigma', \sigma)$ is a factor map, it may be that it is both s-bijective* and *u-bijective. (The case $\Sigma = \Sigma'$ and $\pi = \sigma$ will be of some special interest and qualifies.) In this case, we have four different induced group homomorphisms π^s, π^{s*}, π^u and π^{u*}. The first pair are defined on the D^s invariants and the latter on the D^u invariants. This must be kept in mind particularly in the* *case* $\Sigma = \Sigma'$ *as the domain and range no longer distinguish* π^s *and* π^{s*}.

Example 3.5.3. *We return again to example 2.5.7 and 3.3.4. Now we use the observation that the factor map* π *is u-resolving. Let* n *be in* $\mathbb{Z}$ *and consider the sets* $\{(a,n)\}, \{(b,n)\}, \{+\infty\}$, *each of which is in* $CO^s(\Sigma_G, \sigma)$. *Observe that*

$$\begin{aligned} \pi^{-1}\{(a,n)\} &= \{(a,n)\}, \\ \pi^{-1}\{(b,n)\} &= \{(b,n)\}, \\ \pi^{-1}\{+\infty\} &= \{(a,+\infty),(b,+\infty)\} \end{aligned}$$

and that

$$< \{(a,n)\} >=< \{+\infty\} >=< \{(b,n)\} >\neq 0$$

in $D^s(\Sigma_G, \sigma)$, *while*

$$< \{(a,+\infty),(b,+\infty)\} >=< \{(a,+\infty)\} > + < \{(b,+\infty)\} >$$

in $D^s(\Sigma_H, \sigma)$. *This provides a counter-example to the statement of the last Theorem obtained by replacing u-bijective with u-resolving.*

Having established the result for the functorial properties, we now turn to the issue of having explicit formulae for the computation of these maps if we assume our shifts are given by specific graphs. Again, we begin with a definition.

Definition 3.5.4. *Let* G *and* H *be graphs and let* $\pi : H \to G$ *be a graph homomorphism.*

1. *If* $\pi : (\Sigma_H, \sigma) \to (\Sigma_G, \sigma)$ *is u-bijective, then for any* $K, k \geq 0$, *we define* $\pi^{s*,K}(p) : \mathbb{Z}G^{k+K} \to \mathbb{Z}H^k$ *by*

$$\pi^{s*,K}(p) = Sum\{i^K(q) \mid q \in H^{k+K}, \pi(q) = p\},$$

   for p *in* $\mathbb{Z}G^{k+K}$.

2. *If* $\pi : (\Sigma_H, \sigma) \to (\Sigma_G, \sigma)$ *is s-bijective, then for any* $K, k \geq 0$, *we define* $\pi^{u*,K}(p) : \mathbb{Z}G^{k+K} \to \mathbb{Z}H^k$ *by*

$$\pi^{u*,K}(p) = Sum\{t^K(q) \mid q \in H^{k+K}, \pi(q) = p\},$$

   for p *in* $\mathbb{Z}G^{k+K}$.

The maps $\pi^{s*,K}$ provide symbolic presentations for π^{s*} analogous to those of the last section. This is stated precisely as follows.

Theorem 3.5.5. *Let G and H be graphs and $\pi : H \to G$ a graph homomorphism. Let $k \geq 1$.*

1. *Suppose that the associated $\pi : (\Sigma_H, \sigma) \to (\Sigma_G, \sigma)$ is an s-bijective map and that K satisfies the conclusion of Lemma 2.7.1. If we implicitly identify $D^u(\Sigma_G, \sigma)$ with $D^u(G^{k+K})$ and $D^u(\Sigma_H, \sigma)$ with $D^u(H^k)$ using the isomorphisms of 3.3.3, then we have*

$$\pi^{u*}[a, j] = [\pi^{u*,K}(a), j + K],$$

for all $j \geq 0$ and a in $\mathbb{Z}G^{K+k-1}$.

2. *Suppose that the associated $\pi : (\Sigma_H, \sigma) \to (\Sigma_G, \sigma)$ is a u-bijective map and that K satisfies the conclusion of Lemma 2.7.1. If we implicitly identify $D^s(\Sigma_G, \sigma)$ with $D^s(G^{k+K})$ and $D^s(\Sigma_H, \sigma)$ with $D^s(H^k)$ using the isomorphisms given in 3.3.3, then we have*

$$\pi^{s*}[a, j] = [\pi^{s*,K}(a), j + K]$$

for all $j \geq 0$ and a in $\mathbb{Z}G^{k+K}$.

Proof. We prove only the first statement. It suffices to consider the case $a = p \in G^{K+k-1}$. We find e be in Σ_G such that $e^{[j+1,j+K+k-1]} = p$. It follows from Theorem 3.3.3 that the element $[\Sigma^u_G(e, 2^{-j-K-k+1})]$ in $D^u(\Sigma_G, \sigma)$ is identified with $[p, j]$ in $D^u(G^{K+k})$.

From the Definition 3.5.4, we have $\pi^{u*,K}(p) = SumB$, where $B = \{t^K(q) \mid q \in H^{k+K}, \pi(q) = p\}$. We claim that, for each b in B, we may find f_b is Σ_H such that $\pi(f_b)$ is in $\Sigma^u_G(e, 2^{-j-K-k+1})$ and $f_b^{[j+K+1,j+K+k-1]} = b$. As b is in B, we may find q in H^{k+K-1} such that $\pi(q) = p$ and $t^K(q) = b$. Choose f such that $f^{[j+1,j+K+k-1]} = q$. This means that $e_1 = \sigma^{-j}[\sigma^j(\pi(f_1)), \sigma(e)]$. Moreover, $e_1^l = \pi(f_1)^l$, for all $l > j$. As π is s-bijective, we may find f_b stably equivalent to f such that $\pi(f_b) = e_1$. In addition, by Lemma 2.7.1, we have $f_b^l = f^l$ for $l > j + K$. This implies that

$$f_b^{[j+K+1,j+K+k-1]} = f^{[j+K+1,j+K+k-1]} = t^K(f^{[j+1,j+K+k-1]}) = t^K(q) = b.$$

By definition, for $l \leq j$, $e_1^l = e^l$, while

$$e_1^{[j+1,j+K+k-1]} = \pi(f^{[j+1,j+K+k-1]}) = \pi(q) = p = e^{[j+1,j+K+k-1]}$$

and so $\pi(f_b) = e_1$ is in $\Sigma^u_G(e, 2^{-j-K-k+1})$.

It is clear from the definitions that the sets $\Sigma^u_H(f_b, 2^{-j-K-k+1})$ are pairwise disjoint, for different values of b in B. Now, we claim that

$$\pi^{-1}(\Sigma^u_G(e, 2^{-j-K-k+1})) = \cup_{b\in B}\Sigma^u_H(f_b, 2^{-j-K-k+1}).$$

The containment $\supset$ is clear from the definitions and the choice of f_b. As for the reverse inclusion, let f in Σ_H be such that $\pi(f)$ is in $\Sigma^u_G(e, 2^{-j-K-k+1})$. Let $q = f^{[j+1,j+K+k-1]}$, which is in H^{K+k-1}. Moreover, $\pi(q) = \pi(f)^{[j+1,j+K+k-1]} = e^{[j+1,j+K+k-1]} = p$ and hence $b = t^K(q)$ is in B. It follows that $b = f_b^{[j+1,j+K+k-1]} = f^{[j+1,j+K+k-1]}$. Consider $f_1 = \sigma^{-j}[\sigma^j(f), \sigma^k(f_b)]$, which is defined and stably equivalent to f. For $l > j$, we have $\pi(f_1^l) = \pi(f_1)^l = \pi(f^l)$. On the other hand, for $l \leq j$, we have $\pi(f_1^l) = \pi(f_b^l) = e^l = \pi(f^l)$. We conclude that $\pi(f_1) = \pi(f)$, but as π is s-resolving, it follows that $f_1 = f$ and this implies that f is in $\Sigma^u_H(f_b, 2^{-j-K-k+1})$.

We now have that

$$[\pi^{-1}(\Sigma^u_G(e, 2^{-j-K-k+1})] = \sum_{b\in B}[\pi^{-1}(\Sigma^u_H(f_b, 2^{-j-K-k+1})].$$

Under the isomorphism of Theorem 3.3.3, for each b in B, $[\pi^{-1}(\Sigma^u_H(f_b, 2^{-j-K-k+1})]$ corresponds to $[f_b^{[j+K+1,j+K+k-1]}, j+K]$ in $D^u(H^J)$. Moreover, $f_b^{[j+K+1,j+K+k-1]} = t^K(f_b^{[j+1,j+K+k-1]}) = b$. The conclusion follows since $SumB = \pi^{u*,K}(p)$. □

We want to establish other basic properties analogous to those of the last section. In fact, we can avoid repeating all the proofs by simply noting the following duality.

Lemma 3.5.6. *Let $k, K \geq 0$ and let $\pi : H \to G$ be a graph homomorphism.*

1. *If the induced map $\pi : (\Sigma_H, \sigma) \to (\Sigma_G, \sigma)$ is s-bijective, then*

$$< p, \pi^{s,K}(q) >_{G^{k+K}} = < \pi^{s*,K}(p), q >_{H^k},$$

for all p in $\mathbb{Z}G^{k+K}$ and q in $\mathbb{Z}H^k$.

2. *If the induced map $\pi : (\Sigma_H, \sigma) \to (\Sigma_G, \sigma)$ is u-bijective, then*

$$< p, \pi^{u,K}(q) >_{G^{k+K}} = < \pi^{u*,K}(p), q >_{H^k},$$

for all p in $\mathbb{Z}G^{k+K}$ and q in $\mathbb{Z}H^k$.

Proof. We prove the first statement only. It clearly suffices to consider the case that p is in G^{k+K} and q is in H^k. It follows from the definitions that the left hand side is 1 exactly when there exists q' in H^{k+K} with $t^K(q') = q$ and $\pi(q') = p$ and is zero otherwise. The right hand side has the same value, also from the definition. □

The next result is the analogue of Theorem 3.4.5. In fact, its proof may be deduced from Theorem 3.4.5, the last Lemma and the non-degeneracy of the pairings. We omit the details.

Theorem 3.5.7. *Let G and H be graphs.*

1. *Suppose that $\pi : H \to G$ is a graph homomorphism such that the associated map $\pi : (\Sigma_H, \sigma) \to (\Sigma_G, \sigma)$ is s-bijective and suppose K satisfies the conclusion of Lemma 2.7.1 for this map. We have*

$$\begin{array}{rcccl} \pi^{s*,K} \circ i & = & \pi^{s*,K+1} & = & i \circ \pi^{s*,K} \\ \pi^{s*,K} \circ t^* & = & t^* \circ \pi^{s*,K}, & & \\ \pi^{s*,K} \circ \gamma^s_G & = & \gamma^s_H \circ \pi^{s*,K}. & & \end{array}$$

2. *Suppose that $\pi : H \to G$ is a graph homomorphism such that the associated map $\pi : (\Sigma_H, \sigma) \to (\Sigma_G, \sigma)$ is u-bijective and suppose K satisfies the conclusion of Lemma 2.7.1 for this map. We have*

$$\begin{array}{rcccl} \pi^{u*,K} \circ t & = & \pi^{u*,K+1} & = & t \circ \pi^{u*,K} \\ \pi^{u*,K} \circ i^* & = & i^* \circ \pi^{u*,K}, & & \\ \pi^{u*,K} \circ \gamma^u_G & = & \gamma^u_H \circ \pi^{u*,K}. & & \end{array}$$

We now establish an analogue of Theorem 3.4.6. The same comments made about the proof of the last result apply here as well.

Theorem 3.5.8. *Let I, H and G be graphs, $\eta : I \to H$ and $\pi : H \to G$ graph homomorphisms such that their induced maps on the shift spaces are s-bijective. Let K_η and K_π satisfy the conclusion of Lemma 2.7.1 for these maps. Then we have*

$$(\pi \circ \eta)^{s*,K_\pi+K_\eta} = \eta^{s*,K_\eta} \circ \pi^{s*,K_\pi}.$$

Exactly as in the last section, we take note of the following special case where we regard σ and σ^{-1} a maps from (Σ_G, σ) to itself.

Theorem 3.5.9. *Let G be a graph and (Σ_G, σ) be the associated shift of finite type.*

1. *If we consider $\sigma : (\Sigma_G, \sigma) \to (\Sigma_G, \sigma)$, then it is an u-bijective factor map. For any $J \geq 1$, if we identity $D^s(\Sigma_G, \sigma)$ with $D^s(G^J)$ as in Theorem 3.3.3, then the map σ^{s*} is given by*

$$\sigma^{s*}[p, k] = [\gamma^s(p), k],$$

   for any p in G^{J-1} and $k \geq 1$.

2. *If we consider $\sigma^{-1} : (\Sigma_G, \sigma) \to (\Sigma_G, \sigma)$, then it is an s-bijective factor map. For any $J \geq 1$, if we identity $D^u(\Sigma_G, \sigma)$ with $D^u(G^J)$ as in Theorem 3.3.3, then the map $(\sigma^{-1})^{u*}$ is given by*

$$(\sigma^{-1})^{u*}[p, k] = [\gamma^u(p), k],$$

   for any p in G^{J-1} and $k \geq 1$.

Proof. This follows immediately from the corresponding result Theorem 3.4.7 for the covariant case. Just consider the first statements for a moment. In 3.4.7 we consider the image of local stable sets under the map σ^{-1}. In this case, we must consider pre-images of local stable sets under under σ. As σ is a homeomorphism, these coincide. Or put briefly, $\sigma^{-1} = \sigma^{-1}$. □

Up to this point, this section has been devoted to proving results in the contravariant case which are analogous to those of the last section in the covariant case. The last item here concerns a mix of the two situations, where both type of functorial properties are in play. Specifically, we will consider a commutative diagram of maps

$$\begin{array}{ccc} (Z, \zeta) & \xrightarrow{\eta_1} & (Y_1, \psi_1) \\ \downarrow{\scriptstyle \eta_2} & & \downarrow{\scriptstyle \pi_1} \\ (Y_2, \psi_2) & \xrightarrow{\pi_2} & (X, \varphi) \end{array} .$$

Our interest will be in the situation where the horizontal maps are s-bijective and the vertical maps are u-bijective.

Before getting to the main result, we need to establish the following Lemma. Observe that the map sending z in Z to $(\eta_2(z), \eta_1(z))$ has range in the fibred product of π_2 and π_1. We denote this map by $\eta_2 \times \eta_1$.

Lemma 3.5.10. *Let $(Z,\zeta),(Y_1,\psi_1),(Y_2,\psi_2)$ and (X,φ) be dynamical systems and suppose that*

$$\begin{array}{ccc} (Z,\zeta) & \xrightarrow{\eta_1} & (Y_1,\psi_1) \\ \downarrow{\scriptstyle \eta_2} & & \downarrow{\scriptstyle \pi_1} \\ (Y_2,\psi_2) & \xrightarrow{\pi_2} & (X,\varphi) \end{array}$$

is a commutative diagram of maps. Assume that the map $\eta_2 \times \eta_1$ is a surjection from Z to the fibred product of π_2 and π_1. Then, for any set E in Y_2, we have

$$\pi_1^{-1}(\pi_2(E)) = \eta_1(\eta_2^{-1}(E)).$$

Proof. We first prove the containment $\subset$. Let y be in $\pi_1^{-1}(\pi_2(E))$. This means that $\pi_1(y) = \pi_2(y_2)$, for some y_2 in E. This implies (y_2, y) is in the fibred product so we may find z in Z such that $\eta_2 \times \eta_1(z) = (y_2, y)$. As $\eta_2(z) = y_2$, which is in E, z is in $\eta_2^{-1}(E)$. So $\eta_1(z) = y$ is in $\eta_1(\eta_2^{-1}(E))$.

For the reverse containment, suppose that y is in $\eta_1(\eta_2^{-1}(E))$. This means that $y = \eta_1(z)$ for some z in $\eta_2^{-1}(E)$. Then we have $\pi_2(\eta_2(z)) = \pi_1(\eta_1(z)) = \pi_1(y)$. As $\eta_2(z)$ is in E, y is in $\pi_1^{-1}(\pi_2(E))$. □

Our main result follows more or less immediately from this last Lemma and the definitions given in Theorems 3.4.1 and 3.5.1 and we omit the details.

Theorem 3.5.11. *Suppose that $(\Sigma_1,\sigma),(\Sigma_2,\sigma)$ and (Σ_0,σ) are shifts of finite type and*

$$\pi_i : (\Sigma_i,\sigma) \to (\Sigma_0,\sigma), i = 1,2,$$

are factor maps with π_1 u-bijective and π_2 s-bijective. Let (Σ,σ) be the fibred product:

$$\begin{array}{ccc} (\Sigma,\sigma) & \xrightarrow{\eta_1} & (\Sigma_1,\sigma) \\ \downarrow{\scriptstyle \eta_2} & & \downarrow{\scriptstyle \pi_1} \\ (\Sigma_2,\sigma) & \xrightarrow{\pi_2} & (\Sigma_0,\sigma) \end{array}$$

Then we have

$$\eta_1^s \circ \eta_2^{s*} = \pi_1^{s*} \circ \pi_2^s : D^s(\Sigma_2,\sigma) \to D^s(\Sigma_1,\sigma)$$

and

$$\eta_2^u \circ \eta_1^{u*} = \pi_2^{u*} \circ \pi_1^u : D^u(\Sigma_1,\sigma) \to D^u(\Sigma_2,\sigma).$$

Proof. We prove the first statement only. It suffices for us to consider e in Σ_2 and a compact open subset E of $\Sigma_2^s(e)$. By hypothesis, π_2 maps $\Sigma_2(e)$ bijectively to $\Sigma_0(\pi_2(e))$ and, by Theorem 2.5.12, $\pi_2(E)$ is compact and open. Apply the first part of Theorem 2.5.3 to the map π_1 (while reversing s and u) to obtain $f_1, \ldots, f_K$ in Σ_1 with

$$\pi_1^{-1}(\Sigma_0^s(\pi_2(e))) = \cup_{k=1}^K \Sigma_1^s(f_k).$$

It follows from Theorem 3.5.1 that we may find compact open sets $F_k \subset \Sigma_1^s(f_k)$ satisfying the conditions of 3.5.1 so that

$$\pi_1^{s*} \circ \pi_2^s[E] = [F_1] + \cdots + [F_K].$$

For each $1 \leq k \leq K$, we may find e_k in $\Sigma_2^s(e)$ such that $\pi_2(e_k) = \pi_1(f_k)$. It is easily seen that the points $(e_k, f_k), 1 \leq k \leq K$ satisfy the first part of the conclusion of Theorem 2.5.3 for the factor map η_2 and the point e and also that the sets $\eta_1^{-1}(F_k), 1 \leq k \leq K$ satisfy the conclusion of Theorem 3.5.1 for the map η_2 and the set E. This means that

$$\begin{aligned} \eta_1^s \circ \eta_2^{s*} &= \eta_1^s([\eta_1^{-1}(F_1)] + \cdots + [\eta_1^{-1}(F_K)] \\ &= [F_1] + \cdots + [F_K]. \end{aligned}$$

This completes the proof. □

Chapter 4

The complexes of an s/u-bijective factor map

In this chapter, we consider a shift of finite type, (Σ, σ), a Smale space, (X, φ), and a factor map

$$\rho : (\Sigma, \sigma) \to (X, \varphi).$$

Under the hypothesis that ρ is s-bijective, we associate to this data two chain complexes and two cochain complexes which are based on the dimension group invariants from the last chapter applied to the systems $(\Sigma_N(\pi), \sigma)$ of Section 2.4. There are analogous complexes in the case of a u-bijective map. There is obviously a degree of symmetry in our definitions. Usually, we state main definitions and results in all cases, but on occasion we will just use one case for brevity with the understanding that analogous results hold for the other.

This situation is obviously a rather special one. Given $(X, \varphi), (\Sigma, \sigma)$ and ρ as above, it follows from Theorem 2.5.12 that the stable sets of (Σ, σ) and (X, φ) are homeomorphic and hence the latter must be totally disconnected and this is a severe restriction.

Before proceeding, let us mention some related work. First, Williams [36] defined the notion of an *expanding attractor*. These are basic sets where the stable coordinate is totally disconnected, while the unstable coordinate is a Euclidean space. He provided a construction of such basic sets as inverse limits of branched manifolds and also proved that every expanding attractor could be written in this way. More recently, Wieler [37] gave a generalization of this result to the setting of Smale spaces. While keeping the hypothesis that the stable sets are totally disconnected, there is no hypothesis on the unstable sets. She also provides a construction via inverse limits and also shows that every irreducible Smale space with totally disconnected stable sets can be written in this way.

Let us take a moment to hint, at least, at the later importance of this situation. If we begin with an arbitrary Smale space (X, φ) with an s/u-bijective pair, then, for each fixed $L \geq 0$, we have an s-bijective factor map $\rho_{L,} : (\Sigma_{L,0}(\pi), \sigma) \to (Y_L(\pi_s), \psi)$, from Theorem 2.6.11, whose domain is a shift of finite type. In the next chapter, we will associate double complexes to the s/u-bijective pair π and the restriction of one of these to its Lth row will be one of the complex we consider here for the map $\rho_{L,}$.

The first section will simply give the definitions and establish the important fact that these are indeed chain complexes. In the second section, we assume that we have a graph G which presents the map ρ and describe our complexes in terms of the graph and its dimension group invariants. In the third section, we establish a basic result that the two chain complexes for an s-bijective factor map have isomorphic homologies. There are three other analogous results.

The fourth section elaborates on the functorial properties of the chain complexes and their homologies.

The fifth section is devoted to proving a very important result: that the homology of our complexes do not depend on (Σ, σ) or ρ, but only on (X, φ), in some natural sense. Again, this will be an important ingredient in proving analogous results for the double complexes which follow in the next chapter.

4.1 Definitions of the complexes

In this section, we give the definitions. If $\rho : (\Sigma, \sigma) \to (X, \varphi)$ is an s-bijective factor map, then we may form the systems $(\Sigma_N(\rho), \sigma)$, for each $N \geq 0$ exactly as in Definition 2.4.3. From Theorem 2.4.5, each such system is also a shift of finite type. Moreover, for $0 \leq n \leq N$, there is a factor map $\delta_n : (\Sigma_N(\rho), \sigma) \to (\Sigma_{N-1}(\rho), \sigma)$ which deletes entry n. It follows from Theorem 2.5.14 that each of these maps is also s-bijective and so we may invoke the functorial properties of the last chapter on the invariants. It follows that

$$\begin{aligned}\delta_n^s : D^s(\Sigma_N(\rho), \sigma) &\rightarrow D^s(\Sigma_{N-1}(\rho), \sigma)\\ \delta_n^{u*} : D^u(\Sigma_{N-1}(\rho), \sigma) &\rightarrow D^u(\Sigma_N(\rho), \sigma)\end{aligned}$$

are well-defined by Theorem 3.4.1. There are analogous statements for the case ρ is u-bijective.

Remark 4.1.1. *For the rest of this chapter (and much of what follows later) it will be convenient to drop the σ from our notation and simply write $D^s(\Sigma_N(\pi))$.*

Definition 4.1.2. *Let (Σ, σ) be a shift of finite type, let (X, φ) be a Smale space and let*

$$\rho : (\Sigma, \sigma) \rightarrow (X, \varphi).$$

be a factor map.

1. *Suppose that ρ is s-bijective.*
   We define $d^s(\rho)_N : D^s(\Sigma_N(\rho)) \rightarrow D^s(\Sigma_{N-1}(\rho))$ by

   $$d^s(\rho)_N = \sum_{n=0}^{N} (-1)^n (\delta_n)^s,$$

   for $N \geq 1$. We set $d^s(\rho)_0$ to be the zero map. We also define $d^{u*}(\rho)_N : D^u(\Sigma_N(\rho)) \rightarrow D^u(\Sigma_{N+1}(\rho))$ by

   $$d^{u*}(\rho)_N = \sum_{n=0}^{N+1} (-1)^n (\delta_n)^{u*},$$

   for $N \geq 0$.

2. *Suppose that ρ is u-bijective. We define*
   $d^{s*}(\rho)_N : D^s(\Sigma_N(\rho)) \rightarrow D^s(\Sigma_{N+1}(\rho))$ by

   $$d^{s*}(\rho)_N = \sum_{n=0}^{N+1} (-1)^n (\delta_n)^{s*},$$

   for $N \geq 0$. We also define $d^u(\rho)_N : D^u(\Sigma_N(\rho)) \rightarrow D^u(\Sigma_{N-1}(\rho))$ by

   $$d^u(\rho)_N = \sum_{n=0}^{N} (-1)^n (\delta_n)^u,$$

   for $N \geq 1$. We set $d^u(\rho)_0$ to be the zero map.

The first important property is that these groups with these maps form chain or cochain complexes.

Lemma 4.1.3. *Let*

$$\rho : (\Sigma, \sigma) \to (X, \varphi).$$

be a factor map. If ρ is s-bijective then

$$d^s(\rho)_{N-1} \circ d^s(\rho)_N = 0, \quad d^{u*}(\rho)_{N+1} \circ d^{u*}(\rho)_N = 0,$$

for all integers N. If ρ is u-bijective then

$$d^u(\rho)_{N-1} \circ d^u(\rho)_N = 0, \quad d^{s*}(\rho)_{N+1} \circ d^{s*}(\rho)_N = 0,$$

for all integers N.

Proof. We prove the first part only and we may assume $N \geq 2$. The composition is the sum of terms $(-1)^{m+n}(\delta_m)^s \circ (\delta_n)^s$, ranging over $0 \leq m \leq N-1$ and $0 \leq n \leq N$. The map sending (m, n) to $(n, m+1)$ is a bijection between the set of pairs with $m \geq n$ and those with $m < n$. Moreover, for such a pair (m, n), we have

$$(\delta_m)^s \circ (\delta_n)^s = (\delta_m \circ \delta_n)^s = (\delta_n \circ \delta_{m+1})^s = (\delta_n)^s \circ (\delta_{m+1})^s,$$

where we have used the functorial property of D^s from Theorem 3.4.1 in the first and last steps. So each pair (m, n) with $m \leq n$ now appears twice, once with coefficient $(-1)^{m+n}$ and once with coefficient $(-1)^{n+m-1}$ and hence the sum is zero. □

To summarize, we have proved the following.

Theorem 4.1.4. *Let (Σ, σ) be a shift of finite type, let (X, φ) be a Smale space and let*

$$\rho : (\Sigma, \sigma) \to (X, \varphi).$$

be a factor map.

1. *If ρ is s-bijective then $(D^s(\Sigma_N(\rho)), d^s(\rho)_N)$ is a chain complex.*
2. *If ρ is s-bijective then $(D^u(\Sigma_N(\rho)), d^{u*}(\rho)_N)$ is a cochain complex.*
3. *If ρ is u-bijective then $(D^s(\Sigma_N(\rho)), d^{s*}(\rho)_N)$ is a cochain complex.*

4. *If ρ is u-bijective then $(D^u(\Sigma_N(\rho)), d^u(\rho)_N)$ is a chain complex.*

When considering the homology of a simplicial complex, there are two different complexes, usually called the ordered complex and the alternating complex. It is an important fact that the two yield the same homology groups. In the first case, one considers the N-chains as the free abelian group generated by all $N+1$-tuples $(v_0, v_1, \ldots, v_N)$, whose elements are contained in a simplex. That is, the order of the entries is considered and repeated entries are allowed. In the second case, one ignores all $N+1$-tuples containing a repeated entry and the order is not considered, except for orientation. That is, the generators are the N-simplices. More rigorously, the alternating complex is a quotient of the ordered in the computation of simplicial homology. In the computation of cohomology, the alternating complex has a slightly different definition and it is a sub-complex of the ordered. The ordered complex has various theoretic advantages, especially in terms of the use of the formulae involved in the boundary maps, and is more natural from a combinatorial point of view. The alternating has substantial advantages for performing computations (the groups are "smaller") and is more natural from a geometric point of view. Most modern texts on the subject deal only with the alternating complex.

The complexes above are analogous to the ordered complex. We now introduce a version of the alternating. There is some subtlety here as it is not so obvious exactly what a simplex is in our systems $\Sigma_N(\rho)$. This can be avoided by considering the actions of the permutation groups at the level of the invariant.

We begin, as before, with an s-bijective factor map from a shift of finite type, $\rho : (\Sigma, \sigma) \to (X, \varphi)$. For $N \geq 0$, (Σ_N, σ) has an action of the group S_{N+1}, written on the right, we may regard this action as an s-bijective or u-bijective factor map. For an element α in S_{N+1}, we write α^s for induced map on the invariant $D^s(\Sigma_N)$.

Definition 4.1.5. *Let (Σ_N, σ) be a shift of finite type which has an action of the permutation group S_{N+1} (commuting with the shift).*

1. *We define $D^s_{\mathcal{A}}(\Sigma_N)$ to be the subgroup of $D^s(\Sigma_N)$ of all a such that $\alpha^s(a) = sgn(\alpha)a$, for all α in S_{N+1}, and we let J denote the inclusion map of this subgroup in the group.*

2. *We define $D^s_{\mathcal{B}}(\Sigma_N)$ to be the subgroup of $D^s(\Sigma_N)$ which is generated by all elements:*

(a) *b satisfying $\alpha^s(b) = b$, for some non-trivial transposition α in S_{N+1}, and*

(b) *all elements of the form $b - sgn(\alpha)\alpha^s(b)$.*

3. *We define $D^s_{\mathcal{Q}}(\Sigma_N)$ to be the quotient of $D^s(\Sigma_N)$ by the subgroup $D^s_{\mathcal{B}}(\Sigma_N)$ and we let Q denote the quotient map.*

Our alternating complexes will be formed by the groups $D^s_{\mathcal{A}}(\Sigma_N(\rho))$ and $D^s_{\mathcal{Q}}(\Sigma_N(\rho))$ of the last definition. Let us take a moment to explain why two different complexes are needed. The quotient $D^s_{\mathcal{Q}}(\Sigma_N(\rho))$ is most naturally suited to chain complexes, while $D^s_{\mathcal{A}}(\Sigma_N(\rho))$ is most naturally suited to cochain complexes. We need to see these groups are compatible with our boundary maps.

Lemma 4.1.6. *Let (Σ, σ) be a shift of finite type, (X, φ) be a Smale space and $\rho : (\Sigma, \sigma) \to (X, \varphi)$ be a factor map.*

1. *Suppose ρ is s-bijective. We have*

$$d^s(\rho)_N(D^s_{\mathcal{B}}(\Sigma_N(\rho))) \subset D^s_{\mathcal{B}}(\Sigma_{N-1}(\rho))$$

and, hence, $d^s(\rho)_N$ descends to a well-defined map, denoted $d^s_{\mathcal{Q}}(\rho)_N$, on $D^s_{\mathcal{Q}}(\Sigma_N)$. In addition, we have

$$d^{u*}(\rho)_N(D^u_{\mathcal{A}}(\Sigma_N(\rho))) \subset D^u_{\mathcal{A}}(\Sigma_{N+1}(\rho))$$

and, hence, $d^{u*}(\rho)_N$ restricts to a well-defined map, denoted $d^{u*}_{\mathcal{A}}(\rho)_N$, on $D^u_{\mathcal{A}}(\Sigma_N(\rho))$.

2. *Suppose ρ is u-bijective. We have*

$$d^u(\rho)_N(D^u_{\mathcal{B}}(\Sigma_N(\rho))) \subset D^u_{\mathcal{B}}(\Sigma_{N-1}(\rho))$$

and, hence, $d^u(\rho)_N$ descends to a well-defined map, denoted $d^u_{\mathcal{Q}}(\rho)_N$, on $D^u_{\mathcal{Q}}(\Sigma_N(\rho))$. In addition, we have

$$d^{s*}(\rho)_N(D^s_{\mathcal{A}}(\Sigma_N(\rho))) \subset D^s_{\mathcal{A}}(\Sigma_{N+1}(\rho))$$

and, hence, $d^{s*}(\rho)_N$ restricts to a well-defined map, denoted $d^{s*}_{\mathcal{A}}(\rho)_N$, on $D^s_{\mathcal{A}}(\Sigma_N(\rho))$.

Proof. We prove the first part only. For the first statement, we first consider b in $D^s_{\mathcal{B}}(\Sigma_N(\rho))$ such that $\alpha^s(b) = b$, for some $\alpha = (i\,j)$ in S_{N+1}. We may suppose that $i < j$. We must check that $d^s(\rho)_N(b)$ is in $D^u_{\mathcal{B}}(\Sigma_{N-1}(\rho))$. For $n \neq i, j$, it is clear that $\delta_n \circ \alpha = \beta \circ \delta_n$, for some transposition β in S_N. Then we have

$$\delta^s_n(b) = \delta^s_n \circ \alpha^s(b) = (\delta_n \circ \alpha)^s(b) = (\beta \circ \delta_n)^s(b) = \beta^s \circ \delta^s_n(b)$$

and so $\delta^s_n(b)$ is in $D^s_{\mathcal{B}}(\Sigma_{N-1}(\rho))$. Also, we have $\delta_i \circ \alpha = \beta \circ \delta_j$, where $\beta = (i\,j-1\,j-2\cdots i+1)$ is a cyclic permutation and $sgn(\beta) = (-1)^{j-i+1}$. It follows that

$$\begin{aligned}
(-1)^i\delta^s_i(b) + (-1)^j\delta^s_j(b) &= (-1)^i\delta^s_i(\alpha^s(b)) + (-1)^j\delta^s_j(b) \\
&= (-1)^i\delta^s_i(\alpha^s(b)) + (-1)^j\delta^s_j(b) \\
&= (-1)^i(\delta_i \circ \alpha)^s(b) + (-1)^j\delta^s_j(b) \\
&= (-1)^i(\beta \circ \delta_j)^s(b) + (-1)^j\delta^s_j(b) \\
&= (-1)^i\beta^s \circ \delta^s_j(b) + (-1)^j\delta^s_j(b) \\
&= (-1)^j(\delta^s_j(b) - (-1)^{i-j-1}\beta^s \circ \delta^s_j(b)) \\
&= (-1)^j(\delta^s_j(b) - sgn(\beta)\beta^s \circ \delta^s_j(b))
\end{aligned}$$

is in $D^s_{\mathcal{B}}(\Sigma_{N-1}(\rho))$.

Next, we consider any a in $D^s(\Sigma_N(\rho), \sigma)$, α in S_{N+1} and the element $b = a - sgn(\alpha)\alpha^s(a)$, which is in $D^s_{\mathcal{B}}(\Sigma_N(\rho))$ and show that $d^s(\rho)_N(b)$ is in $D^s_{\mathcal{B}}(\Sigma_{N-1}(\rho))$. It suffices to check $\alpha = (i\,i+1)$, $0 \leq i < N$, since such elements generate S_{N+1}. For $n \neq i, i+1$, we have $\delta_n \circ \alpha = \beta \circ \delta_n$, for some transposition β in S_N and so

$$\begin{aligned}
\delta^s_n(b) &= \delta^s_n(a + \alpha^s(a)) \\
&= \delta^s_n(a) + \delta^s_n \circ \alpha^s(a) \\
&= \delta^s_n(a) + (\delta_n \circ \alpha)^s(a) \\
&= \delta^s_n(a) + (\beta \circ \delta_n)^s(a) \\
&= \delta^s_n(a) + \beta^s(\delta^s_n(a))
\end{aligned}$$

is in $D^s_{\mathcal{B}}(\Sigma_{N-1}(\rho))$. On the other hand, $\delta_i \circ \alpha = \delta_{i+1}$ and $\delta_{i+1} \circ \alpha = \delta_i$ so the remaining terms in the sum giving $d^s(\rho)_N(b)$ are

$$\begin{aligned}(-1)^i\delta_i^s(b)+(-1)^{i+1}\delta_{i+1}^s(b) &= (-1)^i\delta_i^s(a)+(-1)^i\delta_i^s(\alpha^s(a)) \\ &\quad +(-1)^{i+1}\delta_{i+1}^s(a)+(-1)^{i+1}\delta_{i+1}^s(\alpha^s(a)) \\ &= (-1)^i\delta_i^s(a)+(-1)^i(\delta_i\circ\alpha)^s(a)) \\ &\quad +(-1)^{i+1}\delta_{i+1}^s(a)+(-1)^{i+1}(\delta_{i+1}\circ\alpha)^s(a)) \\ &= (-1)^i\delta_i^s(a)+(-1)^i\delta_{i+1}^s(a)) \\ &\quad +(-1)^{i+1}\delta_{i+1}^s(a)+(-1)^{i+1}\delta_i^s(a)) \\ &= 0.\end{aligned}$$

This completes the proof of the first statement.

Now we must check the second statement. Let a be in $D^u(\Sigma_N(\rho))$ and suppose that $\alpha^s(a) = sgn(\alpha)a$, for all α in S_{N+1}. We must check that $\alpha^u(d^{u*}(\rho)_N(a)) = sgn(\alpha)d^{u*}(\rho)_N(a)$, for every α in S_{N+2}. It suffices to check the case $\alpha = (i\, i+1)$, for some $0 \leq i < N+1$, since such elements generate S_{N+2}. We use the conditions observed earlier: for $n \neq i, i+1$, we have $\delta_n \circ \alpha = \beta \circ \delta_n$ for some transposition β in S_{N+1} and

$$\begin{aligned}\alpha^u(\delta_n^{u*}(a)) &= (\delta_n\circ\alpha)^{u*}(a) \\ &= (\beta\circ\delta_n)^{u*}(a) \\ &= \delta_n^{u*}(\beta^u(a)) \\ &= \delta_n^{u*}(-a) \\ &= -\delta_n^{u*}(a).\end{aligned}$$

On the other hand, we also have $\delta_i \circ \alpha = \delta_{i+1}, \delta_{i+1} \circ \alpha = \delta_i$ and

$$\begin{aligned}\alpha^u((-1)^i\delta_i^{u*}(a)+(-1)^{i+1}\delta_{i+1}^{u*}(a)) &= (-1)^i(\delta_i\circ\alpha)^{u*}(a) \\ &\quad +(-1)^{i+1}(\delta_{i+1}\circ\alpha)^{u*}(a) \\ &= (-1)^i\delta_{i+1}^{u*}(a) \\ &\quad +(-1)^{i+1}\delta_i^{u*}(a) \\ &= -((-1)^i\delta_i^{u*}(a) \\ &\quad +(-1)^{i+1}\delta_{i+1}^{u*}(a))).\end{aligned}$$

Putting all this together yields the desired conclusion. □

Theorem 4.1.7. *Let (Σ, σ) be a shift of finite type, let (X, φ) be a Smale space and let*

$$\rho : (\Sigma, \sigma) \to (X, \varphi).$$

be a factor map.

1. *If ρ is s-bijective then $(D^s_{\mathcal{Q}}(\Sigma_N(\rho)), d^s_{\mathcal{Q}}(\rho)_N)$ is a chain complex and the quotient map Q from $D^s(\Sigma_N(\rho))$ to $D^s_{\mathcal{Q}}(\Sigma_N(\rho))$ is a chain map.*

2. *If ρ is s-bijective then $(D^u_{\mathcal{A}}(\Sigma_N(\rho)), d^{u*}_{\mathcal{A}}(\rho)_N)$ is a cochain complex and the inclusion map J from $D^u_{\mathcal{A}}(\Sigma_N(\rho))$ to $D^u(\Sigma_N(\rho))$ is a cochain map.*

3. *If ρ is u-bijective then $(D^s_{\mathcal{A}}(\Sigma_N(\rho)), d^{s*}_{\mathcal{A}}(\rho)_N)$ is a cochain complex and the inclusion map J from $D^s_{\mathcal{A}}(\Sigma_N(\rho))$ to $D^s(\Sigma_N(\rho))$ is a cochain map.*

4. *If ρ is u-bijective then $(D^u_{\mathcal{Q}}(\Sigma_N(\rho)), d^u_{\mathcal{Q}}(\rho)_N)$ is a chain complex and the quotient map Q from $D^u(\Sigma_N(\rho))$ to $D^u_{\mathcal{Q}}(\Sigma_N(\rho))$ is a chain map.*

4.2 Symbolic presentations

We now work under the assumption that our shift of finite type map is presented by a graph and the factor map is regular (as in 2.3.3). That is, we assume that $\rho : (\Sigma_G, \sigma) \to (X, \varphi)$ is s-bijective and regular. We let K satisfy the conclusion of Lemma 2.7.2. We also note that, by Lemma 2.7.3, the same K satisfies the same condition for each map $\delta_n : (\Sigma_{G_N}, \sigma) \to (\Sigma_{G_{N-1}}, \sigma)$, for all $N \geq 1$ and $0 \leq n \leq N$. We recall the maps $\delta^{s,K}_n$ as defined in 3.4.3 and $\delta^{u*,K}_n$ as defined in 3.5.4

We make an obvious analogue of the Definition 4.1.2. For this definition, we also include the case for a u-bijective factor map.

Definition 4.2.1. *Let G be a graph, (X, φ) be a Smale space and $\rho : (\Sigma_G, \sigma) \to (X, \varphi)$ be factor map.*

1. *If ρ is an s-bijective factor map and K satisfies the conclusion of Lemma 2.7.2, for each $N \geq 1$ and $k \geq 0$, we define $d^{s,K}(\rho)_N : \mathbb{Z}G^k_N \to \mathbb{Z}G^{k+K}_{N-1}$ by*
$$d^{s,K}(\rho)_N = \sum_{n=0}^{N} (-1)^n \delta^{s,K}_n,$$
and, for $N \geq 0$ and $k \geq 0$, we define $d^{u*,K}(\rho)_N : \mathbb{Z}G^{k+K}_N \to \mathbb{Z}G^k_{N+1}$ by
$$d^{u*,K}(\rho)_N = \sum_{n=0}^{N+1} (-1)^n \delta^{u*,K}_n.$$

2. *If ρ is a u-bijective factor map and K satisfies the conclusion of Lemma 2.7.2, for each $N \geq 1$ and $k \geq 0$, we define $d^{u,K}(\rho)_N : \mathbb{Z}G^k_N \to \mathbb{Z}G^{k+K}_{N-1}$ by*

$$d^{u,K}(\rho)_N = \sum_{n=0}^{N} (-1)^n \delta^{u,K}_n,$$

and, for $N \geq 0$ and $k \geq 0$, we define $d^{s*,K}(\rho)_N : \mathbb{Z}G^{k+K}_N \to \mathbb{Z}G^k_{N+1}$ by

$$d^{s*,K}(\rho)_N = \sum_{n=0}^{N+1} (-1)^n \delta^{s*,K}_n.$$

We take a moment to state one easy result. It will be used later. The proof is actually an immediate consequence of applying Lemma 3.5.6 to each of the s-bijective factor maps δ_n.

Lemma 4.2.2. *Let G be a graph, (X, φ) be a Smale space, $\rho : (\Sigma_G, \sigma) \to (X, \varphi)$ be an s-bijective factor map and K satisfying the conclusion of Lemma 2.7.2. For each $N \geq 1$ and $k \geq 0$, we have*

$$< p, d^{s,K}(\rho)_N(q) >_{G^{k+K}_{N-1}} = < d^{u*,K}(\rho)_{N-1}(p), q >_{G^k_N},$$

for all p in $\mathbb{Z}G^{k+K}_{N-1}$ and q in $\mathbb{Z}G^k_N$.

For the moment, we will concentrate on the maps $d^{s,K}(\rho)_N$. We summarize their basic properties in the next result. The proof is trivial since $d^{s,K}(\rho)_N$ is a sum of maps $\delta^{s,K}_n$ and the conclusions hold for each of them by virtue of Theorems 3.4.4 and 3.4.5.

Theorem 4.2.3. *Let G be a graph, (X, φ) be a Smale space and $\rho : (\Sigma_G, \sigma) \to (X, \varphi)$ be an s-bijective factor map. Suppose that K satisfies the conclusion of Lemma 2.7.2 for the map ρ.*

1. *We have*

$$\begin{aligned} d^{s,K}(\rho)_N \circ t^* &= d^{s,K+1}(\rho)_N = t^* \circ d^{s,K}(\rho)_N, \\ d^{s,K}(\rho)_N \circ i &= i \circ d^{s,K}(\rho)_N, \\ d^{s,K}(\rho)_N \circ \gamma^s &= \gamma^s \circ d^{s,K}(\rho)_N \end{aligned}$$

Moreover, for any $k \geq 0$, if we identify $D^s(\Sigma_{G_N})$ with $D^s(G^k_N)$ and $D^s(\Sigma_{G_{N-1}})$ with $D^s(G^{k+K}_{N-1})$, by Theorem 3.2.3, then we have

$$d^s(\rho)_N[a, j] = [d^{s,K}(\rho)_N(a), j],$$

for all a in $\mathbb{Z}G^{k-1}_N$ and $j \geq 1$.

2. *We have*

$$\begin{array}{rcl} d^{u*,K}(\rho)_N \circ t &=& d^{u*,K+1}(\rho)_N \quad = \quad t \circ d^{u*,K}(\rho)_N, \\ d^{u*,K}(\rho)_N \circ i^* &=& i^* \circ d^{u*,K}(\rho)_N, \\ d^{u*,K}(\rho)_N \circ \gamma^u &=& \gamma^u \circ d^{u*,K}(\rho)_N \end{array}$$

Moreover, for any $k \geq 0$, if we identify $D^u(\Sigma_{G_N})$ with $D^u(G_N^k)$ and $D^u(\Sigma_{G_N})$ with $D^u(G_N^{k+K})$, by Theorem 3.2.3, then we have

$$d^{u*}(\rho)_N[a, j] = [d^{u*,K}(\rho)_N(a), j],$$

for all a in $\mathbb{Z}G_N^k$ and $j \geq 1$.

We now turn to the issue of local formulae for our alternating complexes.

Definition 4.2.4. *Let $k, N \geq 0$.*

1. *Let*

$$\mathcal{A}(G_N^k, S_{N+1}) = \{a \in \mathbb{Z}G_N^k \mid a \cdot \alpha = sgn(\alpha)a, \text{ for all } \alpha \in S_{N+1}\},$$

which is a subgroup of $\mathbb{Z}G_N^k$. We denote the inclusion map by J.

2. *We let $\mathcal{B}(G_N^k, S_{N+1})$ denote the subgroup of $\mathbb{Z}G_N^k$ by the subgroup generated by all elements p in G_N^k such that $p = p \cdot \alpha$, for some transposition α and all elements $p - sgn(\alpha)p \cdot \alpha$, where p is in G_N^k and α is in S_{N+1}.*

3. *We let $\mathcal{Q}(G_N^k, S_{N+1})$ denote the quotient*

$$\mathbb{Z}G_N^k / \mathcal{B}(G_N^k, S_{N+1}),$$

and Q denote the quotient map.

The following result is standard and we omit the proof, but we state it for future reference. It provides us with a basic idea of duality between the two constructions.

Lemma 4.2.5. *Let $k, N \geq 0$.*

1. *For any a in $\mathbb{Z}G_N^k$, $< a, b >= 0$, for all b in $\mathcal{B}(G_N^k, S_{N+1})$ if and only if a is in $\mathcal{A}(G_N^k, S_{N+1})$.*

2. *For any b in $\mathbb{Z}G_N^k$, $< a, b >= 0$, for all a in $\mathcal{A}(G_N^k, S_{N+1})$ if and only if b is in $\mathcal{B}(G_N^k, S_{N+1})$.*

Lemma 4.2.6. *Let $N \geq 0$.*

1. *For any $k \geq 1$, we have*

$$\begin{aligned} i(\mathcal{A}(G_N^k, S_{N+1})) &\subset \mathcal{A}(G_N^{k-1}, S_{N+1}) \\ t(\mathcal{A}(G_N^k, S_{N+1})) &\subset \mathcal{A}(G_N^{k-1}, S_{N+1}) \\ i(\mathcal{B}(G_N^k, S_{N+1})) &\subset \mathcal{B}(G_N^{k-1}, S_{N+1}) \\ t(\mathcal{B}(G_N^k, S_{N+1})) &\subset \mathcal{B}(G_N^{k-1}, S_{N+1}). \end{aligned}$$

2. *For any $k \geq 0$, we have*

$$\begin{aligned} i^*(\mathcal{A}(G_N^k, S_{N+1})) &\subset \mathcal{A}(G_N^{k+1}, S_{N+1}) \\ t^*(\mathcal{A}(G_N^k, S_{N+1})) &\subset \mathcal{A}(G_N^{k+1}, S_{N+1}) \\ i^*(\mathcal{B}(G_N^k, S_{N+1})) &\subset \mathcal{B}(G_N^{k+1}, S_{N+1}) \\ t^*(\mathcal{B}(G_N^k, S_{N+1})) &\subset \mathcal{B}(G_N^{k+1}, S_{N+1}). \end{aligned}$$

3. *The maps i, t, i^*, t^* descend to the quotient groups $\mathcal{Q}(G_N^k, S_{N+1})$ such that, using the same notation, we have*

$$Q \circ i = i \circ Q, \qquad Q \circ t = t \circ Q,$$
$$Q \circ i^* = i^* \circ Q, \qquad Q \circ t^* = t^* \circ Q$$

Proof. The first item is clear since the action of S_{N+1} commutes with i and t. The second item follows from the first and characterization of these subgroups given in Lemma 4.2.5 above. The third item is immediate from the first two. □

Now we can incorporate the inductive limits as follows.

Definition 4.2.7. *Let $N \geq 0$ and $k \geq 1$. We define $\gamma^s = i \circ t^*$ and $\gamma^u = t \circ i^*$, which we regard as endomorphisms of the groups $\mathcal{A}(G_N^{k-1}, S_{N+1})$ and $\mathcal{Q}(G_N^{k-1}, S_{N+1})$, using the same notation for both cases.*

Furthermore, we define

$$\begin{aligned} D_{\mathcal{A}}^s(G_N^k) &= \lim \mathcal{A}(G_N^{k-1}, S_{N+1}) \xrightarrow{\gamma^s} \mathcal{A}(G_N^{k-1}, S_{N+1}) \xrightarrow{\gamma^s}, \\ D_{\mathcal{A}}^u(G_N^k) &= \lim \mathcal{A}(G_N^{k-1}, S_{N+1}) \xrightarrow{\gamma^u} \mathcal{A}(G_N^{k-1}, S_{N+1}) \xrightarrow{\gamma^u}, \\ D_{\mathcal{Q}}^s(G_N^k) &= \lim \mathcal{Q}(G_N^{k-1}, S_{N+1}) \xrightarrow{\gamma^s} \mathcal{Q}(G_N^{k-1}, S_{N+1}) \xrightarrow{\gamma^s}, \\ D_{\mathcal{Q}}^u(G_N^k) &= \lim \mathcal{Q}(G_N^{k-1}, S_{N+1}) \xrightarrow{\gamma^u} \mathcal{Q}(G_N^{k-1}, S_{N+1}) \xrightarrow{\gamma^u}, \end{aligned}$$

We note the following analogue of Theorem 3.2.3 which relating the alternating invariants for the system with those of the graphs, for different values of k.

Theorem 4.2.8. *Let G be a graph and $\rho : (\Sigma_G, \sigma) \to (X, \varphi)$ be an s-bijective regular factor map. For any $k \geq 1$ and $N \geq 0$, let h denote the isomorphism of Theorem 3.3.3 between $D^s(\Sigma_N(\rho))$ and $D^s(G_N^k)$.*

1. *For any e in $D^s(\Sigma_N(\rho))$ with $h(e) = [a, j]$, for some a in $\mathbb{Z}G_N^{k-1}$ and $j \geq 0$, e is in $D^s_{\mathcal{B}}(\Sigma_N(\rho))$ if and only if $[Q(a), j] = 0$ in $D^s_{\mathcal{Q}}(G_N^k)$. Consequently, this map induces an isomorphism between $D^s_{\mathcal{Q}}(\Sigma_N(\rho))$ and $D^s_{\mathcal{Q}}(G_N^k)$.*
2. *For any e in $D^s(\Sigma_N(\rho))$, e is in $D^s_{\mathcal{A}}(\Sigma_N(\rho))$ if and only if $h(e) = [a, j]$, for some a in $\mathcal{A}(G_N^{k-1}, S_{N+1})$ and $j \geq 0$. Consequently, this map induces an isomorphism between $D^s_{\mathcal{A}}(\Sigma_N(\rho))$ and $D^s_{\mathcal{A}}(G_N^k)$.*

Proof. First, it is clear that the map commutes with the action of S_{N+1}.

In the first part, suppose that e is in $D^s_{\mathcal{Q}}(\Sigma_N(\rho))$. There are two cases to consider, the first being that $e \cdot \alpha = e$, for some transposition α. It follows that, for some $j' \geq 0$, $(\gamma^s)^{j'}(a) = (\gamma^s)^{j'}(a \cdot \alpha) = (\gamma^s)^{j'}(a) \cdot \alpha$. We write the element $(\gamma^s)^{j'}(a)$ as a sum over the generating set G_N^{k-1}. For each p in G_N^{k-1} with $p \cdot \alpha = p$, the term in $(\gamma^s)^{j'}(a)$ for p is in $\mathcal{B}(G^{k-1}, S_{N+1})$. As for other p, the equation above means that the coefficient of p and of $p \cdot \alpha$ are equal and so their sum is a multiple of $p - p \cdot \alpha$ and is in $\mathcal{B}(G^{k-1}, S_{N+1})$. In total, we conclude $(\gamma^s)^{j'}(a)$ is in $\mathcal{B}(G^{k-1}, S_{N+1})$. This means that $[Q(a), j] = [Q((\gamma^s)^{j'}(a)), j + j']$ is zero in $D^s_{\mathcal{Q}}(G_N^k)$.

The second case is when $e = e' - sgn(\alpha)e' \cdot \alpha$, for some e' in $D^s(\Sigma_N(\rho))$. Let $h(e') = [b, j]$. It is then immediate that $[a, j] = [b, j] - sgn(\alpha)[b, j] \cdot \alpha = [b - sgn(\alpha)b \cdot \alpha, j]$ and $Q(a) = 0$.

Next, we suppose that $[Q(a), j] = 0$. It follows that a may be written as a sum of terms of the form p, where p is in G_N^{k-1} satisfies $p \cdot \alpha = p$, for some transposition α and terms of the form $q - sgn(\beta)q \cdot \beta$, where q is in G_N^{k-1} and β is in S_{N+1}. In the former case, $h^{-1}[p, j] \cdot \alpha = h^{-1}[p, j]$ is in $D^s_{\mathcal{B}}(\Sigma_N(\rho))$ and in the latter

$$h^{-1}[q - sgn(\beta)q \cdot \beta, j] = h^{-1}[q, j] - sgn(\beta)h^{-1}[q, j] \cdot \beta$$

which is also in $D^s_{\mathcal{B}}(\Sigma_N(\rho))$. We conclude that e is in $D^s_{\mathcal{B}}(\Sigma_N(\rho))$ as desired.

For the second part, the 'if' part is clear. We suppose that e is in $D^s_{\mathcal{A}}(\Sigma_N(\rho))$. For each α in S_{N+1}, we know that

$$[a \cdot \alpha, j] = [a, j] \cdot \alpha = h(e) \cdot \alpha = h(e \cdot \alpha) = sgn(\alpha)h(e) = sgn(\alpha)[a, j].$$

We find J_α such that, for all $j' \geq J_\alpha$, we have $(\gamma^s)^{j'}(a \cdot \alpha) = sgn(\alpha)(\gamma^s)^{j'}(a))$. Letting J be the maximum of J_α, over all α in S_{N+1}, we see that $(\gamma^s)^{j'}(a)$ is in $\mathcal{A}(G_N^{k-1}, S_{N+1})$ and $h(e) = [a, j] = [(\gamma^s)^{j'}(a), j + j']$, for any $j' \geq J$. □

Our next ingredient is to verify that our local version of the boundary operator preserves the subgroups of interest. Its proof is similar to that of Lemma 4.1.6.

Lemma 4.2.9. *Suppose that (X, φ) is a Smale space, G is a graph and that $\rho : (\Sigma_G, \sigma) \to (X, \varphi)$ is a regular factor map. Let $k \geq 0$.*

1. *If ρ is s-bijective and and K satisfies the condition of Lemma 2.7.2, then we have*

$$\begin{aligned} d^{s,K}(\rho)_N(\mathcal{B}(G_N^k, S_{N+1})) &\subset \mathcal{B}(G_{N-1}^{k+K}, S_N), \\ d^{u*,K}(\rho)_N(\mathcal{A}(G_N^{k+K}, S_{N+1})) &\subset \mathcal{A}(G_{N+1}^{k+K}, S_{N+2}). \end{aligned}$$

2. *If ρ is u-bijective and and K satisfies the condition of Lemma 2.7.2, then we have*

$$\begin{aligned} d^{u,K}(\rho)_N(\mathcal{B}(G_N^k, S_{N+1})) &\subset \mathcal{B}(G_{N-1}^{k+K}, S_N), \\ d^{s*,K}(\rho)_N(\mathcal{A}(G_N^{k+K}, S_{N+1})) &\subset \mathcal{A}(G_{N+1}^{k}, S_{N+2}). \end{aligned}$$

Proof. To prove the first statement, we first consider the case that p in G_N^k has a repeated entry, say $p_i = p_j$, where $i < j$. We will actually show that the desired containment actually holds replacing $d^{s,K}(\rho)_N$ by $\delta_n^{s,K}$, for each $0 \leq n \leq N$. The desired conclusion follows.

For $i, j < n$, it is clear that the set $\{\delta_n(q) \mid t^K(q) = p\}$ is invariant under the permutation switching entries i and j. If we eliminate those elements which are fixed by this permutation and hence are in Q, the remaining sum can be written as $Sum(A) + Sum(A \cdot (i\, j))$, for some set A, and hence is in the kernel of Q since $sgn(i\, j) = -1$. A similar argument deals with the other cases $n \neq i, j$. Finally, it is easy to see that if $A_i = \{\delta_i(q) \mid t^K(q) = p\}$ and

$A_j = \{\delta_j(q) \mid t^K(q) = p\}$, then $A_i = A_j \cdot \sigma$, where σ is the cyclic permutation $\sigma = (j-1 \ldots i+1\, i)$. Notice that $sgn(\sigma) = (-1)^{j-1-i}$. It follows that

$$(-1)^i Sum(A_i) + (-1)^j Sum(A_j) = (-1)^i (Sum(A_i) - sgn(\sigma) Sum(A_i \cdot \sigma)),$$

which is in $\mathcal{B}(G^{k+K}_{N-1}, S_N)$.

Next, we must consider a generator of $\mathcal{B}(G^k_N, S_{N+1})$ of the form $p - sgn(\alpha) p \cdot \alpha$, where p is in G^k_N and α is in S_{N+1}. In fact, it suffices to consider α to be the transposition which switches entries i and $i+1$, $0 \leq i < N$, since such elements generate S_{N+1}. It is fairly easy to see that

$$\begin{aligned} \{\delta_i(q) \mid t^K(q) = p \cdot (i\, i+1)\} &= \{\delta_{i+1}(q) \mid t^K(q) = p\}, \\ \{\delta_{i+1}(q) \mid t^K(q) = p \cdot (i\, i+1)\} &= \{\delta_i(q) \mid t^K(q) = p\}. \end{aligned}$$

From this it follows that

$$(-1)^i \delta^{s,K}_i (p + p \cdot (i\, i+1)) + (-1)^{i+1} \delta^{s,K}_{i+1} (p + p \cdot (i\, i+1)) = 0,$$

which is certainly in $\mathcal{B}(G^{k+K}_{N-1}, S_N)$. We are left to consider the remaining terms $\delta_n(p + p \cdot (i\, i+1))$, where $n \neq i, i+1$. Here, it is easy to see that

$$\delta^{s,K}_n (p + p \cdot (i\, i+1)) = \delta^{s,K}_n (p) + \delta^{s,K}_n (p) \cdot (i-1\, i)$$

if $n < i$ and

$$\delta^{s,K}_n (p + p \cdot (i\, i+1)) = \delta^{s,K}_n (p) + \delta^{s,K}_n (p) \cdot (i\, i+1)$$

if $n > i+1$. In all cases, the right hand side is in $\mathcal{B}(G^{k+K}_{N-1}, S_N)$. This completes the proof of the first part.

The proof of the second part is obviously the same. The third and fourth parts follow from the first two and the characterization of $\mathcal{A}(G^{k+K}_{N-1}, S_N)$ given in Lemma 4.2.5. □

Definition 4.2.10. *Suppose that (X, φ) is a Smale space, G is a graph and that $\rho : (\Sigma_G, \sigma) \to (X, \varphi)$ is a regular factor map.*

1. *If ρ is s-bijective, then for any K which satisfies the conclusion of Lemma 2.7.2 and $k \geq 0$, we define $d^{s,K}_{\mathcal{Q}}(\rho)_N$ to be the map from $\mathcal{Q}(G^k_N, S_{N+1})$ to $\mathcal{Q}(G^{k+K}_N, S_{N+1})$ such that*

$$d^{s,K}_{\mathcal{Q}}(\rho)_N \circ Q = Q \circ d^{s,K}(\rho)_N.$$

   We also define, for $k \geq K$, $d^{u*,K}_{\mathcal{A}}(\rho)_N$ to be the restriction of $d^{u*,K}(\rho)_N$ to $\mathcal{A}(G^k_N, S_{N+1})$.

2. *If ρ is u-bijective, then for any K which satisfies the conclusion of Lemma 2.7.2 and $k \geq 0$, we define $d_{\mathcal{Q}}^{u,K}(\rho)_N$ to be the map from $\mathcal{Q}(G_N^k, S_{N+1})$ to $\mathcal{Q}(G_N^{k+K}, S_{N+1})$ such that*

$$d_{\mathcal{Q}}^{u,K}(\rho)_N \circ Q = Q \circ d^{u,K}(\rho)_N.$$

We also define, for $k \geq K$, $d_{\mathcal{A}}^{s*,K}(\rho)_N$ to be the restriction of $d^{s*,K}(\rho)_N$ to $\mathcal{A}(G_N^k, S_{N+1})$.

Finally, we note that the definitions above provide a method for computation of the boundary operator for the alternating complex. Its proof is immediate from the definitions and Theorem 4.2.3.

Theorem 4.2.11. *Suppose that (X, φ) is a Smale space, G is a graph and that $\rho : (\Sigma_G, \sigma) \to (X, \varphi)$ is a regular factor map. Suppose that K satisfies the conclusion of Lemma 2.7.2 for the map ρ.*

1. *Suppose that ρ is s-bijective. Using implicitly the identifications of Theorem 4.2.8, we have*

$$d_{\mathcal{Q}}^{s}(\rho)_N[Q(a), j] = [d_{\mathcal{Q}}^{s,K}(\rho)_N(Q(a)), j],$$

for a in $\mathbb{Z}G_N^k$ and $j \geq 0$, and

$$d_{\mathcal{A}}^{u*}(\rho)_N[a, j] = [d_{\mathcal{A}}^{u*,K}(\rho)_N(a), j]$$

for all a in $\mathcal{A}(G_N^k, S_{N+1})$ and $j \geq 0$.

2. *Suppose that ρ is u-bijective. Using implicitly the identifications of Theorem 4.2.8, we have*

$$d_{\mathcal{Q}}^{u}(\rho)_N[Q(a), j] = [d_{\mathcal{Q}}^{u,K}(\rho)_N(Q(a)), j],$$

for a in $\mathbb{Z}G_N^k$ and $j \geq 0$, and

$$d_{\mathcal{A}}^{s*}(\rho)_N[a, j] = [d_{\mathcal{A}}^{s*,K}(\rho)_N(a), j]$$

for all a in $\mathcal{A}(G_N^k, S_{N+1})$ and $j \geq 0$.

We finish this section with a useful consequence of the symbolic presentations.

Theorem 4.2.12. *Suppose that (X, φ) is a Smale space, (Σ, σ) is a shift of finite type and that $\rho : (\Sigma_G, \sigma) \to (X, \varphi)$ is a factor map which is either s-bijective or u-bijective. Let $N_0 \geq 1$ be such that $\#\rho^{-1}\{x\} \leq N_0$, for all x in X (whose existence follows from Theorem 2.5.3). For any $N \geq N_0$, we have*

$$D^s_{\mathcal{Q}}(\Sigma_N(\rho)) = D^s_{\mathcal{A}}(\Sigma_N(\rho)) = D^u_{\mathcal{Q}}(\Sigma_N(\rho)) = D^u_{\mathcal{A}}(\Sigma_N(\rho)) = 0.$$

Proof. Select a graph G which presents ρ. We suppress the isomorphisms between $\Sigma_N(\rho)$ and Σ_{G_N} given in Theorem 2.4.5. Fix $k \geq 0$. If p is in G^k_N, then $p = e^{[1,k]}$, for some e in $\Sigma_N(\rho)$. From the choice of N and N_0, we see that $e_i = e_j$, for some $i \neq j$. Then we have $p = p \cdot (i\,j)$ and so p is in $\mathcal{B}(G^k_N, S_{N+1})$. Since p was arbitrary, we see that

$$D^s_{\mathcal{Q}}(\Sigma_N(\rho)) = D^u_{\mathcal{Q}}(\Sigma_N(\rho)) = 0.$$

Now suppose that a is in $\mathcal{A}(G^k_N, S_{N+1})$. For any p in G^k_N, we find a transposition $(i\,j)$ as above. As $a = -a \cdot (i\,j)$, the coefficient of p in the expression for a must be zero. But as p was arbitrary, $a = 0$. It follows that

$$D^s_{\mathcal{A}}(\Sigma_N(\rho)) = D^u_{\mathcal{A}}(\Sigma_N(\rho)) = 0.$$

□

In principle, Theorem 4.2.8 provides a method for computation of the alternating invariants $D^s_{\mathcal{A}}(\Sigma_N)$ and $D^s_{\mathcal{Q}}(\Sigma_N)$, but there is a more concrete form available for each.

Recall that the isotropy subgroup of S_{N+1} at an element p in G^k_N is the set of all α such that $p \cdot \alpha = p$ and we say that p has trivial isotropy if this group consists of only the identity element. In this case, the isotropy subgroup of every point in the orbit of p is also trivial.

For each $k \geq 0$ and $N \geq 0$, suppose that B^k_N is a subset of G^k_N which meets each orbit having trivial isotropy exactly once and does not meet any orbit having non-trivial isotropy. For p in B^k_N and $j \geq 1$, let

$$\begin{aligned} t^*_{\mathcal{A}}(p, j) &= \{(q, \alpha) \in G^{k+j}_N \times S_{N+1} \mid t^j(q) = p, i^j(q) \cdot \alpha \in B^k_N\} \\ i^*_{\mathcal{A}}(p, j) &= \{(q, \alpha) \in G^{k+j}_N \times S_{N+1} \mid i^j(q) = p, t^j(q) \cdot \alpha \in B^k_N\}. \end{aligned}$$

We define $\gamma^s_B : \mathbb{Z}B^{k-1}_N \to \mathbb{Z}B^{k-1}_N$ by

$$\begin{aligned}\gamma^s_{\mathcal{A}}(p) &= \sum_{(q,\alpha)\in t^*_{\mathcal{A}}(p,1)} sgn(\alpha) i(q)\cdot\alpha,\\ \gamma^u_{\mathcal{A}}(p) &= \sum_{(q,\alpha)\in i^*_{\mathcal{A}}(p,1)} sgn(\alpha) t(q)\cdot\alpha.\end{aligned}$$

These maps are well-known; they are exactly the *signed subset matrices* as used in the computation of the zeta function for a sofic shift, as described in Theorem 6.4.8 of [23].

Theorem 4.2.13. *Suppose that (X,φ) is a Smale space, (Σ,σ) is a shift of finite type and that $\rho : (\Sigma_G,\sigma) \to (X,\varphi)$ is a regular factor map and is either s-bijective or u-bijective. Let $k \geq 1$ and $N \geq 0$, and suppose $B^k_N \subset G^k_N$ is as above.*

1. *For any $j \geq 1$, we have*

$$\begin{aligned}(\gamma^s_B)^j(p) &= \sum_{(q,\alpha)\in t^*_{\mathcal{A}}(p,j)} sgn(\alpha) i^j(q)\cdot\alpha,\\ (\gamma^u_B)^j(p) &= \sum_{(q,\alpha)\in i^*_{\mathcal{A}}(p,j)} sgn(\alpha) t^j(q)\cdot\alpha.\end{aligned}$$

2. *The restriction of Q from $\mathbb{Z}B^k_N$ to $\mathcal{Q}(G^k_N, S_{N+1})$ is an isomorphism and*

$$Q\circ\gamma^s_B = \gamma^s\circ Q, \qquad Q\circ\gamma^u_B = \gamma^u\circ Q.$$

3. *The map A from $\mathbb{Z}B^k_N$ to $\mathcal{A}(G^k_N, S_{N+1})$ defined by*

$$A(p) = \sum_{\alpha\in S_{N+1}} sgn(\alpha) p\cdot\alpha,$$

for p in B^k_N, is an isomorphism and

$$A\circ\gamma^s_B = \gamma^s\circ A, \qquad A\circ\gamma^u_B = \gamma^u\circ A.$$

Proof. The first statement we show the first part only. The proof is by induction on j, with the case $j = 1$ being the definition. Now assume the result is true for j. By direct computation, for any p in B_N^k, we have

$$\begin{aligned}(\gamma_B^s)^{j+1}(p) &= \gamma_B^s((\gamma_B^s)^j(p)) \\ &= \gamma_B^s(\sum_{(q,\alpha)\in t^*_{\mathcal{A}}(p,j)} sgn(\alpha) i^j(q)) \\ &= \sum_{(q',\alpha')\in t^*_{\mathcal{A}}(i^j(q)\cdot\alpha,1)} \sum_{(q,\alpha)\in t^*_{\mathcal{A}}(p,j)} sgn(\alpha) i(q')\cdot\alpha'.\end{aligned}$$

Suppose that (q'',α'') is in $t^*_{\mathcal{A}}(p,j+1)$. We claim that $i^j\circ t(q'')$ has trivial isotropy. Suppose the contrary. If ρ is s-bijective, then it follows from Lemma 2.5.15 that $i^j(q'')$ also has non-trivial isotropy and hence so does $i^{j+1}(q'')$. But this contradicts the assumption that q'' is in $t^*_{\mathcal{A}}(p,j+1)$. Similarly, if ρ is u-bijective, then it follows from Lemma 2.5.15 that $t(q'')$ also has non-trivial isotropy and hence so does $t^{j+1}(q'')=p$ and we reach another contradiction.

Therefore, there exists a unique α such that $i^j\circ t(q'')\cdot\alpha$ is in B_N^k. Now $(t(q''),\alpha)$ is in $t^*_{\mathcal{A}}(p,j)$ and $(i^j(q'')\cdot\alpha,\alpha^{-1}\alpha'')$ is in $t^*_{\mathcal{A}}(t\circ i^j(q'')\cdot\alpha,1)$. It is fairly easy to check that the map described which associates to $(q'',alpha'')$ the pair $(q,\alpha)=(t(q''),\alpha)$ in $t^*_{\mathcal{A}}(p,j)$ and $(i^j(q'')\cdot\alpha,\alpha^{-1}\alpha'')=(q',\alpha')$ in $t^*_{\mathcal{A}}(q,1)$ is a bijection. Moreover, $i(q')=i^{j+1}(q'')$ and $sgn(\alpha)sgn(\alpha')=sgn(\alpha)sgn(\alpha^{-1}\alpha'')=sgn(\alpha'')$. This means that the sum we have above is equal to

$$(\gamma_B^s)^{j+1}(p)=\sum_{(q'',\alpha'')\in t^*_{\mathcal{A}}(p,j+1)} sgn(\alpha'') i^{j+1}(q'')\cdot\alpha''.$$

The fact that Q is an isomorphism is evident: the inverse may be described as follows. If p is any element of G_N^k having to repeated entries, then $Q^{-1}(p)=sgn(\alpha)p\cdot\alpha$, where α is the unique element of S_{N+1} such that $p\cdot\alpha$ is in B_N^k. For any p in B_N^k, we have

$$\begin{aligned}Q\circ\gamma_B^s(p) &= = Q(\sum_{(q,\alpha)\in t^*_{\mathcal{A}}(p,1)} sgn(\alpha) i(q)\cdot\alpha)) \\ &= \sum_{(q,\alpha)\in t^*_{\mathcal{A}}(p,1)} Q(sgn(\alpha) i(q)\cdot\alpha)) \\ &= \sum_{(q,\alpha)\in t^*_{\mathcal{A}}(p,1)} Q(i(q)).\end{aligned}$$

On the other hand, if we consider the set of all q with $t(q)=p$, this may be divided into two groups: those for which $i(q)$ contains no repeated entry and those for which it does. For each q in the former class, there is a unique α for which (q, α) is in $t^*_{\mathcal{A}}(p, 1)$. For each q in the latter class, $Q(i(q)) = 0$. So we see the sum above is equal to

$$Q \circ \gamma^s_B(p) = \sum_{t(q)=p} Q(i(q)) = Q(i \circ t^*(p)) = Q \circ \gamma^s(p).$$

For the third part, consider the map, R, defined on $\mathbb{Z}G^k_N$ which is the identity on each generator in B^k_N and zero on all others. It is easy to verify this, when restricted to $\mathcal{A}(G^k_N, S_{N+1})$, is an inverse to the map A.

Let p be in B^k_N. First, the map sending q to $q \cdot \alpha^{-1}$ provides a bijection from the set of q with $t(q) = p \cdot \alpha$ and the set of q' with $t(q') = p$. Secondly, consider the set of q in G^{k+1}_N with $t(q) = p$ and let C be the set of such q where $i(q)$ has trivial isotropy and D the those q where $i(q)$ has non-trivial isotropy. For q in D, there exists a transposition β that fixes $i(q)$. Then if the sum of $sgn(\alpha)i(q) \cdot \alpha$ over all α in a coset of the subgroup $\{1, \beta\}$ is zero and hence we conclude that the sum over all α in S_{N+1} is also zero. Putting these things together we obtain

$$\begin{aligned}
\gamma^s \circ A(p) &= \gamma^s(\sum_{\alpha \in S_{N+1}} sgn(\alpha) p \cdot \alpha) \\
&= i(\sum_{t(q)=p\cdot\alpha} \sum_{\alpha \in S_{N+1}} sgn(\alpha) q) \\
&= i(\sum_{\alpha \in S_{N+1}} \sum_{t(q)=p} sgn(\alpha) q \cdot \alpha^{-1}) \\
&= \sum_{t(q)=p} \sum_{\alpha \in S_{N+1}} sgn(\alpha) i(q \cdot \alpha^{-1}) \\
&= \sum_{q \in C} \sum_{\alpha \in S_{N+1}} sgn(\alpha) i(q \cdot \alpha^{-1}) + \sum_{q \in D} \sum_{\alpha \in S_{N+1}} sgn(\alpha) i(q \cdot \alpha^{-1}) \\
&= \sum_{q \in C} \sum_{\alpha \in S_{N+1}} sgn(\alpha) i(q \cdot \alpha^{-1}).
\end{aligned}$$

For each q in C, there exists a unique α_q in S_{N+1} such that $i(q) \cdot \alpha_q$ is in B^k_N. In this case, (q, α_q) is in $t^*_{\mathcal{A}}(p, 1)$ and each element of this sets occurs uniquely in this way. Then we have

$$\begin{aligned}
\gamma^s \circ A(p) &= \sum_{q \in C} \sum_{\alpha \in S_{N+1}} sgn(\alpha\alpha_q^{-1}) i(q \cdot (\alpha\alpha_q^{-1})^{-1}) \\
&= \sum_{(q,\alpha_q) \in t^*_{\mathcal{A}}(p,1)} sgn(\alpha_q) \sum_{\alpha \in S_{N+1}} sgn(\alpha)(i(q) \cdot \alpha_q) \cdot \alpha^{-1} \\
&= \sum_{(q,\alpha_q) \in t^*_{\mathcal{A}}(p,1)} sgn(\alpha_q) A(i(q) \cdot \alpha_q) \\
&= \gamma^s_B \circ A(p).
\end{aligned}$$

□

Corollary 4.2.14. *The groups $D^s_{\mathcal{A}}(\Sigma_N(\rho))$ and $D^s_{\mathcal{Q}}(\Sigma_N(\rho))$ are both isomorphic with the inductive of the system*

$$\mathbb{Z}B^k_N \xrightarrow{\gamma^s_B} \mathbb{Z}B^k_N \xrightarrow{\gamma^s_B} \cdots$$

4.3 Equivalence of the complexes

With the relevant definitions and the methods of computation via the graphs in place, we are ready to state our main theorem of this section, which relates the homology of the ordered and alternating complexes.

Theorem 4.3.1. *Suppose that (X, φ) is a Smale space, G is a graph and that $\rho : (\Sigma_G, \sigma) \to (X, \varphi)$ is a regular factor map.*

1. *If ρ is s-bijective then the map Q is a chain map from $(D^s(\Sigma_*(\rho)), d^s(\rho))$ to $(D^s_{\mathcal{Q}}(\Sigma_*(\rho)), d^s_{\mathcal{Q}}(\rho))$ which induces an isomorphism on homology.*

2. *If ρ is s-bijective then the map J is a chain map from $(D^u_{\mathcal{A}}(\Sigma_*(\rho)), d^{u*}_{\mathcal{A}}(\rho))$ to $(D^u(\Sigma_*(\rho)), d^{u*}(\rho))$ which induces an isomorphism on homology.*

3. *If ρ is u-bijective then the map Q is a chain map from $(D^u(\Sigma_*(\rho)), d^u(\rho))$ to $(D^u_{\mathcal{Q}}(\Sigma_*(\rho)), d^u_{\mathcal{Q}}(\rho))$ which induces an isomorphism on homology.*

4. *If ρ is u-bijective then the map J is a chain map from $(D^s_{\mathcal{A}}(\Sigma_*(\rho)), d^{s*}_{\mathcal{A}}(\rho))$ to $(D^s(\Sigma_*(\rho)), d^{s*}(\rho))$ which induces an isomorphism on homology.*

The proof of this result requires a series of lemmas and will occupy the rest of the section.

We follow the usual idea by finding a type of inverse for the maps Q and J, which we will denote by $Q^\#$ and $J^\#$. Some care must be taken since these will not actually be chain maps. Both of them are done by choosing a linear order, denoted $<$, on the vertex set G^0. We define $Q^\#$ from $\mathcal{Q}(G^k_N, S_{N+1})$ to $\mathbb{Z}G^k_N$ as follows. First, we define a function F on G^k_N. If $p = (p_0, \ldots, p_N)$ is in G^k_N and all entries are distinct, it follows from Lemma 2.5.15 that the entries of $t^k(p)$ are also distinct. We define

$$F(p) = sgn(\alpha) p \cdot \alpha,$$

where α is the unique element of S_{N+1} such that $t^k(p_{\alpha(0)}) < \ldots < t^k(p_{\alpha(N)})$. On the other hand, if two entries of p are equal, then we set $F(p) = 0$. This map extends to a unique group homomorphism on $\mathbb{Z}G^k_N$. We check that it is zero on the subgroup $\mathcal{B}(G^k_N, S_{N+1})$. First, if $p_i = p_j$, for some $i \neq j$, then it is clear that $F(p) = 0$ and also that $F(p - sgn(\alpha')p \cdot \alpha') = 0$, for any α' in S_{N+1}. Finally, suppose that all entries of p are distinct and that α is as in the definition of F. We again want to check that $F(p - sgn(\alpha')p \cdot \alpha') = 0$, for any α' in S_{N+1}. It follows at once that $(\alpha')^{-1} \circ \alpha$ is the element of S_{N+1} that puts the entries of $t^k(p \cdot \alpha')$ into ascending order. This means that

$$F(p \cdot \alpha') = sgn(\alpha') sgn(\alpha)(p \cdot \alpha'_{.}((\alpha')^{-1} \cdot \alpha) = sgn(\alpha') sgn(\alpha) p \cdot \alpha.$$

All together, we have

$$F(p - sgn(\alpha')p \cdot \alpha') = sgn(\alpha) p \cdot \alpha - sgn(\alpha') sgn(\alpha') sgn(\alpha) p \cdot \alpha = 0.$$

We conclude, as claimed, that F is identically zero on $\mathcal{B}(G^k_N, S_{N+1})$ and so we define $Q^\#(Q(p)) = F(p)$, for all p in G^k_N. It is easily seen that $Q \circ Q^\#$ is the identity on $\mathcal{Q}(G^k_N, S_{N+1})$.

We also define $J^\# : \mathbb{Z}G^k_N \to \mathcal{A}(G^k_N, S_{N+1})$ by setting

$$J^\#(p) = \begin{cases} \sum_{\alpha \in S_{N+1}} sgn(\alpha) p \cdot \alpha & \text{if } t^k(p_0) < \cdots < t^k(p_N) \\ 0 & \text{otherwise} \end{cases}$$

for p in G^k_N. It is easily seen that $J^\# \circ J$ is the identity map on $\mathcal{A}(G^k_N, S_{N+1})$.

We will need the following result.

Lemma 4.3.2. *For all $N \geq 0$, we have*

1.
$$t \circ Q^{\#} = Q^{\#} \circ t, \quad t \circ J^{\#} = J^{\#} \circ t,$$
if $k \geq 1$,

2.
$$t^* \circ Q^{\#} = Q^{\#} \circ t^*, \quad t^* \circ J^{\#} = J^{\#} \circ t^*,$$
if $k \geq 0$,

3.
$$< Q^{\#} \circ Q(p), q >=< p, J \circ J^{\#}(q) >,$$
for $k \geq 0$ and p, q in G_N^k.

Proof. For the first statement, for p in G_N^k, to compute $Q^{\#}(t(Q(p))) = Q^{\#}(Q(t(p))$, we consider the vertices of $t^{k-1}(t(p)) = t^k(p)$. If one is repeated, then the result is zero. Otherwise, the result is $sgn(\alpha)t(p)\cdot\alpha = t(sgn(\alpha)p\cdot\alpha)$, where α puts the vertices in increasing order. In either case, this is the same as $t(Q^{\#}(Q(p)))$. The proof of the second equation is the same.

The second part follows from the third, the first and Lemma 4.2.6.

For the third part, we first suppose that there exists α in S_{N+1} so that $p \cdot \alpha = q$ and that the entries of $t^k(q)$ are in ascending order. It follows that $Q^{\#} \circ Q(p) = sgn(\alpha)p \cdot \alpha$ and the left hand side is $sgn(\alpha)$. On the other hand, $J \circ J^{\#}(q) = \sum_{\beta \in S_{N+1}} sgn(\beta)q \cdot \beta$. Pairing this sum with p, all terms are zero except for $\beta = \alpha^{-1}$ and we get $sgn(\beta) = sgn(\alpha)$ and the desired equality.

For the next case, suppose that there exists an α such that the entries of $t^k(p) \cdot \alpha$ are in ascending order, but $p \cdot \alpha \neq \beta$. It follows that the left hand side is zero. If the entries of $t^k(q)$ are not in ascending order, then the right hand side is also zero. If they are, since this is true also of $t^k(p) \cdot \alpha$, $t^k(q)$ and $t^k(p)$ must have distinct S_{N+1} orbits. It follows the same is true of q and p and this implies the right hand side is again zero.

Finally, we consider the case that $t^k(p)$ contains a repeated entry. Then the left hand side is zero. On the right, for $J^{\#}(q)$ to be non-zero, then entries of $t^k(q)$ must all be distinct, so $t^k(q) \cdot \alpha$ cannot equal $t^k(p)$ and hence $q \cdot \alpha \neq p$, for any α in S_{N+1}. In any case, the right hand side is zero. □

Lemma 4.3.3. *For any $N \geq 0$, we have*

$$\begin{aligned} Q^{\#} \circ d_{\mathcal{Q}}^{s,K}(\rho)_N &= d^{s,K}(\rho)_N \circ Q^{\#}, \\ d_{\mathcal{A}}^{u*,K}(\rho)_N \circ J^{\#} &= J^{\#} \circ d^{u*,K}(\rho)_N. \end{aligned}$$

Proof. We begin with the first equation. Let p be in G_N^k which is in the range of $Q^{\#}$. This means that $t^k(p_0) < t^k(p_1) < \cdots < t^k(p_N)$. Suppose q is in G_N^{k+K} and $t^K(q) = p$. It follows that $t^{k+K}(q_0) < t^{k+K}(q_1) < \cdots < t^{k+K}(q_N)$ and the same inequalities hold if we delete any entry. It follows then that for such q and any $0 \leq n \leq N$, we have $Q^{\#} \circ Q(q) = q$. It then follows from the definition of $d^{s,K}(\rho)_N$ that

$$Q^{\#} \circ Q \circ d^{s,K}(\rho)_N \circ Q^{\#} = d^{s,K}(\rho)_N \circ Q^{\#}.$$

But from part 3 of Lemma 4.2.6, the left hand side is just

$$Q^{\#} \circ Q \circ d^{s,K}(\rho)_N \circ Q^{\#} = Q^{\#} \circ d_{\mathcal{Q}}^{s,K}(\rho)_N \circ Q \circ Q^{\#} = Q^{\#} \circ d_{\mathcal{Q}}^{s,K}(\rho)_N$$

as desired.

For the second equation, let a, b be in $\mathbb{Z}G_N^k$. As $J^{\#} \circ d^{u*,K}(\rho)_N(a)$ is in $\mathcal{A}(G_{N+1}^{k-K}, S_{N+2})$, we may consider

$$\begin{aligned} < J \circ J^{\#} \circ d^{u*,K}(\rho)_N(a), b > &= < d^{u*,K}(\rho)_N(a), Q^{\#} \circ Q(b) > \\ &= < a, d^{s,K}(\rho)_N \circ Q^{\#} \circ Q(b) > \\ &= < a, Q^{\#} \circ Q \circ d^{s,K}(\rho)_N(b) > \\ &= < J \circ J^{\#}(a), d^{s,K}(\rho)_N(b) > \\ &= < d^{u*,K}(\rho)_N \circ J \circ J^{\#}(a), b > \\ &= < J \circ d_{\mathcal{A}}^{u*,K}(\rho)_N \circ J^{\#}(a), b > . \end{aligned}$$

As b was arbitrary and J is injective, the conclusion follows. □

We now concentrate our attention on the proof of the first of the four parts of Theorem 2.7.2. In the end, the proof of the second will follow from what we establish to prove the first and some duality arguments. We shall continue to assume that ρ is s-bijective.

The technique is an adaptation of the method of acyclic carriers found in [14, 34] We introduce a new complex, first fixing a positive integer K which satisfies the conclusion of 2.7.2 for the map ρ. We define $E_N = \oplus_{k \geq 0} \mathbb{Z}G_N^k$ with map $d_N = \oplus_k d^{s,K}(\rho)_N$.

For each $0 \leq I \leq \#G^0$, let $K(M) = K(\#G^0 - I)$ and

$$\begin{aligned} \mathcal{V}_I &= \{(v_0, v_1, \ldots, v_I) \in G_I^{K(I)} \mid \\ & t^{K(I)}(v_0) < t^{K(I)}(v_1) < \cdots < t^{K(I)}(v_I)\}. \end{aligned}$$

Notice, in particular, that the entries of any v in $\mathcal{V}_I$ are all distinct.

For v in $\mathcal{V}_I$, first, for every $k \geq K(I)$, $N \geq 0$, let

$$\begin{aligned} \tilde{G}_N^k(v) &= \{p \in G_N^k \mid t^{k-K(I)}(p_0) = v_0, \\ & t^{k-K(I)}(p_n) \in \{v_0, \ldots, v_I\}, 1 \leq n \leq N\}. \end{aligned}$$

and secondly, let

$$G_N^k(v) = \delta_0(\tilde{G}_{N+1}^k(v)).$$

We make some elementary observations.

1. Whether p in G_N^k is in $G_N^k(v)$ or $\tilde{G}_N^k(v)$ is completely determined by $t^{k-K(I)}(p)$.
2. We have $t(\tilde{G}_N^k(v)) \subset \tilde{G}_N^{k-1}(v)$ and hence $t(G_N^k(v)) \subset G_N^{k-1}(v)$.
3. If p is in $\tilde{G}_N^k(v)$, then $\epsilon_0(p) = (p_0, p_0, p_1, \ldots, p_N)$ is in $\tilde{G}_{N+1}^k(v)$ and $\delta_0(\epsilon_0(p)) = p$. This shows that $\tilde{G}_N^k(v) \subset G_N^k(v)$.
4. The set $G_N^k(v)$ is invariant under the action of S_{N+1} (but $\tilde{G}_N^k(v)$ is not).
5. For any $1 \leq n \leq N$, we have $\delta_n(\tilde{G}_N^k(v)) \subset \tilde{G}_{N-1}^k(v)$. For any $0 \leq n \leq N$, we have $\delta_n(G_N^k(v)) \subset G_{N-1}^k(v)$.

We define $E_N(v) = \oplus_{k \geq K(M)} \mathbb{Z} G_N^k(v)$. The first subtle point is the following.

Lemma 4.3.4. *1. The subsets $E_*(v)$ define a subcomplex of (E_*, d_*).*

2. Suppose $I \geq 1$, v is in $\mathcal{V}_I$, w is in $G_N^{K(M)+K}$ with $t^K(w) = v$. Then for all $0 \leq i \leq I$, $\delta_i(w)$ is in $\mathcal{V}_{I-1}$ and

$$E_N(\delta_i(w)) \subset E_N(v),$$

for all N.

Proof. We consider the first statement. It suffices to show that, for any $0 \leq n \leq N$ and $k \geq K$, we have $\delta_n^{s,K}(\mathbb{Z}G_N^k(v)) \subset \mathbb{Z}G_{N-1}^{k+K}(v)$. To do this, we let q be in $\tilde{G}_{N+1}^k(v)$ and compute $\delta_n^{s,K}(\delta_0(q)) = Sum(A)$, where

$$A = \{\delta_n(p) \mid p \in G_N^{k+K}, t^K(p) = \delta_0(q)\}.$$

Let p be in G_N^{k+K} with $t^K(p) = \delta_0(q)$. We apply Lemma 2.7.5 to the map $\delta_0 : G_{N+1} \to G_N$. We may find q' in G_{N+1}^{k+K} such that $t^{2K}(q') = t^K(q)$ and $\delta_0(q') = p$. From the first observation above and the fact that q is in $\tilde{G}_{N+1}^k(v)$, it follows that q' is in $\tilde{G}_{N+1}^{k+K}(v)$. Moreover, we have

$$\delta_n(p) = \delta_n(\delta_0(q')) = \delta_0(\delta_{n+1}(q')) \in \delta_0(\tilde{G}_N^{k+K}(v)) = G_{N-1}^{k+K}(v)$$

using the final observation above. We have shown that A is contained in $G_{N-1}^{k+K}(v)$. This completes the proof.

For the second statement, the fact that $\delta_i(w)$ is in $\mathcal{V}_{I-1}$ is clear. For the last part, we first consider the case $i = 0$. Let p be in $\tilde{G}_{N+1}^k(\delta_0(w))$ so that $\delta_0(p)$ is a typical element of $G_N^k(\delta_0(w))$. As p is in $\tilde{G}_{N+1}^k(\delta_0(w))$, for each $0 \leq n \leq N+1$, $t^{k-K(I)-K}(p_n)$ is in $\{w_1, \ldots, w_I\}$. Define $w' = (w_0, t^{k-K(I)-K}(p))$ which is in $G_{N+1}^{K(I)+K}$ and $\delta_0(w') = t^{k-K(I)-K}(p)$. By Lemma 2.7.5, there exists p' in G_{N+1}^k such that $\delta_0(p') = p$ and $t^{k-K(I)}(p') = t^K(w')$. We claim that p' is in $\tilde{G}_{N+1}^k(v)$; from this it follows that $p = \delta_0(p')$ is in $G_N^k(v)$ as desired. For any $0 \leq n \leq N+1$, we have

$$t^{k-K(I)}(p'_n) = (t^{k-K(I)}(p'))_n = t^K(w')_n = t^K(w'_n).$$

For $n = 0$, we know that $t^K(w'_0) = t^K(w_0) = v_0$. For $1 \leq n \leq N+1$, we have

$$t^K(w'_n) = t^K(t^{k-K(I)-K}(p_{n-1})) \in t^K\{w_0, w_1, \ldots, w_I\} = \{v_0, v_1, \ldots, v_I\}.$$

Finally, we consider the case $0 < i \leq I$. Again, let p be in $\tilde{G}_{N+1}^k(\delta_0(w))$ so that $\delta_0(p)$ is a typical element of $G_N^k(\delta_0(w))$. This means that $t^{k-K(I)-K}(p_0) = \delta_i(w)_0 = w_0$ and

$$t^{k-K(I)-K}(p_n) \in \{\delta_i(w)_0, \delta_i(w)_1, \ldots, \delta_i(w)_{I-1}\} \subset \{w_0, w_1, \ldots, w_I\}.$$

It follows at once that $t^{k-K(I)}(p_0) = t^K(w_0) = v_0$ and

$$t^{k-K(I)}(p_n) \subset t^K\{w_0, w_1, \ldots, w_I\} = \{v_0, v_1, \ldots, v_I\}$$

From this, we see that p is in $\tilde{G}_{N+1}^k(v)$ and the conclusion follows. ∎

We now define, for v in $\mathcal{V}_I$ fixed, a group homomorphism $h_v : \mathbb{Z}G^k_N(v) \to \mathbb{Z}G^k_{N+1}$, for all $k \geq K(I), N \geq 0$, by $h_v(p) = Sum(A)$, where

$$A = \{q \in \tilde{G}^k_{N+1}(v) \mid \delta_0(q) = p\},$$

for any p in $G^k_N(v)$. It follows from the definition of $G^k_N(v)$ that A above is non-empty. We also note that the range of h_v is contained in $\mathbb{Z}\tilde{G}^k_{N+1}(v)$, which in turn is contained in $\mathbb{Z}G^k_N(v)$.

Lemma 4.3.5. *We have*

$$\delta^{s,K}_0 \circ h_v = t^{K*}$$

and, for $1 \leq n \leq N$*,*

$$\delta^{s,K}_n \circ h_v = h_v \circ \delta^{s,K}_{n-1}.$$

Moreover, we have

$$d^{s,K}(\rho)_{N+1} \circ h_v + h_v \circ d^{s,K}(\rho)_N = t^{K*}.$$

Proof. First, we compute $\delta^{s,K}_n \circ h_v(p)$, for p in $G^k_N(v)$, $k \geq K$. From the definition, we have

$$h_v(p) = Sum\{q \in \tilde{G}^k_{N+1}(v) \mid \delta_0(q) = p\}.$$

For a q in the set, define

$$A(q) = \{q' \in G^{k+K}_{N+1} \mid t^K(q') = q\}$$

and we note that $A(q)$ is contained in $\tilde{G}^k_{N+1}(v)$.

Next, we claim that, for $0 \leq n \leq N+1$, the sets $\delta_n(A(q))$ are pairwise disjoint for distinct values of q. Suppose that q' is in $A(q)$ and $\bar{q}'$ is in $A(\bar{q})$ and $\delta_n(q') = \delta_n(\bar{q}')$. We have

$$\delta_0 \circ t^{k+K}(q') = \delta_0 \circ t^k(q) = t^k(p) = \delta_0 \circ t^k(\bar{q}) = \delta_0 \circ t^{k+K}(\bar{q}')$$

and so, for all $m \geq 1$, we have $t^{k+K}(q'_m) = t^{k+K}(\bar{q}'_m)$. On the other hand for $m = 0$, we have

$$t^{k+K-K(I)}(q'_0) = t^{k-K(I)}(q_0) = v_0 = t^{k-K(I)}(\bar{q}_0) = t^{k+K-K(I)}(\bar{q}'_0)$$

since q and q' are in $\tilde{G}^k_{N+1}(v)$. We conclude that $t^{k+K-K(I)}(q') = t^{k+K-K(I)}(\bar{q}')$. In addition, we have $\delta_n(q') = \delta_n(\bar{q}')$ and the fact that

$$q = t^K(q') = t^K(\bar{q}') = \bar{q}$$

follows from the uniqueness statement of Lemma 2.7.5. This completes the proof of the claim.

It follows that $\delta_n^{s,K} \circ h_v(p) = Sum(A)$, where

$$A = \{\delta_n(q) \mid q \in \tilde{G}_{N+1}^{k+K}(v), \delta_0 \circ t^K(q) = p\}.$$

We first consider the case $n = 0$. Of course, we have $t^{K*}(p) = Sum(B)$, where

$$B = \{q \mid q \in G_N^{k+K}, t^K(q) = p\}.$$

It is clear that $A \subset B$. For the reverse inclusion, let q be in B. We know that $p = \delta_0(\tilde{p})$ for some $\tilde{p}$ in $\tilde{G}_{N+1}^k(v)$. Applying part one of Lemma 2.7.5. we may find $\tilde{q}$ in G_{N+1}^{k+K} such that $\delta_0(\tilde{q}) = q$ and $t^{2K}(\tilde{q}) = t^K(\tilde{p})$. The latter implies that $\tilde{q}$ is in $\tilde{G}_{N+1}^{k+K}(v)$ and hence q is in A.

We now turn to the case $n \geq 1$. It follows from the definitions that $h_v \circ \delta_{n-1}(p) = Sum(B)$, where

$$B = \{q \in \tilde{G}_N^{k+K}(v) \mid \delta_0(q) = \delta_{n-1}(q'), \text{ for some } q' \in G_N^{k+K}, t^K(q') = p\}.$$

Suppose that q is in $\tilde{G}_{N+1}^{k+K}(v)$ with $\delta_0 \circ t^K(q) = p$. Since $n \geq$, we have $\delta_n(q)$ is in $\tilde{G}_N^{k+K}(v)$ and

$$\delta_0(\delta_n(q)) = \delta_{n-1} \circ \delta_0(q).$$

Moreover, we have $t^K(\delta_0(q)) = \delta_0(t^K(q)) = p$. This implies that $\delta_n(q)$ is in B and we have shown $A \subset B$. For the reverse conclusion, let q be in B and q' be as in the definition. Since p is in $G_N^k(v)$, we may find $\tilde{p}$ in $\tilde{G}_{N+1}^k$ such that $\delta_0(\tilde{p}) = p$. We also have $t^K(q') = p$ and so we may apply the existence part of Lemma 2.7.5 to find $\tilde{p}'$ in G_{N+1}^{k+K} such that $\delta_0(\tilde{p}') = q'$ and $t^{2K}(\tilde{p}') = t^K(\tilde{p})$. We have

$$t^{k+K}(\tilde{p}_0') = t^k(\tilde{p}_0) = v_0 = t^{k+K}(q_0)$$

since $\tilde{p}$ is in $\tilde{G}_{N+1}^{k+K}(v)$ and q is in $\tilde{G}_N^{k+K}(v)$. We also have

$$\delta_0(q) = \delta_{n-1}(q') = \delta_{n-1} \circ \delta_0(\tilde{p}') = \delta_0 \circ \delta_n(\tilde{p}').$$

The uniqueness part of Lemma 2.7.5 implies that $t^K(q) = t^K(\delta_n(\tilde{p}')) = \delta_n \circ t^K(\tilde{p}')$. We again apply the existence part of Lemma 2.7.5 to find $\tilde{q}$ in G_{N+1}^{k+K} such that $\delta_n(\tilde{q}) = q$ and $t^{2K}(\tilde{q}) = t^K \circ t^K(\tilde{p}') = t^K(\tilde{p})$. Now, we have

$$\delta_{n-1}(\delta_0(\tilde{q})) = \delta_0 \circ \delta_n(\tilde{q}) = \delta_0(q) = \delta_{n-1}(q').$$

Moreover, we also have

$$t^{2K}(\delta_0(\tilde{q})) = \delta_0 \circ t^K(\tilde{p}) = t^K \circ \delta_0(\tilde{p}) = t^K(p),$$

and

$$t^{2K}(q') = t^K \circ t^K(q') = t^K(p).$$

One more application of the uniqueness part of 2.7.5 implies that $t^K(\delta_0(\tilde{q})) = t^K(q') = p$. It follows that $\tilde{q}$ satisfies the condition so that $\delta_n(\tilde{q}) = q$ is in A. This completes the proof.

The final statement is an immediate consequence of the first two. □

Lemma 4.3.6. *let p be in G_N^k, with $k \geq K\#G^0$. There is a unique minimal $0 \leq I \leq N$ and v in $\mathcal{V}_I$ such that p is in $G_N^k(v)$.*

Proof. Consider the vertices $t^k(p)$ and assume there are $I+1$ distinct ones. We choose distinct entries of $t^{k-K(M)}(p)$ and order them appropriately. That is, we find $t^k(p_{n_0}) < \cdots < t^k(p_{n_I})$. Let $v = t^{k-K(I)}(p_{n_0}, \ldots, p_{n_I})$. It is clear that $(p_{n_0}, p_0, \ldots, p_N)$ is in $\tilde{G}_{N+1}^k(v)$ and hence p is in $G_N^k(v)$. The minimality of I and the uniqueness of v are clear. □

Lemma 4.3.7. *There exists a homomorphisms $H_N : E_N \to E_{N+1}, N \in \mathbb{Z}$ satisfying*

$$t^{K*} \circ H_{N-1} \circ d^{s,K}(\rho)_N - d^{s,K}(\rho)_{N+1} \circ H_N = t^{(N+1)K*} \circ (1 - Q^{\#} \circ Q).$$

Proof. Our proof is by induction on N, but we had an extra hypothesis to the induction statement. Suppose p is in G_N^k, for some $k \geq K\#G^0$. Let v in $\mathcal{V}_I$ be as in Lemma 4.3.6 so that p is in $G_N^k(v)$. We require that $H_N(p)$ is in $E_{N+1}(v)$.

We begin by setting $H_N = 0$, for all $N \leq 0$ and then proceed by induction. These satisfy the induction hypothesis for $N \leq 0$ (since $Q^{\#} \circ Q$ is the identity map on E_0). Assume that H_{N-1} has been defined, where $N > 0$, satisfying the desired conclusion.

Let p be in G_N^k and find the minimal M and v in $\mathcal{V}_I$ as in 4.3.6. We define

$$H_N(p) = h_v[H_{N-1} \circ d_N^{s,K}(\rho)(p) + t^{NK*}(p) - t^{NK*} \circ Q^{\#} \circ Q(p)].$$

In doing so, we note that since p is in $G_N^k(v)$, so is $\#Q \circ Q(p)$, since it is either zero or a permutation of p. This means that $t^{NK*}(p)$ and $t^{NK*}(Q^{\#} \circ Q(p))$ are in $E_N(v)$. As for the other term, if q is any element of G_N^{k+K} such that

$t^K(q) = p$ and $0 \leq n \leq N$, if $t^k(p_n) = t^{k+K}(q_n)$ is a repeated entry of $t^k(p) = t^{k+K}(q)$ then v is the minimal element such that $\delta_n(q)$ is in $G_N^{k+K}(v)$ and hence by lemma 4.3.4, $H_{N-1}(\delta_n(q))$ is in $E_N(v)$. On the other hand, if this is not a repeated entry, then it is equal to $t^{K(I)}(v_m)$, for some m. We form w in $\mathcal{V}_{I-1}$ by taking the distinct entries of $t^{k-K(I)}(q)$ arranging them in the proper order. It follows that the desired minimal element for $\delta_n(q)$ is $\delta_m(w)$. This w satisfies the conditions of part 2 of Lemma 4.3.4 and it follows that $E_N(\delta_m(w)) \subset E_N(v)$. Then by our induction hypothesis, we have $H_{N-1}(\delta_n(q))$ is in $E_N(v)$. In this way, we see that $H_N(p)$ is well-defined and in $E_{N+1}(v)$, as desired.

To prove that the formula holds, we proceed as follows. First, we apply Lemma 4.3.5 to see

$$\begin{aligned}
d_{N+1}^{s,K} \circ H_N &= d_{N+1}^{s,K} \circ h_v \circ (H_{N-1} \circ d_N^{s,K} + t^{NK*} - t^{NK*} \circ Q^\# \circ Q) \\
&= (-h_v \circ d_N^{s,K} + t^{K*}) \\
&\quad \circ (H_{N-1} \circ d_N^{s,K} + t^{NK*} - t^{NK*} \circ Q^\# \circ Q) \\
&= -h_v \circ d_N^{s,K} \circ H_{N-1} \circ d_N^{s,K} \\
&\quad + t^{K*} \circ H_{N-1} \circ d_{N-1}^{s,K} \\
&\quad - h_v \circ d_N^{s,K} \circ (t^{NK*} - t^{NK*} \circ Q^\# \circ Q) \\
&\quad + t^{(N+1)K*} - t^{(N+1)K*} \circ Q^\# \circ Q.
\end{aligned}$$

Considering just the first term on the right and applying the induction hypothesis gives

$$\begin{aligned}
h_v \circ d_N^{s,K} \circ H_{N-1} \circ d_N^{s,K} &= -h_v \circ (t^{K*} \circ H_{N-2} \circ d_{N-1}^{s,K} - t^{NK*} \\
&\quad + t^{NK*} \circ Q^\# \circ Q) \circ d_N^{s,K} \\
&= -h_v \circ t^{K*} \circ H_{N-2} \circ d_{N-1}^{s,K} \circ d_N^{s,K} \\
&\quad + h_v \circ (t^{NK*} - t^{NK*} \circ Q^\# \circ Q) \circ d_N^{s,K} \\
&= 0 + h_v \circ (t^{NK*} - t^{NK*} \circ Q^\# \circ Q) \circ d_N^{s,K} \\
&= h_v \circ d_N^{s,K} \circ (t^{NK*} - t^{NK*} \circ Q^\# \circ Q),
\end{aligned}$$

where in the last step we have used Lemma 4.3.3. Returning to the first display above, we now see that the first and third terms on the right cancel. This gives the desired conclusion. □

We are now ready to give a proof of Theorem 4.3.1.

Proof. We begin by proving the first statement. As Q is a chain map, it induces a well-defined homomorphism at the level of homologies. Let us first check that it is onto. Suppose that b is in $\mathcal{Q}(G_N^k, S_{N+1})$ and j is $\mathbb{N}$ with $d_{\mathcal{Q}}^s(\rho)_N[b, j] = 0$. This means that for some $j' \geq 0$, $(\gamma^s)^j \circ d_{\mathcal{Q}}^{s,K}(\rho)_N(b) = 0$. Now consider $a = Q^{\#} \circ (\gamma^s)^{j'}(b)$. First, we note that

$$\begin{aligned} d^{s,K}(\rho)_N(a) &= d^{s,K}(\rho)_N \circ Q^{\#} \circ (\gamma^s)^{j'}(b) \\ &= Q^{\#} \circ d_{\mathcal{Q}}^{s,K}(\rho)_N \circ (\gamma^s)^{j'}(b) \\ &= Q^{\#} \circ (\gamma^s)^{j'} \circ d_{\mathcal{Q}}^{s,K}(\rho)_N(b) \\ &= 0. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} Q[a, j + j'] &= [Q(a), j + j'] \\ &= [Q \circ Q^{\#} \circ (\gamma^s)^{j'}(b), j + j'] \\ &= [(\gamma^s)^{j'}(b), j + j'] \\ &= [b, j]. \end{aligned}$$

It follows that the map on homology is onto.

Next, we show that the map induced by Q is injective. Let a be in $\mathbb{Z}G_N^k$ and $j \geq 1$. We may assume that $k \geq \#G^0 K$. Further suppose that $d^s(\rho)_N[a, j] = 0$ and $[Q(a), j]$ is zero in the homology of $(C_{\mathcal{Q}}^s(\rho), d_{\mathcal{Q}}^s(\rho))$. We may then find $j' \geq 0$ such that $(\gamma^s)^{j'} \circ d^{s,K}(\rho)_N(a) = 0$ and b in $\mathcal{Q}(G_{N+1}^{k-K}, S_{N+2})$ such that $[Q(a), j] = [d_{\mathcal{Q}}^{s,K}(\rho)_{N+1}(b), j]$. The latter means that we may find $j'' \geq 0$ such that $(\gamma^s)^{j''}(Q(a)) = (\gamma^s)^{j''} \circ d_{\mathcal{Q}}^{s,K}(\rho)_{N+1}(b)$. By replacing j' and j'' by the larger of the two, we may assume they are equal.

Now, we apply both sides of the equation of Lemma 4.3.7 to the element $(\gamma^s)^{j'} \circ i^{(N+1)K}(a)$. On the left hand side, we get two terms. The first is

$$\begin{aligned} t^{K*} \circ H_{N-1} \circ d^{s,K}(\rho)_N \circ (\gamma^s)^{j'} \circ i^{(N+1)K}(a) &= \\ t^{K*} \circ H_{N-1} \circ i^{(N+1)K} \circ (\gamma^s)^{j'} \circ d^{s,K}(\rho)_N(a) &= 0. \end{aligned}$$

We do not need to know the second, except that it is in the image of $d^{s,K}(\rho)_{N+1}$. On the right hand side, the first term is

$$t^{(N+1)K*} \circ (\gamma^s)^{j'} \circ i^{(N+1)K}(a) = (\gamma^s)^{j'+(N+1)K}(a).$$

The second term on the right (ignoring the negative sign) is

$$\begin{array}{rl} t^{(N+1)K*} \circ Q^{\#} \circ Q \circ (\gamma^s)^{j'} \circ i^{(N+1)K}(a) & = \\ t^{(N+1)K*} \circ Q^{\#} \circ i^{(N+1)K} \circ (\gamma^s)^{j'} \circ Q(a) & = \\ t^{(N+1)K*} \circ Q^{\#} \circ i^{(N+1)K} \circ (\gamma^s)^{j'} \circ d^{s,K}_{\mathcal{Q}}(\rho)_{N+1}(b) & = \\ Q^{\#} \circ (\gamma^s)^{j'+(N+1)K} \circ d^{s,K}_{\mathcal{Q}}(\rho)_{N+1}(b) & = \\ d^{s,K}(\rho)_{N+1} \circ Q^{\#} \circ (\gamma^s)^{j'+(N+1)K}(b). & \end{array}$$

Putting this all together we conclude that

$$(\gamma^s)^{j'+(N+1)K}(a) = d^{s,K}(\rho)_{N+1}(c),$$

for some c. It follows that $[a, j] = [(\gamma^s)^{j'+(N+1)K}(a), j + j' + (N + 1)K]$ is in the image of $d^s(\rho)_{N+1}$ and hence is zero in homology. This completes the proof of the first part of 4.3.1.

For the second part, we have already noted that $J : D^u_{\mathcal{A}}(\Sigma_*(\rho)) \to D^u(\Sigma_*(\rho))$ is a chain map. Let us first show that it surjective an homology. Suppose that b is in $\mathbb{Z}G^k_N$, $j \geq 1$ and $d^{u*}(\rho)_N[b, j] = 0$. It follows that for some $l \geq 0$, we have $(\gamma^u)^l \circ d^{u*,K}(\rho)_N(b) = 0$. Without loss of generality, we may assume that $l > (N + 1)K$.Consider $a = J^{\#} \circ (\gamma^u)^l(b)$. Then, using Lemma 4.3.3, we have

$$\begin{aligned} d^{u*,K}_{\mathcal{A}}(\rho)_N(a) &= d^{u*,K}_{\mathcal{A}}(\rho)_N \circ J^{\#} \circ (\gamma^u)^l(b) \\ &= J^{\#} \circ (\gamma^u)^l \circ d^{u*,K}_{\mathcal{A}}(\rho)_N(b) \\ &= 0. \end{aligned}$$

This means that that . Next, we claim that $c = (\gamma^u)^{l+(N+1)K}(b) - J(a)$ is in the image of $d^{u*}(\rho)_{N-1}$. Once this is shown, then we know the class of $[b, j] = [(\gamma^u)^{l+(N+1)K}(b), j+l+(N+1)K]$ is equal to the class of $[J(a), j+l+(N+1)K]$ in homology and our proof of surjectivity is done.

As for the claim, let p be any element of $\mathbb{Z}G^k_N$ and we use Lemma 4.3.2 to see that

$$\begin{aligned} < p, c > &= < p, (1 - J \circ J^{\#}) \circ (\gamma^u)^{l+(N+1)K}(b) > \\ &= < (1 - Q^{\#} \circ Q)(p), (\gamma^u)^{l+(N+1)K}(b) > \\ &= < (1 - Q^{\#} \circ Q)(p), t^{l+(N+1)K} \circ i^{(l+(N+1)K)*}(b) > \\ &= < t^{(N+1)K*} \circ (1 - Q^{\#} \circ Q)(p), t^l \circ i^{(l+(N+1)K)*}(b) > . \end{aligned}$$

We may now apply Lemma 4.3.7, then Lemma 4.2.2 and finally the second part of Theorem 4.2.3:

$$\begin{aligned}
< p, c > &= < t^{(N+1)K*} \circ (1 - Q^{\#} \circ Q)(p), t^l \circ i^{(l+(N+1)K)*}(b) > \\
&= < t^{K*} \circ H_{N-1} \circ d^{s,K}(\rho)_N(p), t^l \circ i^{(l+(N+1)K)*}(b) > \\
&\quad - < d^{s,K}(\rho)_{N+1} \circ H_N(p), t^l \circ i^{(l+(N+1)K)*}(b) > \\
&= < p, d^{u*,K}(\rho)_N H^*_{N-1} \circ t^{l+K} \circ i^{(l+(N+1)K)*}(b) > \\
&\quad - < H_N(p), d^{u*,K}(\rho)_{N+1} \circ t^l \circ i^{(l+(N+1)K)*}(b) > \\
&= < p, d^{u*,K}(\rho)_N H^*_{N-1} \circ t^{l+K} \circ i^{(l+(N+1)K)*}(b) > \\
&\quad - < p, H^*_N \circ i^{(N+1)K*} \circ d^{u*,K}(\rho)_{N+1} \circ (\gamma^u)^l(b) > \\
&= < p, d^{u*,K}(\rho)_N H^*_{N-1} \circ t^{l+K} \circ i^{(l+(N+1)K)*}(b) > -0.
\end{aligned}$$

As this is valid for any p in $\mathbb{Z}G^k_N$, the claim is proved.

We now prove that the map J is injective on homology. Suppose that a is in $\mathcal{A}(G^k_N, S_{N+1})$, $j \geq 1$, that $[a, j]$ is in the kernel of $d^{u*}(\rho)_N$ and $[J(a), j]$ is zero in the homology of $D^u(\Sigma_*(\rho))$. The last condition means that $[a, j] = d^{u*}(\rho)_{N-1}[b, j']$, for some b in $\mathbb{Z}G^{k+K}_{N-1}$. It follows that, for some $l \geq 0$, we have $(\gamma^u)^{j+l} \circ d^{u*,K}(\rho)_{N-1}(b) = (\gamma^u)^{j'+l}(a)$. Then $J^{\#}(b)$ is in $\mathcal{A}(G^{k+K}_{N-1}, S_N)$. Moreover, using Lemma 4.3.3, we have

$$\begin{aligned}
d^{u*,K}(\rho)_{N-1} \circ J^{\#} \circ (\gamma^u)^{j+l}(b) &= J^{\#} \circ d^{u*,K}(\rho)_{N-1} \circ (\gamma^u)^{j+l}(b) \\
&= J^{\#} \circ (\gamma^u)^{j+l} \circ d^{u*,K}(\rho)_{N-1}(b) \\
&= J^{\#} \circ (\gamma^u)^{j'+l}(a) \\
&= (\gamma^u)^{j'+l}(a)
\end{aligned}$$

since a and hence $(\gamma^u)^{j'+l}(a)$ are in $\mathcal{A}(G^k_N, S_{N+1})$. From this it follows that

$$d^{u*}(\rho)_{N-1}[J^{\#} \circ (\gamma^u)^{j+l}(b), j + j' + l] = [(\gamma^u)^{j'+l}(a), j + j' + l] = [a, j]$$

and hence $[a, j]$ is zero in homology of $D^u_{\mathcal{A}}(\Sigma_*(\rho))$.

The third and fourth parts are immediate consequences of the first two.

□

4.4 Functorial properties

In this section, we establish some simple functorial properties of the complexes introduced in Section 4.1.

Theorem 4.4.1. *Let (X, φ) and (X', φ') be Smale spaces, (Σ', σ) be a shift of finite type and $\eta_X : (X, \varphi) \to (X', \varphi')$ and $\rho' : (\Sigma', \sigma) \to (X', \varphi')$ be factor maps. Let (Σ, σ) be the fibred product::*

$$\begin{array}{ccc} (\Sigma, \sigma) & \xrightarrow{\rho} & (X, \varphi) \\ \downarrow{\scriptstyle \eta_\Sigma} & & \downarrow{\scriptstyle \eta_X} \\ (\Sigma', \sigma) & \xrightarrow{\rho'} & (X', \varphi') \end{array}$$

1. *If ρ' and η_X are s-bijective, then, for each $N \geq 0$, η_Σ induces a natural s-bijective factor map $\eta_N : (\Sigma_N(\rho), \sigma) \to (\Sigma_N(\rho'), \sigma)$. The induced maps η_N^s are chain maps from the complex $(D^s(\Sigma_*(\rho)), d^s(\rho))$ to $(D^s(\Sigma_*(\rho')), d^s(\rho'))$.*

2. *If ρ' is s-bijective while η_X is u-bijective, then, for each $N \geq 0$, η_Σ induces a natural u-bijective factor map $\eta_N : (\Sigma_N(\rho), \sigma) \to (\Sigma_N(\rho'), \sigma)$. The induced maps η_N^{u*} are chain maps from the complex $(D^u(\Sigma_*(\rho')), d^{u*}(\rho'))$ to $(D^u(\Sigma_*(\rho)0, d^{u*}(\rho))$.*

3. *If ρ' is u-bijective while η_X is s-bijective, then, for each $N \geq 0$, η_Σ induces a natural s-bijective factor map $\eta_N : (\Sigma_N(\rho), \sigma) \to (\Sigma_N(\rho'), \sigma)$. The induced maps η_N^{s*} are chain maps from the complex $(D^s(\Sigma_*(\rho')), d^{s*}(\rho'))$ to $(D^s(\Sigma_*(\rho)), d^{s*}(\rho))$.*

4. *If ρ' and η_X are u-bijective, then, for each $N \geq 0$, η_Σ induces a natural u-bijective factor map $\eta_N : (\Sigma_N(\rho), \sigma) \to (\Sigma_N(\rho'), \sigma)$. The induced maps η_N^u are chain maps from the complex $(D^s(\Sigma_*(\rho)), d^u(\rho))$ to $(D^u(\Sigma_*(\rho')), d^u(\rho'))$.*

Proof. We prove the first statement. It follows from Theorem 2.5.13 that both η_Σ and ρ are s-bijective factor maps.

It is a simple matter to check that for any $N \geq 0$, $\Sigma_N(\rho)$ is simply the fibred product of η_X and $\rho' : (\Sigma_N(\rho'), \sigma) \to (X', \varphi')$. This establishes the first statement of the conclusion.

It follows from Theorem 3.4.1 that η_N induces a homomorphism between the chain groups $D^s(\Sigma_N(\rho), \sigma))$ and $D^s(\Sigma_N(\rho'), \sigma))$. In addition, it is clear that $\delta_n \circ \eta_N = \eta_{N-1} \circ \delta_n$, for any $0 \leq n \leq N$ and $N \geq 1$. From this, the functorial property of the dimension group invariant and the definitions, it follows that

$$d^s(\rho')_N \circ \eta_N^s = \eta_{N-1}^s \circ d^s(\rho)_N$$

and hence is a chain map.

The other three parts are done in an almost identical fashion except in the mixed cases (2 and 3), we need to apply Theorem 3.5.11 to conclude that we have a chain map. For that, it is a simple matter to fix $0 \leq n \leq N$ and begin with $\delta_n : (\Sigma_N(\rho'), \sigma) \to (\Sigma_{N-1}(\rho'), \sigma)$ and $\eta_{N-1} : (\Sigma_{N-1}(\rho), \sigma) \to (\Sigma_{N-1}(\rho'), \sigma)$ and see that the fibred product is exactly

$$\begin{array}{ccc} (\Sigma_N(\rho), \sigma) & \xrightarrow{\delta_n} & (\Sigma_{N-1}(\rho)) \\ \downarrow \eta_N & & \downarrow \eta_{N-1} \\ (\Sigma_N(\rho'), \sigma) & \xrightarrow{\delta_n} & (\Sigma_{N-1}(\rho'), \sigma) \end{array}$$

This allows us to apply Theorem 3.5.11. □

4.5 Independence of resolution

Here, we show that the homology (or cohomology) of the chain complex (cochain complex, respectively) of Section 4.1 is independent of the shift of finite type, (Σ, σ) and the factor map ρ. This is done in a sequence of three Theorems. The first treats the special case where we have graphs G and H and s-bijective factor maps

$$(\Sigma_H, \sigma) \xrightarrow{\theta} (\Sigma_G, \sigma) \xrightarrow{\rho} (X, \varphi)$$

and the map θ is induced by a left-covering graph homomorphism $\theta : H \to G$. Then we show that the complexes for ρ and $\theta \circ \rho$ yield naturally isomorphic homologies. The second theorem drops the hypothesis that there are specific presentations for the two shifts of finite type above. The third Theorem treats the general case.

Of these three, the first is the most difficult; the other two follow in fairly easy fashion from the first. We actually delay the proof of the first until we have stated all three. It is then done in a series of Lemmas. The basic ideas are standard ones from the proofs of analogous results in algebraic topology. However, they are not completely routine because some of the maps involved which function as chain homotopies are only defined via the symbolic presentations of the invariants and are not chain maps in the usual sense because they do not pass through to the inductive limit groups.

Theorem 4.5.1. *Suppose that G, H are graphs, $\theta : H \to G$ is a left-covering graph homomorphism, (X, φ) is a Smale space and $\rho : (\Sigma_G, \sigma) \to (X, \varphi)$ is a regular s-bijective factor map.*

1. *The map θ induces an isomorphism between the homologies of the chain complexes $(D^s(\Sigma_*(\rho \circ \theta)), d^s(\rho \circ \theta))$ and $(D^s(\Sigma_*(\rho), d^s(\rho))$.*

2. *The map θ^* (3.1) induces an isomorphism between the cohomologies of the cochain complexes $(D^u(\Sigma_*(\rho)), d^{u*}(\rho))$ and $(D^u(\Sigma_*(\rho \circ \theta)), d^{u*}(\rho \circ \theta))$.*

The proof will be done in a series of Lemmas, but we postpone the start while noting some more important consequences.

Theorem 4.5.2. *Let (X, φ) be a Smale space and (Σ, σ) and (Σ', σ) be shifts of finite. Suppose that $\rho : (\Sigma, \sigma) \to (X, \varphi)$ and $\theta : (\Sigma', \sigma) \to (\Sigma, \sigma)$ are s-bijective factor maps.*

1. *The map θ induces an isomorphism between the homologies of the chain complexes $(D^s(\Sigma_*(\rho \circ \theta)), d^s(\rho \circ \theta))$ and $(D^s(\Sigma_*(\rho)), d^s(\rho))$.*

2. *The map θ^* (3.1) induces an isomorphism between the cohomologies of the cochain complexes $(D^u(\Sigma_*(\rho)), d^{u*}(\rho))$ and $(D^u(\Sigma_*(\rho \circ \theta)), d^{u*}(\rho \circ \theta))$.*

Proof. First, we may find a graph G which presents the map ρ; that is, (Σ, σ) is conjugate to (Σ_G, σ) and, after identifying them, the map ρ is regular. By Proposition 1 of [11], we may find a graph H and a left-covering graph homomorphism, also denoted θ, from H to G such that (Σ', σ) is conjugate to (Σ_H, σ) and, after suitable identifications, the factor map θ is induced by the graph homomorphism θ. The result now follows from the last Theorem. □

Finally, our main result is the following.

Theorem 4.5.3. *Let (X, φ) be a Smale space and suppose that (Σ, σ) and (Σ', σ) are shifts of finite type with factor maps $\rho : (\Sigma, \sigma) \to (X, \varphi)$ and $\rho' : (\Sigma', \sigma) \to (X, \varphi)$.*

1. *If ρ and ρ' are s-bijective, then there are canonical isomorphisms between the homologies of the complexes $(D^s(\Sigma_*(\rho)), d^s(\rho))$ and $(D^s(\Sigma_*(\rho')), d^s(\rho'))$ and between the cohomologies of the cochain complexes $(D^u(\Sigma_*(\rho)), d^{u*}(\rho))$ and $(D^u(\Sigma_*(\rho')), d^{u*}(\rho'))$.*

2. *If ρ and ρ' are u-bijective, then there are canonical isomorphisms between the homologies of the complexes $(D^u(\Sigma_*(\rho)), d^u(\rho))$ and $(D^u(\Sigma_*(\rho')), d^u(\rho'))$ and between the cohomologies of the cochain complexes $(D^s(\Sigma_*(\rho)), d^{s*}(\rho))$ and $(D^s(\Sigma_*(\rho')), d^{s*}(\rho'))$.*

Proof. Let (Σ'', σ) denote the fibred product of ρ and ρ'. It comes with natural factor maps θ to (Σ, σ) and θ' to (Σ', σ). The conclusion now follows from two applications of the last Theorem. □

We now turn to the proof of Theorem 4.5.1. Of course, we consider the C^s complex only. First, we fix a $K \geq 1$ such that the conclusion of Lemma 2.7.2 is satisfied for the map ρ. We will assume throughout the rest of the section that this holds. It follows from Theorem 4.4.1 (using $(X', \varphi') = (X, \varphi)$ and $\eta = id$) that θ induces a chain map.

Since θ is a graph homomorphism, we have $t \circ \theta = \theta \circ t, \qquad i \circ \theta = \theta \circ i.$ Moreover, since θ is left-covering, we also have $t^* \circ \theta = \theta \circ t^*$. From Theorem 3.4.6, we also know that $\delta_n^{s,K} \circ \theta = \theta \circ \delta_n^{s,K}$, for any $0 \leq n \leq N$, and hence $d^{s,K}(\rho)_N \circ \theta = \theta \circ d^{s,K}(\rho \circ \theta)_N$.

Now, we choose a lifting for the function θ between the vertex sets of the two graphs. Specifically, since θ is surjective, we find a function $\lambda : G^0 \to H^0$ such that $\theta(\lambda(v)) = v$, for all v in G^0. From the fact that θ is left-covering, for any $k \geq 1$ and p in G^k, there is a unique path, denoted $\lambda(p)$, in H^k such that $\theta(\lambda(p)) = p$ and $\lambda(p)$ terminates at the vertex $\lambda(t^k(p))$; that is, $t^k(\lambda(p)) = \lambda(t^k(p))$. It is important to note that λ is *not*, in general, a graph homomorphism as the condition $i \circ \lambda = \lambda \circ i$ may fail. However, the important properties of λ are summarized in the two lemmas below.

Lemma 4.5.4. *Let $N \geq 0$, $k \geq 0$. If p' is in H_N^{k+K} and $t^K(p') = \lambda(p)$, for some p is in G_N^k, then $p' = \lambda \circ \theta(p')$.*

Proof. It is clear from the fact that λ is a lift of θ that the paths p' and $\lambda \circ \theta(p')$ have the same image under θ. Moreover, we have

$$t^K(\lambda \circ \theta(p')) = \lambda \circ \theta(t^K(p')) = \lambda \circ \theta(\lambda(p)) = \lambda(p) = t^K(p').$$

As θ is left-covering, it follows that $p' = \lambda \circ \theta(p')$. □

Lemma 4.5.5. *For $N \geq 0$ and $0 \leq n \leq N$, we have*

$$\begin{aligned} t \circ \lambda &= \lambda \circ t, \\ t^* \circ \lambda &= \lambda \circ t^*, \\ \delta_n^{s,K}(\rho \circ \theta) \circ \lambda &= \lambda \circ \delta_n^{s,K}(\rho), \\ d^{s,K}(\rho \circ \theta)_N \circ \lambda &= \lambda \circ d^{s,K}(\rho)_N. \end{aligned}$$

Proof. The first two properties follow from the definitions. For the third part, let p be in G_N^k. It follows from the definitions that $\delta_n^{s,K}(\rho) \circ \lambda(p) = Sum(A)$, where

$$A = \{\delta_n(q) \mid q \in H_N^{k+K}, t^K(q) = \lambda(p)\}.$$

It also follows from the definitions and the fact that λ is injective that $\lambda \circ \delta_n^{s,K}(\rho)(p) = Sum(B)$, where

$$B = \{\lambda(\delta_n(p')) \mid p' \in G_N^{k+K}, t^K(p') = p\}.$$

We claim that $A = B$ and the conclusion follows. Suppose that q is in H_N^{k+K} with $t^K(q) = \lambda(p)$ so that $\delta_n(q)$ is in A. Applying θ to both sides, we have $t^K(\theta(q)) = p$. Then applying λ, we have $t^K(\lambda(\theta(q))) = \lambda(p) = t^K(q)$. But this implies that $\lambda(\theta(q)) = q$. Then $p' = \theta(q)$ is in G_N^{k+K} and satisfies $t^K(p') = t^K(\theta(q)) = p$ and

$$\lambda(\delta_n(p')) = \delta_n(\lambda(p')) = \delta_n(\lambda(\theta(q))) = \delta_n(q).$$

It follows that $\delta_n(q)$ is in B.

Conversely, suppose that p' is in G_N^{k+K} with $t^K(p') = p$ so that $\lambda(\delta_n(p'))$ is in B. Let $q = \lambda(p')$, which is in H_N^{k+K}. Moreover, $t^K(q) = t^K(\lambda(p')) = \lambda(t^K(p')) = \lambda(p)$ so that $\delta_n(q) = \lambda(\delta_n(p'))$ is in A.

The fourth formula follows from the third. □

We now define maps $\kappa_n : \mathbb{Z}H_N^k \to \mathbb{Z}H_{N+1}^k$, for all $k \geq 0$ and $0 \leq n \leq N$, by

$$\kappa_n(p_0, \ldots, p_N) = (\lambda \circ \theta(p_0), \ldots, \lambda \circ \theta(p_n), p_n, \ldots, p_N),$$

for all $(p_0, \ldots, p_N)$ in H_N^k. We set $\kappa = \sum_{n=0}^{N} (-1)^n \kappa_n$. Since t commutes with both θ and λ (4.5.5), we have the following.

Lemma 4.5.6. *For $0 \leq n \leq N$, we have $t^k \circ \kappa_n = \kappa_n \circ t^k$ and $t^k \circ \kappa = \kappa \circ t^k$.*

Next, we establish an analogue of the property that κ is a chain homotopy from $\lambda \circ \theta$ to the identity.

Lemma 4.5.7. *For $N \geq 0$, we have*

$$d^{s,K}(\theta \circ \rho)_{N+1} \circ \kappa - \kappa \circ d^{s,K}(\theta \circ \rho)_N = t^{K*} \circ \lambda \circ \theta - t^{K*}.$$

Proof. Let p be in H^k_N and fix $0 \leq n \leq N$. We claim that for p' in H^{k+K}_{N+1}, $t^K(p') = \kappa_n(p)$ if and only if there exists p'' in H^{k+K}_N such that $t^K(p'') = p$ and $p' = \kappa_n(p'')$.

For the 'if' direction, if such a p'' exists, then we have

$$t^K(p') = t^K(\kappa_n(p'')) = \kappa_n(t^K(p'')) = \kappa_n(p).$$

Conversely, suppose that $t^K(p') = \kappa_n(p)$. For $0 \leq i \leq n$, we know that $t^K(p'_i) = \lambda \circ \theta(p_i)$. It follows from Lemma 4.5.4 that $p'_i = \lambda \circ \theta(p'_i)$. We also have that

$$\theta(t^K(p'_i)) = \theta(\lambda \circ \theta(p_i)) = \theta(p_i).$$

From the fact that θ is left-covering, there is a unique path p''_i in H^{K+k} such that $\theta(p''_i) = \theta(p'_i)$ and $t^K(p''_i) = p_i$. Notice that, in particular, from considering the $n+1$ entries of $t^K(p') = \kappa_n(p)$, we have $t^K(p'_{n+1}) = p_n$ and so, by uniqueness, $p''_n = p'_{n+1}$. We let $p''_i = p'_{i+1}$ for $n \leq i \leq N$. We claim that p'' is in H^{K+k}_N. This follows from the fact that $\theta(p'') = \theta(p')$ and $\theta(p')$ is in G^{K+k}_N. It is clear that $t^K(p'') = p$. Finally, we see that $\kappa_n(p'') = p'$ as follows. For $n < i \leq N+1$, we have $(\kappa_n(p''))_i = p''_{i-1} = p'_i$, by definition. For $0 \leq i \leq n$, we have

$$(\kappa_n(p''))_i = \lambda \circ \theta(p''_i) = \lambda \circ \theta(p'_i).$$

We also know that $t^K(p'_i) = \lambda \circ \theta(p_i)$ and so it follows from Lemma 4.5.4 that $\lambda \circ \theta(p'_i) = p'_i$. This completes the proof of the claim.

We are now ready to prove the result. We compute directly from the definitions, for p in H^k_N, $0 \leq m \leq N+1$ and $0 \leq n \leq N$,

$$\delta^{s,K}_m \circ \kappa_n(p) = Sum\{\delta_m(p') \mid p' \in H^{k+K}_{N+1}, t^K(p') = \kappa_n(p)\}.$$

The claim above shows that this is equal to

$$\delta^{s,K}_m \circ \kappa_n(p) = Sum\{\delta_m(\kappa_n(p'')) \mid p'' \in H^{k+K}_N, t^K(p'') = p\}.$$

Now we use the standard, easily computed facts:

$$\begin{array}{rcll} \delta_m \circ \kappa_n & = & \kappa_{n-1} \circ \delta_m & \text{if} \quad m < n \\ \delta_m \circ \kappa_n & = & \kappa_n \circ \delta_{m-1} & \text{if} \quad m > n+1 \\ \delta_{n+1} \circ \kappa_n & = & \delta_{n+1} \circ \kappa_{n+1}. & \end{array}$$

Summing over all $0 \leq m \leq N+1$ and $0 \leq n \leq N$ with a coefficient of $(-1)^{m+n}$ yields

$$d^{s,K}(\theta \circ \rho)_{N+1} \circ \kappa(p) = \sum_{p'' \in B} \left(\sum_{m,n=0}^{N} (-1)^{m+n} \kappa_n \circ \delta_m(p'') + \lambda \circ \theta(p'') - p'' \right),$$

where $B = \{p'' \in H_N^{k+K} \mid t^K(p'') = p\}$. □

We are now ready to prove Theorem 4.5.1.

Proof. Considering the first statement of 4.5.1, we first show that the map on homology induced by θ is injective. Let a be in $\mathbb{Z}H_N^K$ and $j \geq 0$. Suppose that $[a, j]$ as an element of $D^s(H_N^{K+1})$ is a cocycle. That means that $0 = d^s(\rho \circ \theta)_N[a, j]$. From 4.2.3, we know that $d^s(\rho \circ \theta)_N[a, j] = [d^{s,K}(\rho \circ \theta)_N(a), j]$ and it follows that, for some $j' \geq 0$, $(\gamma_{H_n}^s)^{j'}(d^{s,K}(\rho \circ \theta)_N(a)) = 0$. By Theorem 4.2.3, $0 = (\gamma_{H_{N-1}}^s)^{j'}(d^{s,K}(\rho \circ \theta)_N(a)) = d^{s,K}(\rho \circ \theta)_N((\gamma_{H_N}^s)^{j'}(a))$. Let $a_1 = (\gamma_{H_N}^s)^{j'}(a)$ and $j_1 = j + j'$. We have $[a, j] = [a_1, j_1]$ and $d^{s,K}(\rho \circ \theta)_N(a_1) = 0$. The fact that θ maps the class of the cocycle to zero means that in the group $D^s(G_N)$, $[\theta(a_1), j_1] = d^s(\rho)_N[b, j'']$, for some b in $\mathbb{Z}G_{N+1}^0$. We know from Theorem 4.2.3 that $d^s(\rho)_N[b, j''] = [d^{s,K}(\rho)_{N+1}(b), j'']$ and so we know there is $l \geq 0$ such that

$$(\gamma_{G_N}^s)^{j''+l}(\theta(a_1)) = (\gamma_{G_N}^s)^{j_1+l}(d_{N+1}^{s,K}(\rho)(b)).$$

Letting $a_2 = (\gamma_{H_N}^s)^{j''+l}(a_1)$ and $b_2 = (\gamma_{G_{N+1}}^s)^{j_1+l}(b)$, we have $[a_2, j_1 + j'' + l] = [a, j]$, $d^{s,K}(\rho \circ \theta)_N(a_2) = 0$ and

$$\theta(a_2) = d^{s,K}(\rho)_{N+1}(b_2).$$

Now we apply both sides of Lemma 4.5.7 to the element a_2 and obtain

$$\begin{aligned} d^{s,K}(\rho \circ \theta)_{N+1}(\kappa(a_2)) &= t^{*K} \circ \lambda \circ \theta(a_2) - t^{*K}(a_2) \\ &= t^{*K} \circ \lambda \circ d_{N+1}^{s,K}(\rho)(b_2) - t^{*K}(a_2) \\ &= d^{s,K}(\rho \circ \theta)_{N+1} \circ t^{*K} \circ \lambda(b_2) - t^{*K}(a_2). \end{aligned}$$

Applying i^K to both sides and using the fact that i commutes with $d^{s,K}(\rho \circ \theta)_{N+1}$, we see that $(\gamma_N^s)^K(a_2)$ is in the image $d^{s,K}(\rho \circ \theta)_{N+1}$ and it follows that $[a, j]$ is in the image of $d^s(\rho \circ \theta)_{N+1}$ and hence is zero in homology.

Now, we must show the map is surjective. Let b be in $\mathbb{Z}G_N^K$ and $j \geq 1$. We assume that $d^s(\rho)_N[b, j] = 0$. Arguing in much the same fashion as above, we may assume that $d^{s,K}(\rho)_N(b) = 0$. It follows from Lemma 4.5.5 that

$$d^s(\rho \circ \theta)_{N+1}(\lambda(b)) = \lambda(d^s(\rho)_{N+1}(b)) = 0,$$

so that $[\lambda(b), j]$ is a cocycle and $\theta[\lambda(b), j] = [\theta \circ \lambda(b), j] = [b, j]$ and we are done.

The proof of the second part regarding the cochain complexes is done in much the same way. We just observe that analogues of the four Lemmas 4.5.4, 4.5.5, 4.5.6, 4.5.7 all hold for θ^* by simply using the duality of 3.1.1 and Lemma 3.5.6. We omit the details. $\square$

Chapter 5

The double complexes of an s/u-bijective pair

This chapter represents the heart of our theory: the definition of the homology theories for a Smale space and their basic properties. In the first section, we consider a Smale space (X, φ) and an s/u-bijective pair π for it. From this we construct several double complexes. There are eight in all, four based on Krieger's invariant D^s and four on D^u. For most of what follows, we restrict our attention to the first four. The differences between these four are analogous to the ordered verses alternating complexes we saw in the last chapter. Indeed, much of this chapter runs parallel to the developments in the last chapter. In addition, for many of the proofs, the difficult part has already been established in the special cases done in the last chapter. Our invariants, denoted $H^s(\pi)$ and $H^u(\pi)$ are defined as the homology of two of these double complexes. We stress that, at this point, it is defined for an s/u-bijective pair.

The second section develops the symbolic versions of the objects we defined in the first section. These will be crucial in working with the definitions.

The third section is devoted to comparing the alternating and ordered complexes of the first section. In particular, we shall show that three of these all have the same homology, in a natural way. This is a natural extension of Theorem 4.3.1. At this point, it is likely that the fourth complex also has the same homology, but this seems out of the reach of standard techniques.

The fourth section outlines the functorial properties of our invariants. Put briefly, H^s is covariant for s-bijective factor maps and contravariant for u-bijective factor maps. Not surprisingly, H^u is covariant for u-bijective factor maps and contravariant for s-bijective factor maps. These are exactly the same as for Krieger's invariants.

The constructions of the first section will clearly be dependent on the choice of s/u-bijective pair. In the last section, we establish the crucial fact that while the double complexes depend on this choice, their homologies do not. That is, our invariant depends only on the Smale space and not the s/u-bijective pair. With this result, we write the invariants as $H^s(X,\varphi)$ and $H^u(X,\varphi)$.

5.1 Definitions of the complexes

We begin with a Smale space (X,φ) and assume that it has an s/u-bijective pair, π. Based on π, we construct first two double complexes: the groups in the first are Krieger's invariant D^s, applied to the shifts of finite type $\Sigma_{L,M}(\pi)$ defined in 2.6.4. The second complex simply replaces D^s with D^u. These complexes will be denoted by $(C^s(\pi), d^s(\pi))$ and $(C^u(\pi), d^u(\pi))$.

We note one important result: if we restrict our attention to a single row or column in the double complex, the result is a complex of the type we considered in the last chapter. This will allow us ultimately to transfer a number of results from the last chapter to the situation here.

Following this, we introduce six other double complexes, three associated with the invariant D^s and three with D^u. These will be analogues of the alternating complexes from the last chapter. They correspond to the natural actions of the groups $S_{L+1}\times 1$, $1\times S_{M+1}$ and $S_{L+1}\times S_{M+1}$ on the spaces $\Sigma_{L,M}(\pi)$. Ignoring the D^u case for the moment, these three new complexes will be denoted $(C^s_{\mathcal{Q},}(\pi), d^s_{\mathcal{Q},}(\pi))$, $(C^s_{,\mathcal{A}}(\pi), d^s_{,\mathcal{A}}(\pi))$ and $(C^s_{\mathcal{Q},\mathcal{A}}(\pi), d^s_{\mathcal{Q},\mathcal{A}}(\pi))$. Our homology, denoted $H^s(\pi)$, will be defined as the homology of the last.

We recall the definition from section 2.6: for each $L, M \geq 0$,

$$\begin{aligned}\Sigma_{L,M}(\pi) &= \{(y_0,\ldots,y_L,z_0,\ldots,z_M) \mid y_l\in Y, z_m\in Z,\\ &\qquad \pi_s(y_l)=\pi_u(z_m), 0\leq l\leq L, 0\leq m\leq M\},\end{aligned}$$

which is a shift of finite type from Theorem 2.6.6.

We also recall the maps $\delta_{l,}$ and $\delta_{,m}$ of Definition 2.6.4, which delete entries y_l and z_m, respectively, and the fact from Theorem 2.6.13 that the former are all s-bijective and the latter are all u-bijective. This fact is at the heart of the following definition.

Definition 5.1.1. *Let π be an s/u-bijective pair for the Smale space (X, φ).*

1. *For each $L, M \geq 0$, we define*

$$C^s(\pi)_{L,M} = D^s(\Sigma_{L,M}(\pi), \sigma).$$

We define these groups to be zero for all other integral values of L, M. We define also

$$d^s(\pi)_{L,M} = \sum_{0 \leq l \leq L} (-1)^l \delta^s_{l,} + \sum_{0 \leq m \leq M+1} (-1)^{L+m} \delta^{s*}_{,m}$$

on the summand $C^s(\pi)_{L,M}$. Notice that the first sum has range $C^s(\pi)_{L-1,M}$, while the second has range $C^s(\pi)_{L,M+1}$. The first term only appears when $L \geq 1$ and the second only for $M \geq 0$.

2. *For each $L, M \geq 0$, we define*

$$C^u(\pi)_{L,M} = D^u(\Sigma_{L,M}(\pi), \sigma).$$

We define these groups to be zero for all other integral values on L, M. We define

$$d^u(\pi)_{L,M} = \sum_{0 \leq l \leq L+1} (-1)^l \delta^{u*}_{l,} + \sum_{0 \leq m \leq M} (-1)^{L+m} \delta^{u}_{,m}$$

on the summand $C^u(\pi)_{L,M}$. Notice that the first sum has range $C^u(\pi)_{L+1,M}$, while the second has range $C^u(\pi)_{L,M-1}$. The first term only appears when $L \geq 0$ and the second only for $M \geq 1$.

For the moment, we concentrate on $C^s(\pi)$. For any integer N, we may consider $\oplus_{L-M=N} C^s(\pi)_{L,M}$ and observe that the map

$$\oplus_{L-M=N} d^s(\pi)_{L,M} : \oplus_{L-M=N} C^s(\pi)_{L,M} \to \oplus_{L-M=N-1} C^s_{L,M}(\pi).$$

Suppose for a moment that we fix an integer M and consider $C^s(\pi)_{L,M}, L \in \mathbb{Z}$, as a $\mathbb{Z}$-graded subgroup of $C^s(\pi)$. We refer to the restriction of $(C^s(\pi), d^s(\pi))$ to row M to be this graded group with the map $\sum_{0 \leq l \leq L} (-1)^l \delta^s_{l,}$. This is exactly the restriction of $d^s(\pi)$ to $C^s(\pi)_{L,M}, L \in \mathbb{Z}$, in the usual sense, followed by the projection onto $C^s(\pi)_{L,M}, L \in \mathbb{Z}$. There is an analogous definition of the restriction of $(C^s(\pi), d^s(\pi))$ to column L.

Recall from Definition 2.6.10 and Theorem 2.6.11 that, for each $L \geq 0$, we have a u-bijective factor map

$$\rho_{L,} : (\Sigma_{L,0}(\pi), \sigma) \to (Y_L(\pi_s), \psi)$$

and, for each $M \geq 0$, we have an s-bijective factor map

$$\rho_{,M} : (\Sigma_{0,M}(\pi), \sigma) \to (Z_M(\pi_u), \zeta)$$

From Theorem 2.6.12, $\Sigma_{L,M}(\pi)$ is the same as $\Sigma_{0,M}(\pi)_L(\rho_{,M})$. Moreover, under this identification, the map $\delta_{,m}$, which deletes entry m, is the same for these two systems. This leads us to the first part below; the others follow in a similar way.

Lemma 5.1.2. *1. For each L, M, as maps defined on $C^s_{L,M}(\pi)$, we have*

$$\begin{aligned} \sum_{0 \leq l \leq L} (-1)^l (\delta_{l,})^s &= d^s(\rho_{,M})_L \\ \sum_{0 \leq m \leq M+1} (-1)^m (\delta_{,m})^{s*} &= d^{s*}(\rho_{L,})_M \\ d^s(\pi)_{L,M} &= d^s(\rho_{,M})_L + (-1)^L d^{s*}(\rho_{L,})_M \end{aligned}$$

2. For each L, M, as maps defined on $C^u_{L,M}(\pi)$, we have

$$\begin{aligned} \sum_{0 \leq l \leq L+1} (-1)^l (\delta_{l,})^{u*} &= d^{u*}(\rho_{,M})_L \\ \sum_{0 \leq m \leq M} (-1)^m (\delta_{,m})^{u} &= d^{u}(\rho_{L,})_M \\ d^u(\pi)_{L,M} &= d^{u*}(\rho_{,M})_L + (-1)^L d^{u}(\rho_{L,})_M \end{aligned}$$

The first important consequence is that $(C^s(\pi), d^s(\pi))$ and $(C^u(\pi), d^u(\pi))$ do indeed form complexes. We state the relevant result for the former only.

Lemma 5.1.3. *Let π be an s/u-bijective pair. For all N in $\mathbb{Z}$, we have*

$$(\oplus_{L-M=N-1} d^s(\pi)_{L,M}) \circ (\oplus_{L-M=N} d^s(\pi)_{L,M}) = 0.$$

Proof. Fix L, M. We must show that the composition is zero on $C^s(\pi)_{L,M}$. The only non-trivial cases are for $L \geq 1$ and $M \geq 0$. We first apply $d^s(\pi)_{L,M} = d^s(\rho_{,M})_L + (-1)^L d^{s*}(\rho_{L,})_M$. The first term has its image in $C^s(\pi)_{L-1,M}$ and the second in $C^s(\pi)_{L,M+1}$. On the first, we apply $d^s(\pi)_{L-1,M}$ and, on the second, $d^s(\pi)_{L,M+1}$. We summarize: the restriction of the composition to $C^s(\pi)_{L,M}$ is

$$\begin{aligned} d^s(\rho_{,M})_{L-1} \circ d^s(\rho_{,M})_L \quad &+ \quad (-1)^{L-1} d^{s*}(\rho_{L-1,})_M \circ d^s(\rho_{,M})_L \\ +(-1)^L d^s(\rho_{,M+1})_L \circ d^{s*}(\rho_{L,})_M \quad &+ \quad d^{s*}(\rho_{L,})_{M+1} \circ d^{s*}(\rho_{L,})_M. \end{aligned}$$

The first and last terms are zero by Lemma 4.1.3 and we are left to consider the two middle terms. Here, we get

$$\begin{aligned} &\left((-1)^{L-1} \sum_m (-1)^m (\delta_{,m})^{s*}\right) \quad \circ \quad \left(\sum_l (-1)^l (\delta_{l,})^s\right) \\ &\qquad + \left(\sum_l (-1)^l (\delta_{l,})^s\right) \quad \circ \quad \left((-1)^L \sum_m (-1)^m (\delta_{,m})^{s*}\right) \\ &= (-1)^{L-1} \sum_{l,m} (-1)^{l+m} ((\delta_{l,})^s \circ (\delta_{,m})^{s*} \quad - \quad (\delta_{,m})^{s*} \circ (\delta_{l,})^s). \end{aligned}$$

It is clear that, for every l and m, the maps $\delta_{l,}$ and $\delta_{,m}$ commute. Consider the diagram

$$\begin{array}{ccc} \Sigma_{L,M+1} & \xrightarrow{\delta_{l,}} & \Sigma_{L-1,M+1} \\ \downarrow{\scriptstyle \delta_{,m}} & & \downarrow{\scriptstyle \delta_{,m}} \\ \Sigma_{L,M} & \xrightarrow{\delta_{l,}} & \Sigma_{L-1,M} \end{array}$$

We apply Theorem 3.5.11, using $\eta_1 = \delta_{l,} = \pi_2$ and $\eta_2 = \delta_{,m} = \pi_1$ (there is a slight abuse of notation since $\delta_{l,}$ and $\delta_{,m}$ each have two different domains) to conclude that each term in the sum above is zero and our proof is complete. Notice that it is a trivial matter to see that the diagram above is actually the fibred product. □

We have already observed the following result, but we state it for emphasis.

Theorem 5.1.4. *1. For fixed $M \geq 0$, row M of the complex $(C^s(\pi), d^s(\pi)$ is the same as the complex $(D^s(\Sigma_{*,M}(\rho_{,M})), d^s(\rho_{,M}))$.*

2. *For fixed $L \geq 0$, column L of the complex $(C^s(\pi), d^s(\pi)$ is the same as the complex $(D^s(\Sigma_{L,*}(\rho_{L,})), d^s(\rho_{L,}))$.*

Next, we need to consider analogues of the alternating complexes in this setting. Of course, there are two different permutation groups acting, S_{L+1} and S_{M+1}. For the former, we are using a chain complex in that variable, so we need to consider the quotient complex and for the latter we are using a cochain complex and need to consider the subcomplex.

Definition 5.1.5. *Let π be an s/u-bijective pair for the Smale space (X, φ). Let $L, M \geq 0$.*

1. *We define $D^s_{\mathcal{B},}(\Sigma_{L,M}(\pi))$ to be the subgroup of $D^s(\Sigma_{L,M}(\pi))$ which is generated by*
   - *(a) all elements b such that $b = b \cdot (\alpha, 1)$, for some transposition α in S_{L+1}, and*
   - *(b) all elements of the form $a - sgn(\alpha) a \cdot (\alpha, 1)$, where a is in $D^s(\Sigma_{L,M}(\pi))$ and α is in S_{L+1}.*
2. *We define $D^s_{\mathcal{Q},}(\Sigma_{L,M}(\pi))$ to be the quotient of $D^s(\Sigma_{L,M}(\pi))$ by the subgroup $D^s_{\mathcal{B},}(\Sigma_{L,M}(\pi))$ and we let Q denote the quotient map.*
3. *We define $D^s_{,\mathcal{A}}(\Sigma_{L,M}(\pi))$ to be the subgroup of all elements a in $D^s(\Sigma_{L,M}(\pi))$ satisfying $a = sgn(\beta) a \cdot (1, \beta)$, for all β in S_{M+1} and we let J denote the inclusion map.*
4. *We define $D^s_{\mathcal{Q},\mathcal{A}}(\Sigma_{L,M}(\pi))$ to be the image in $D^s_{\mathcal{Q},}(\Sigma_{L,M}(\pi))$ of $D^s_{,\mathcal{A}}(\Sigma_{L,M}(\pi))$ under Q. We let $Q_{\mathcal{A}}$ denote the restriction of Q to $D^s_{,\mathcal{A}}(\Sigma_{L,M}(\pi))$ and $J_{\mathcal{Q}}$ denote the inclusion of $D^s_{\mathcal{Q},\mathcal{A}}(\Sigma_{L,M}(\pi))$ in $D^s_{\mathcal{Q},}(\Sigma_{L,M}(\pi))$.*

There are analogous definitions of $D^u_{\mathcal{A},}(\Sigma_{L,M}(\pi)), D^u_{,\mathcal{Q}}(\Sigma_{L,M}(\pi))$ and $D^u_{\mathcal{A},\mathcal{Q}}(\Sigma_{L,M}(\pi))$.

Lemma 5.1.6. *For $L, M \geq 0$, as maps on $D^s(\Sigma_{L,M}(\pi))$, we have*

$$\begin{aligned}
d^s(\rho_{,M})_L(D^s_{\mathcal{B},}(\Sigma_{L,M}(\pi))) &\subset D^s_{\mathcal{B},}(\Sigma_{L-1,M}(\pi)) \\
d^{s*}(\rho_{L,})_M(D^s_{\mathcal{B},}(\Sigma_{L,M}(\pi))) &\subset D^s_{\mathcal{B},}(\Sigma_{L,M+1}(\pi)) \\
d^{s,K}(\rho_{,M})_L(D^s_{,\mathcal{A}}(\Sigma_{L,M}(\pi))) &\subset D^s_{,\mathcal{A}}(\Sigma_{L-1,M}(\pi)) \\
d^{s*,K}(\rho_{L,})_M(D^s_{,\mathcal{A}}(\Sigma_{L,M}(\pi))) &\subset D^s_{,\mathcal{A}}(\Sigma_{L,M+1}(\pi))
\end{aligned}$$

Proof. For the first part, the value M is held constant and the result here is an immediate consequence of the first statement in part 1 of Lemma 4.1.6, applied to the s-bijective factor map $\rho_{,M}$. Similarly, the fourth part is obtained from the second statement in part 2 of 4.1.6 applied to the u-bijective factor map $\rho_{L,}$. The second part follows from the fact that the action of $S_{L+1} \times 1$ commutes with $d^{s*,K}(\rho_{L,})_M$. The third is analogous to the second. □

Definition 5.1.7. *Let π be an s/u-bijective pair for the Smale space (X, φ). For $L, M \geq 0$, we define*

$$\begin{aligned} C^s_{\mathcal{Q},}(\pi)_{L,M} &= D^s_{\mathcal{Q},}(\Sigma_{L,M}(\pi)), \\ C^s_{,\mathcal{A}}(\pi)_{L,M} &= D^s_{,\mathcal{A}}(\Sigma_{L,M}(\pi)), \\ C^s_{\mathcal{Q},\mathcal{A}}(\pi)_{L,M} &= D^s_{\mathcal{Q},\mathcal{A}}(\Sigma_{L,M}(\pi)). \end{aligned}$$

We also let

1. *$d^s_{\mathcal{Q},}(\pi)_{L,M}$ be the map induced by $d^s(\pi)_{L,M}$ on the quotient $D^s_{\mathcal{Q},}(\Sigma_{L,M}(\pi))$,*
2. *$d^s_{,\mathcal{A}}(\pi)_{L,M}$ to be the restriction of $d^s(\pi)_{L,M}$ to $D^s_{,\mathcal{A}}(\Sigma_{L,M}(\pi))$, and*
3. *$d^s_{\mathcal{Q},\mathcal{A}}(\pi)_{L,M}$ to be the restriction of $d^s_{\mathcal{Q},}(\pi)_{L,M}$ to $D^s_{\mathcal{Q},\mathcal{A}}(\Sigma_{L,M}(\pi))$.*

The following simply summarizes these last definitions and a simple consequence of Lemma 5.1.6.

Theorem 5.1.8. *Let π be an s/u-bijective pair for the Smale space (X, φ). We have a commutative diagram of chain complexes and chain maps as shown:*

$$\begin{array}{ccc} (C^s_{,\mathcal{A}}, d^s_{,\mathcal{A}}) & \xrightarrow{J} & (C^s, d^s) \\ {\scriptstyle Q_{\mathcal{A}}}\downarrow & & \downarrow{\scriptstyle Q} \\ (C^s_{\mathcal{Q},\mathcal{A}}, d^s_{\mathcal{Q},\mathcal{A}}) & \xrightarrow{J_{\mathcal{Q}}} & (C^s_{\mathcal{Q},}, d^s_{\mathcal{Q},}) \end{array}$$

Before proceeding further towards our main definition, we want to make some important (although fairly simple) observations about the four complexes of the last theorem.

Theorem 5.1.9. *Let π be an s/u-bijective pair for the Smale space (X, φ).*

1. *For a fixed $M \geq 0$, row M of the complex $C^s_{\mathcal{Q},}(\pi), d^s_{\mathcal{Q},}(\pi))$ is the same as $(D^s_{\mathcal{Q}}(\Sigma_{*,M}(\rho_{,M})), d^s_{\mathcal{Q}}(\rho_{,M}))$ of Theorem 4.1.4.*

2. *For a fixed $L \geq 0$, column L of the complex $C^s_{\mathcal{A},}(\pi), d^s_{\mathcal{A},}(\pi))$ is the same as $(D^s_{\mathcal{A}}(\Sigma_{L,*}(\rho_{L,})), d^{s*}_{\mathcal{A}}(\rho_{L,}))$ of Theorem 4.1.4.*

Let us now make a preliminary observation as to why these alternating complexes may be useful and/or simpler than our original.

Theorem 5.1.10. *Let π be an s/u-bijective pair for the Smale space (X, φ). Let L_0 and M_0 be such that $\#\pi_s^{-1}\{x\} \leq L_0$ and $\#\pi_u^{-1}\{x\} \leq M_0$, for all x in X (which exist by Theorem 2.5.3).*

1. *If $L \geq L_0$, then $C^s_{\mathcal{Q},}(\pi)_{L,M} = 0$.*

2. *If $M \geq M_0$, then $C^s_{,\mathcal{A}}(\pi)_{L,M} = 0$.*

3. *If either $L \geq L_0$ or $M \geq M_0$, then $C^s_{\mathcal{Q},\mathcal{A}}(\pi)_{L,M} = 0$.*

Proof. The first two parts follow immediately from the last Theorem and the analogous result for the map complexes given in Theorem 4.2.12.

As for the third part, for any $L, M \geq 0$, the map $Q_{\mathcal{A}}$ defines a homomorphism from $C^s_{,\mathcal{A}}(\pi)_{L,M}$ to $C^s_{\mathcal{Q},}(\pi)_{L,M}$ whose image is $C^s_{\mathcal{Q},\mathcal{A}}(\pi)_{L,M}$. So if either its domain or range is zero, then so is its image. □

This brings us to our main definition, but some comments are in order before we give it. To an s/u-bijective pair π, we have associated four double complexes, $(C^s(\pi), d^s(\pi)), (C^s_{\mathcal{Q},}(\pi), d^s_{\mathcal{Q},}(\pi)), (C^s_{,\mathcal{A}}(\pi), d^s_{,\mathcal{A}}(\pi))$ and $(C^s_{\mathcal{Q},\mathcal{A}}(\pi), d^s_{\mathcal{Q},\mathcal{A}}(\pi))$. The first of these stands alone as having the simplest definition, in a certain sense. On the other hand, it is the most troublesome of the complexes because it is non-zero in infinitely many positions as either L or M tend to infinity. In addition, because its boundary has components of degree $(-1, 0)$ and $(1, 0)$, it is really a second (or fourth) quadrant double complex, which make it less accessible to standard techniques of spectral sequences.

The next two complexes are both non-zero in infinitely many coordinates, but each in only a single direction. This means they are much more amenable to study by the use of spectral sequences. Finally, the last complex is actually only non-zero in finitely many positions, which makes it the simplest, in some sense.

Ultimately (in the next section), we will show that the second, third and fourth all yield isomorphic homologies. At this point, we believe the first also yields the same homology, but this does not follow easily from standard methods. So we choose as our definition, the homology of the last complex. It is this which is the simplest to compute and which gives the right answer for the next chapter.

Definition 5.1.11. *Let π be an s/u-bijective pair for the Smale space (X, φ).*

1. *We define $H^s_*(\pi)$ to be the homology of the double complex $(C^s_{\mathcal{Q},\mathcal{A}}(\pi), d^s_{\mathcal{Q},\mathcal{A}}(\pi))$. That is, for each integer N, we have*

$$H^s_N(\pi) = Ker(\oplus_{L-M=N} d^s_{\mathcal{Q},\mathcal{A}}(\pi)_{L,M})/Im(\oplus_{L-M=N+1} d^s_{\mathcal{Q},\mathcal{A}}(\pi)_{L,M}).$$

2. *We define $H^u_*(\pi)$ to be the homology of the double complex $(C^u_{\mathcal{A},\mathcal{Q}}(\pi), d^u_{\mathcal{A},\mathcal{Q}}(\pi))$. That is, for each integer N, we have*

$$H^u_N(\pi) = Ker(\oplus_{L-M=N} d^u_{\mathcal{A},\mathcal{Q}}(\pi)_{L,M})/Im(\oplus_{L-M=N-1} d^u_{\mathcal{A},\mathcal{Q}}(\pi)_{L,M}).$$

It is worth observing that the homology groups are actually defined for an s/u-bijective pair. Our main result of section 5.5 will show that it actually only depends on (X, φ) and we will ultimately use the notation $H^s_*(X, \varphi)$ and $H^u_*(X, \varphi)$. (The definition does rely on the *existence* of an s/u-bijective pair.)

The bounds on the complexes given in Theorem 5.1.10, particularly for $C^s_{\mathcal{Q},\mathcal{A}}(\pi)_{L,M}$, will be very useful. Recall that an abelian group H is finite rank if the rational vector space $H \otimes_{\mathbb{Z}} \mathbb{Q}$ is finite dimensional. If we consider the case of a finitely generated free abelian group, such as $\mathbb{Z}G^k_{L,M}$, we note that we may identify $\mathbb{Z}G^k_{L,M} \otimes_{\mathbb{Z}} \mathbb{Q}$ with the vector space of formal linear combinations of the generating set, $G^k_{L,M}$, which we denote by $\mathbb{Q}G^k_{L,M}$. Here, $G^k_{L,M}$ is a basis and this vector space is finite dimensional.

Continuing, we note that $D^s(G^k_{L,M}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is the inductive limit of the system

$$\mathbb{Q}G^{k-1}_{L,M} \otimes \xrightarrow{\gamma^s} \mathbb{Q}G^{k-1}_{L,M} \xrightarrow{\gamma^s} \cdots$$

Now suppose V is any finite dimensional (rational) vector space and $\alpha : V \to V$ is any linear transformation. Let W denote the inductive limit of the system

$$V \xrightarrow{\alpha} V \xrightarrow{\alpha} \cdots .$$

The sequence of subspaces $\alpha^n(V), n \geq 0$ is decreasing and hence there exists some $N \geq 1$ with $\dim(\alpha^N(V)) = \dim(\alpha^{N+1}(V))$. It follows that α restricts to an isomorphism between $\alpha^N(V)$ and $\alpha^{N+1}(V)$. It is then a simple matter to prove that the map sending v in $\alpha^N(V)$ to $[v, 1]$ in W is an isomorphism and hence $\dim(W) = \dim(\alpha^N(V)) \leq \dim(V)$. So we conclude from this that $D^s(G^k_{L,M})$ (and also $D^u(G^k_{L,M})$) is finite rank. The following result is an immediate consequence.

Theorem 5.1.12. *Let π be an s/u-bijective pair for the Smale space (X, φ) and let L_0 and M_0 be as in Theorem 5.1.10.*

1. *The homology groups $H^s_N(\pi)$ are finite rank, for each integer N.*
2. *The homology groups $H^s_N(\pi)$ are zero for $N \leq -M_0$ and for $N \geq L_0$.*

5.2 Symbolic presentations

With the assumption that we are given a graph which represents an s/u-bijective pair for the Smale space (X, φ), we develop appropriate symbolic presentations for computations of our invariants. Most of the material in this section is an immediate adaptation of the results in Section 4.2.

Definition 5.2.1. *For any $k, L, M \geq 0$, we define:*

1. *$\mathcal{B}(G^k_{L,M}, S_{L+1} \times 1)$ is the subgroup of $\mathbb{Z}G^k_{L,M}$ generated by all p in $G^k_{L,M}$ such that $p \cdot ((l\, l'), 1) = p$, for some $0 \leq l \neq l' \leq L$ and all $p \cdot (\alpha, 1) - sgn(\alpha)p$, where p is in $G^k_{L,M}$ and α is in S_{L+1},*
2. *$\mathcal{Q}(G^k_{L,M}, S_{L+1} \times 1)$ is the quotient of $\mathbb{Z}G^k_{L,M}$ by $\mathcal{B}(G^k_{L,M}, S_{L+1} \times 1)$ and Q is the quotient map,*
3. *$\mathcal{A}(G^k_{L,M}, 1 \times S_{M+1})$ is the subgroup of $\mathbb{Z}G^k_{L,M}$ of all elements a satisfying $a \cdot (1, \beta) = sgn(\beta)a$, for all β in $G^k_{L,M}$.*

Just as in Section 4.2, the groups $\mathcal{B}(G^k_{L,M}, S_{L+1} \times 1)$ and $\mathcal{A}(G^k_{L,M}, 1 \times S_{M+1})$ are invariant under the map γ^s. We use the same notation for the map induced on the quotient, $\mathcal{Q}(G^k_{L,M}, S_{L+1} \times 1)$.

Definition 5.2.2. *For any $k, L, M \geq 0$, we define:*

1.

$$D^s_{\mathcal{Q},}(G^k_{L,M}) = \lim \mathcal{Q}(G^k_{L,M}, S_{L+1} \times 1) \xrightarrow{\gamma^s} \mathcal{Q}(G^k_{L,M}, S_{L+1} \times 1) \xrightarrow{\gamma^s}$$

2.

$$D^s_{,\mathcal{A}}(G^k_{L,M}) = \lim \mathcal{A}(G^k_{L,M}, 1 \times S_{M+1}) \xrightarrow{\gamma^s} \mathcal{A}(G^k_{L,M}, 1 \times S_{M+1}) \xrightarrow{\gamma^s}$$

3.

$$D^s_{\mathcal{Q},\mathcal{A}}(G^k_{L,M}) = \lim Q(\mathcal{A}(G^k_{L,M}, 1 \times S_{M+1})) \xrightarrow{\gamma^s} Q(\mathcal{A}(G^k_{L,M}, 1 \times S_{M+1})) \xrightarrow{\gamma^s}$$

We will (also) let $Q_{\mathcal{A}}$ be the map from $D^s_{,\mathcal{A}}(G^k_{L,M})$ to $D^s_{\mathcal{Q},\mathcal{A}}(G^k_{L,M})$ which is the restriction of Q. We let J denote the inclusion of $D^s_{,\mathcal{A}}(G^k_{L,M})$ into $D^s(G^k_{L,M})$ and $J_{\mathcal{Q}}$ be the inclusion of $D^s_{\mathcal{Q},\mathcal{A}}(G^k_{L,M})$ in $D^s_{\mathcal{Q},}(G^k_{L,M})$. Thus, we have a commutative diagram

$$\begin{array}{ccc} D^s_{,\mathcal{A}}(G^k_{L,M}) & \xrightarrow{J} & D^s(G^k_{L,M}) \\ \downarrow Q_{\mathcal{A}} & & \downarrow Q \\ D^s_{\mathcal{Q},\mathcal{A}}(G^k_{L,M}) & \xrightarrow{J_{\mathcal{Q}}} & D^s_{\mathcal{Q},}(G^k_{L,M}). \end{array}$$

We have an analogue of Theorem 4.2.8 that allows to compute the groups in our alternating complexes using the presentation G.

Theorem 5.2.3. *Let π be an s/u-bijective pair for the Smale space (X, φ) and let G be a presentation of π. Let $L, M \geq 0$ and $k \geq 1$. The isomorphism of Theorem 3.3.3 between $D^s(\Sigma_{L,M}(\pi))$ and $D^s(G^k_{L,M})$ induces isomorphisms*

1. *$D^s_{\mathcal{Q},}(\Sigma_{L,M}(\pi))$ and $D^s_{\mathcal{Q},}(G^k_{L,M})$,*
2. *$D^s_{,\mathcal{A}}(\Sigma_{L,M}(\pi))$ and $D^s_{,\mathcal{A}}(G^k_{L,M})$, and*
3. *$D^s_{\mathcal{Q},\mathcal{A}}(\Sigma_{L,M}(\pi))$ and $D^s_{\mathcal{Q},\mathcal{A}}(G^k_{L,M})$.*

We now bring in the boundary maps.

Lemma 5.2.4. *Suppose that K satisfies the conclusion of Lemma 2.7.2 for ρ_s and ρ_u. Then for any $k \geq K, L, M \geq 0$, we have*

$$\begin{aligned} d^{s,K}(\rho_{,M})_L(\mathcal{B}(G^k_{L,M}, S_{L+1} \times 1)) &\subset \mathcal{B}(G^{k+K}_{L-1,M}, S_L \times 1), \\ d^{s*,K}(\rho_{L,})_M(\mathcal{B}(G^{k-K}_{L,M}, S_{L+1} \times 1)) &\subset \mathcal{B}(G^k_{L,M+1}, S_{L+1} \times 1), \\ d^{s,K}(\rho_{,M})_L(\mathcal{A}(G^{k+K}_{L,M}, 1 \times S_{M+1})) &\subset \mathcal{A}(G^k_{L-1,M}, 1 \times S_{M+1}), \\ d^{s*,K}(\rho_{L,})_M(\mathcal{A}(G^{k-K}_{L,M}, 1 \times S_{M+1})) &\subset \mathcal{A}(G^k_{L,M+1}, 1 \times S_{M+2}). \end{aligned}$$

Proof. For the first part, the value M is held constant and the result here is an immediate consequence of the first conclusion of Lemma 4.2.9, applied to the s-bijective factor map $\rho_{,M}$. Similarly, the fourth part is obtained from the third part of 4.2.9 applied to the u-bijective factor map $\rho_{L,}$. The second part follows from the fact that the action of $S_{L+1} \times 1$ commutes with $d^{s*,K}(\rho_{L,})_M$. The third is analogous to the second. □

As a consequence of the first two parts, we obtain well-defined boundary maps for our three new complexes.

Theorem 5.2.5. *Let π be an s/u-bijective pair for the Smale space (X, φ) and let G be a presentation of π. Let K satisfy the conclusion of Lemma 2.7.2 for ρ_s and ρ_u. With the identifications given in Theorem 5.2.3, we have*

1.
$$\begin{aligned} d^s_{\mathcal{Q},}(\pi)_{L,M}[Q(a), j] &= [Q(d^{s,K}(\rho_{,M})_L(a)), j] \\ &\quad +(-1)^L[Q(d^{s*,K}(\rho_{L,})_M(a)), j], \end{aligned}$$
for $k \geq K$, $j \geq 1$ and a in $\mathbb{Z}G^k_{L,M}$,

2.
$$\begin{aligned} d^s_{,\mathcal{A}}(\pi)_{L,M}[a, j] &= [d^{s,K}(\rho_{,M})_L(a), j] \\ &\quad +(-1)^L[d^{s*,K}(\rho_{L,})_M(a), j], \end{aligned}$$
for $j \geq 1$, $k \geq K$ and a in $\mathcal{A}(G^k_{L,M}, 1 \times S_{M+1})$ and

3.
$$\begin{aligned} d^s_{\mathcal{Q},\mathcal{A}}(\pi)_{L,M}[Q(a), j] &= [Q(d^{s,K}(\rho_{,M})_L(a)), j] \\ &\quad +(-1)^L[Q(d^{s*,K}(\rho_{L,})_M(a)), j], \end{aligned}$$
for $j \geq 1$, $k \geq K$ and a in $\mathcal{A}(G^k_{L,M}, 1 \times S_{M+1})$.

Let us remark that if we look just at the first boundary map, the term $Q(d^{s,K}(\rho_{,M})_L(a))$ actually lies in $\mathcal{Q}(G^{k+K}_{L,M}, S_{L+1} \times 1)$. We implicitly identify its class in $D^s_{\mathcal{Q},}(G^{k+K}_{L,M})$ with an element of $D^s_{\mathcal{Q},}(G^k_{L,M})$. We leave it as an exercise to see that each is well-defined and independent of the choice of k and K.

Let us now fix k, L, M. We choose k sufficiently large so that $k \geq 2K_0$, as in Lemma 2.7.6. We choose $B^k_{L,M}$, a subset of $G^k_{L,M}$, which meets $S_{L+1} \times S_{M+1}$-orbit having trivial isotropy exactly once and does not meet any other orbits.

We note the following definition for future reference.

Definition 5.2.6. *For each p in $B^k_{L,M}$ and $j \geq 1$, define*

$$t^*_{\mathcal{Q},\mathcal{A}}(p,j) = \{(q,\alpha,\beta) \in G^{k+j}_{L,M} \times S_{L+1} \times S_{M+1} \mid t^j(q) = p, i^j(q)\cdot(\alpha,\beta) \in B^k_{L,M}\}.$$

Define a homomorphism on the vector space $\mathbb{Z}B^k_{L,M}$ by setting, for q in $B^k_{L,M}$,

$$\gamma^s_B(p) = \sum_{q,\alpha,\beta)\in t^*_{\mathcal{Q},\mathcal{A}}(p,j)} sgn(\alpha) sgn(\beta) i(q') \cdot (\alpha,\beta).$$

Lemma 5.2.7. *Let $j \geq 1$. Then, we have*

$$(\gamma^s_B)^j(p) = \sum_{(q,\alpha,\beta)\in A(p,n)} sgn(\alpha) sgn(\beta) i^n(q) \cdot (\alpha,\beta).$$

Proof. The proof is by induction and the step $j = 1$ is simply the definition of γ^s_B. Assume the result is true for j and let us consider $(\gamma^s_B)^j(p)$.

Let (q,α,β) be in $t^*_{\mathcal{Q},\mathcal{A}}(p, j+1)$. First, we claim that $i^j(t(q'))$ is in B. Suppose there exist α', β' such that $i^j(t(q))\cdot(\alpha',\beta') = i^j(t(q'))$. Then by part 2 of Proposition 2.7.7, we know that $i^j(t(q'))\cdot(\alpha',1) = i^j(t(q'))$. Then by part 3 of the same result, we have $i^j(q')\cdot(\alpha',1) = i^j(q')$ and hence $i^{j+1}(q')\cdot(\alpha',1) = i^{j+1}(q')$. From our choice of q', it follows that $\alpha' = 1$. A similar argument shows that $\beta' = 1$ as well and so our claim is established.

We may find a unique (α',β') such that $i^n(t(q))\cdot(\alpha',\beta')$ is in B_0. It follows immediately that that $(t(q),\alpha',\beta')$ is in $A(q,n)$.

We also claim that $(i^n(q)\cdot(\alpha',\beta'), (\alpha')^{-1}\alpha, (\beta')^{-1}\beta)$ is in $t^*_{\mathcal{Q},\mathcal{A}}(t(i^n(q)), 1)$. The first condition is trivially satisfied. For the second, we have

$$i(i^j(q)\cdot(\alpha',\beta'))\cdot((\alpha')^{-1}\alpha, (\beta')^{-1}\beta) = i^{j+1}(q)\cdot(\alpha,\beta) \in B^k_{L,M},$$

from the fact (q,α,β) is in $t^*_{\mathcal{Q},\mathcal{A}}(p,j+1)$.

Let A denote the set of all sextuples $(q',\alpha',\beta',q'',\alpha'',\beta'')$ where (q',α',β') is in $t^*_{\mathcal{Q},\mathcal{A}}(p,j)$ and (q'',α'',β'') is in $t^*_{\mathcal{Q},\mathcal{A}}(i^j(q'),1)$. We claim that the map $f : t^*_{\mathcal{Q},\mathcal{A}}(p,j+1) \to A$ defined by

$$f(q,\alpha,\beta) = (t(q),\alpha',\beta', i^j(q)\cdot(\alpha',\beta'), (\alpha')^{-1}\alpha, (\beta')^{-1}\beta).$$

The inverse may be written explicitly as follows.

Let $(q', \alpha', \beta', q'', \alpha'', \beta'')$ be in A. Since (q'', α'', β'') is in $t^*_{\mathcal{Q},\mathcal{A}}(i^j(q'), 1)$, we know that $t(q'') = i^j(q') \cdot (\alpha', \beta')$. It follows that there is a unique path q with $t(q) = q'$ and $i^n(\tilde{q}) \cdot (\alpha', \beta') = q''$. We check that $(q, \alpha'\alpha'', \beta'\beta'')$ is in $t^*_{\mathcal{Q},\mathcal{A}}(q, j+1)$. First, we have $t^{j+1}(q) = t^j(q') = p$ since (q', α', β') is in $t^*_{\mathcal{Q},\mathcal{A}}(i^j(q'), 1)$. Secondly, we have

$$i^{j+1}(q) \cdot (\alpha'\alpha'', \beta'\beta'') = i(i^j(q) \cdot (\alpha', \beta')) \cdot (\alpha'', \beta'') = i(q'') \cdot (\alpha'', \beta'') \in B^k_{L,M}.$$

Let us set $g(q', \alpha', \beta', q'', \alpha'', \beta'') = (q, \alpha'\alpha'', \beta'\beta'')$.

It is a simple matter to verify that f and g are inverses and we omit the details.

It follows that

$$\begin{aligned}
(\gamma^s_B)^{j+1}(p) &= \gamma^s_B \left(\sum_{(q',\alpha',\beta') \in t^*_{\mathcal{Q},\mathcal{A}}(p,j)} sgn(\alpha') sgn(\beta') i^j(q') \cdot (\alpha', \beta') \right) \\
&= \sum_{(q',\alpha',\beta') \in t^*_{\mathcal{Q},\mathcal{A}}(p,j)} \; \sum_{(q'',\alpha'',\beta'') \in t^*_{\mathcal{Q},\mathcal{A}}(i^j(q') \cdot (\alpha',\beta'),1)} \\
&\quad sgn(\alpha') sgn(\beta') sgn(\alpha'') sgn(\beta'') i(q'') \cdot (\alpha', \beta') \cdot (\alpha'', \beta'') \\
&= \sum_{A} sgn(\alpha'\alpha'') sgn(\beta'\beta'') i(q'') \cdot (\alpha'\alpha'', \beta'\beta'') \\
&= \sum_{(q,\alpha,\beta) \in t^*_{\mathcal{Q},\mathcal{A}}(p,j+1)} sgn(\alpha) sgn(\beta) i^{j+1}(q) \cdot (\alpha, \beta).
\end{aligned}$$

□

Lemma 5.2.8. *Define a homomorphism* $A : \mathbb{Z}G^k_{L,M} \to \mathcal{A}(G^k_{L,M}, 1 \times S_{M+1})$ *by*

$$A(p) = \sum_{\beta \in S_{M+1}} sgn(\beta) p \cdot (1, \beta),$$

for all p *in* $G^k_{L,M}$. *The map*

$$QA : \mathbb{Z}B^k_{L,M} \to Q(\mathcal{A}(G^k_{L,M}, 1 \times S_{M+1}))$$

is an isomorphism and the following diagram commutes.

$$\begin{array}{ccc}
\mathbb{Z}B^k_{L,M} & \xrightarrow{\gamma^s_B} & \mathbb{Z}B^k_{L,M} \\
\downarrow{\scriptstyle QA} & & \downarrow{\scriptstyle QA} \\
Q(\mathcal{A}(G^k_{L,M}, 1 \times S_{M+1})) & \xrightarrow{\gamma^s} & Q(\mathcal{A}(G^k_{L,M}, 1 \times S_{M+1})).
\end{array}$$

Proof. We first show the map is onto. It is clear that the map A maps $\mathbb{Z}G^K_{L,M}$ onto $\mathcal{A}(G^K_{L,M}, 1 \times S_{M+1})$ and, by definition, Q maps the latter onto $Q(\mathcal{A}(G^K_{L,M}, 1 \times S_{M+1}))$. Consider q in $G^K_{L,M}$ and $QA(q)$. If q is in B, then there is a unique (α, β) such that $q \cdot (\alpha', \beta')$ is in $B^k_{L,M}$. Moreover, we have

$$\begin{aligned}
QA(q) &= Q(\sum_{\beta \in S_{M+1}} sgn(\beta) q \cdot (1, \beta)) \\
&= Q(\sum_{\beta \in S_{M+1}} sgn(\beta) sgn(\beta') q \cdot (1, \beta'\beta)) \\
&= Q(\sum_{\beta \in S_{M+1}} sgn(\alpha') sgn(\beta) sgn(\beta') q \cdot (\alpha', \beta'\beta)) \\
&= sgn(\alpha') sgn(\beta') Q(\sum_{\beta \in S_{M+1}} sgn(\beta)(q \cdot (\alpha', \beta')) \cdot (1, \beta) \\
&= sgn(\alpha') sgn(\beta') QA(q \cdot (\alpha', \beta')) \\
&\in QA(\mathbb{Z}B^k_{L,M}).
\end{aligned}$$

On the other hand, if q is not in B, then $q \cdot (\alpha, \beta) = q$, for some $(\alpha, \beta) \neq (1, 1)$ and it follows from the second part of Proposition 2.7.7 that $q \cdot (\alpha, 1) = q = q \cdot (1, \beta)$. First, suppose that $\beta \neq 1$. Then we may find a transposition β' such that $q \cdot (1, \beta')$. Let C be a subset of S_{M+1} which contains exactly one entry of each right coset of the subgroup $\{1, \beta'\}$. It follows that

$$\begin{aligned}
A(q) &= \sum_{\beta'' \in S_{M+1}} sgn(\beta'') q \cdot (1, \beta'') \\
&= \sum_{\beta'' \in C} sgn(\beta'') q \cdot (1, \beta'') + sgn(\beta'\beta'') q \cdot (1, \beta'\beta'') \\
&= \sum_{\beta'' \in C} sgn(\beta'') q \cdot (1, \beta'') - sgn(\beta'') q \cdot (1, \beta'') \\
&= 0.
\end{aligned}$$

Hence, we have $QA(q)$ is in $QA(\mathbb{Q}B_0)$. Now we consider the case $\alpha \neq 1$. Here, we have $Q(q) = 0$. But it is also the case that $Q(q \cdot (1, \beta)) = 0$, for all β in S_{M+1} and so we have $QA(q) = 0$. Again,we have $QA(q)$ is in $QA(\mathbb{Q}B_0)$. This completes the proof that QA is surjective.

We define a variant of the map $Q^{\#}$ of Section 4.3. We choose an order on the set of S_{M+1} orbits of elements in $G^k_{0,M}$. Let p be in $G^k_{L,M}$. From part

2 of Proposition 2.7.7, two rows of p are either equal or have distinct S_{M+1} orbits. So we define $Q^{\#}(Q(p)) = 0$ or two rows are equal and $Q^{\#}(Q(p)) = sgn(\alpha)p \cdot (\alpha, 1)$ if the rows are all distinct, where α is the unique element of S_{L+1} which puts the S_{M+1} orbits of the rows into increasing order. (Of course, to do this correctly, we should proceed as in Section 4.3 by defining a version on $\mathbb{Z}G^k_{L,M}$ and showing it is zero on $\mathcal{B}(G^k_{L,M}, S_{L+1} \times 1)$, but this is very similar to what was done earlier and we skip this.)

We make the following two claims, which are both quite clear. First, $Q^{\#} \circ Q$ commutes with the action of $1 \times S_{M+1}$ and hence also with the map A. Secondly, for any p in $B^k_{L,M}$, $Q^{\#} \circ Q(p)$ is in $\mathbb{Z}p \cdot S_{L+1} \times 1$. For different elements p, these subgroups have trivial intersection and it follows that $Q^{\#} \circ Q(p)$ is injective on $\mathbb{Z}B^k_{L,M}$. In addition, we see that $Q^{\#} \circ Q(p)$ maps $B^k_{L,M}$ into $B^k_{L,M} \cdot S_{L+1} \times 1$. It is a simple matter to see that the restriction of A to $\mathbb{Z}B^k_{L,M} \cdot S_{L+1} \times 1$ is injective.

Now, if a is in $\mathbb{Z}B^k_{L,M}$ and $QA(a) = 0$, then we have $0 = Q^{\#} \circ Q \circ A(a) = A \circ Q^{\#} \circ Q(a)$. It follows from the last paragraph that $Q^{\#} \circ Q(a) = 0$ and hence that $a = 0$, as desired.

We now prove the commutativity of the diagram. We first consider p in $B^k_{L,M}$ and

$$\gamma^s(p) = i \circ t^*(p) = \sum i(q),$$

where the sum is over all q in $G^{k+1}_{L,M}$ such that $t(q) = p$. We divide the set of all such q into two groups: one with $i(q)$ having trivial isotropy and the others with $i(q)$ having non-trivial isotropy. Consider q in the latter class for the moment. This means that $i(q) \cdot (\alpha, \beta) = i(q)$, for some α in S_{L+1} and β in S_{M+1}, not both the identity. It follows from the fourth part of Proposition 2.7.7 that we have

$$i(q) \cdot (\alpha, 1) = i(q) = i(q) \cdot (1, \beta).$$

The second equality and part five of 2.7.7 then imply that $q = q \cdot (1, \beta)$ and hence $t(q) \cdot (1, \beta) = t(q)$ and as $t(q) = p$ is in $B^k_{L,M}$, we conclude that $\beta = 1$. This means that $\alpha \neq 1$. It follows that $i(t(p)) \cdot ((l\, l'), 1) = i(t(p))$, for some $l \neq l'$ and this implies that $Q(i(t(p))) = 0$. We conclude from this that

$$Q(\gamma^s(p)) = \sum Q(i(q)),$$

where the sum s over all q with $t(q) = q$ and $i(q)$ having trivial isotropy. As each such element is in the $S_{L+1} \times S_{M+1}$ orbit of a point in $B^k_{L,M}$, we may re-write this as

$$Q(\gamma^s(p)) = \sum_{(q,\alpha,\beta)\in A(p,1)} Q(i(q)).$$

Now let p be in $B^k_{L,M}$ and β' in S_{M+1} and consider $Q(\gamma^s(p\cdot(1,\beta'))) = Q(\gamma^s(p)\cdot(1,\beta'))$. The action of $(1,\beta)$ does not affect $i(q)$ having repeated rows and so we conclude that

$$Q(\gamma^s(p\cdot(1,\beta'))) = \sum_{(q,\alpha,\beta)\in t^*_{\mathcal{Q},\mathcal{A}}(p,1)} Q(i(q)\cdot(1,\beta')).$$

Summing over β' with a $sgn(\beta')$, we have

$$\gamma^s \circ QA(p) = \sum_{\beta'\in S_{M+1}} \sum_{(q,\alpha,\beta)\in t^*_{\mathcal{Q},\mathcal{A}}(p,1)} sgn(\beta) Q(i(q)\cdot(1,\beta')).$$

On the other hand, we may use the fact that $sgn(\alpha)Q(q\cdot(\alpha,1)) = Q(q)$ to compute

$$\begin{aligned}
QA\circ\gamma^s_B(p) &= QA\left(\sum_{(q,\alpha,\beta)\in t^*_{\mathcal{Q},\mathcal{A}}(p,1)} sgn(\alpha)sgn(\beta)i(q)\cdot(\alpha,\beta)\right)\\
&= Q\left(\sum_{\beta'\in S_{M+1}}\sum_{(q,\alpha,\beta)} sgn(\alpha)sgn(\beta)sgn(\beta')i(q)\cdot(\alpha,\beta\beta')\right)\\
&= Q\left(\sum_{(q,\alpha,\beta)}\sum_{\beta'\in S_{M+1}} sgn(\beta\beta')i(q)\cdot(1,\beta\beta')\right)\\
&= Q\left(\sum_{(q,\alpha,\beta)}\sum_{\beta'\in S_{M+1}} sgn(\beta')i(q)\cdot(1,\beta')\right)\\
&= \gamma^s\circ QA(p)
\end{aligned}$$

as desired. □

Theorem 5.2.9. *For any $k \geq 2K_0$ and $L, M \geq 0$, $D^s_{\mathcal{Q},\mathcal{A}}(\Sigma_{L,M}(\pi))$ is isomorphic to the inductive limit of the system*

$$\mathbb{Z}B^k_{L,M} \xrightarrow{\gamma^s_B} \mathbb{Z}B^k_{L,M} \xrightarrow{\gamma^s_B} \cdots$$

Moreover, this isomorphism intertwines the automorphism $(\sigma^{-1})^s$ with the automorphism of the inductive system defined by $\gamma^s_B[a,j] = [\gamma^s_B(a), j]$.

5.3 Equivalence of the complexes

The main objective of this section is to show that, for a Smale space (X, φ) with s/u-bijective pair π, the complexes $(C^s_{\mathcal{Q},}(\pi), d^s_{\mathcal{Q},}(\pi)), (C^s_{,\mathcal{A}}(\pi), d^s_{,\mathcal{A}}(\pi))$ and $(C^s_{\mathcal{Q},\mathcal{A}}(\pi), d^s_{\mathcal{Q},\mathcal{A}}(\pi))$, all have the same homology. Specifically, we prove the following two results.

Theorem 5.3.1. *Let π be an s/u-bijective pair for the Smale space (X, φ). The chain map*

$$Q_{\mathcal{A}} : (C^s_{,\mathcal{A}}(\pi), d^s_{,\mathcal{A}}(\pi)) \to (C^s_{\mathcal{Q},\mathcal{A}}(\pi), d^s_{\mathcal{Q},\mathcal{A}}(\pi))$$

induces an isomorphism on homology.

Theorem 5.3.2. *Let π be an s/u-bijective pair for the Smale space (X, φ). The chain map*

$$J_{\mathcal{Q}} : (C^s_{\mathcal{Q},\mathcal{A}}(\pi), d^s_{\mathcal{Q},\mathcal{A}}(\pi)) \to (C^s_{\mathcal{Q},}(\pi), d^s_{\mathcal{Q},}(\pi))$$

induces an isomorphism on homology.

Let us begin the proof of Theorem 5.3.1. We consider the two filtrations:

$$\begin{aligned} F^p C^s_{,\mathcal{A}}(\pi) &= \oplus_{L \geq 0, M \geq p} C^s_{,\mathcal{A}}(\pi)_{L,M} \\ F^p C^s_{\mathcal{Q},\mathcal{A}}(\pi) &= \oplus_{L \geq 0, M \geq p} C^s_{\mathcal{Q},\mathcal{A}}(\pi)_{L,M} \end{aligned}$$

Notice that the first filtration is decreasing and bounded: that is,

$$C^s_{,\mathcal{A}}(\pi) = F^0 C^s_{,\mathcal{A}}(\pi) \supset F^1 C^s_{,\mathcal{A}}(\pi) \supset \cdots \supset F^{M_0} C^s_{,\mathcal{A}}(\pi) = 0,$$

where M_0 is as in Theorem 5.1.10. The same holds for the second.

The first page of the spectral sequence associated to each, $E_1^{*,*}$, is obtained by computing the homology of each row of the complex. We will show that the chain map $Q_{\mathcal{A}}$ induces an isomorphism between the resulting homologies. The conclusion of Theorem 5.3.1 then follows from standard results on spectral sequences, for example Theorem 3.5 of [25].

Restricting our attention to row M of the two complexes, they are subcomplexes of $(D^s(\Sigma_{*,M}(\rho_{,M})), d^s(\rho_{,M}))$ and $(D^s_{\mathcal{Q},}(\Sigma_{*,M}(\rho_{,M})), d^s_{\mathcal{Q},}(\rho_{,M}))$, respectively. We know from part one of Theorem 4.3.1 that the quotient map Q induces an isomorphism on the homology of these complexes. It is true that Q is also equivariant for the action of $1 \times S_{M+1}$ and so passes to a map between the subcomplexes, but it does not follow at once that it is also an isomorphism on homology. Instead, we must adapt the proof given in Theorem 4.3.1. Fortunately, the adjustments are fairly minor.

We want to show that the arguments in the proof of Theorem 4.3.1 showing that, for fixed M, q induces an isomorphism between the homologies of the complexes $(D^s(\Sigma_{*,M}(\rho_{,M})), d^s(\rho_{,M}))$ and $(D^s_{\mathcal{Q},}(\Sigma_{*,M}(\rho_{,M})), d^s_{\mathcal{Q},}(\rho_{,M}))$, may be adapted to reach the same conclusion for the subcomplexes $(D^s_{,\mathcal{A}}(\Sigma_{*,M}(\rho_{,M})), d^s(\rho_{,M}))$ and $(D^s_{\mathcal{Q},\mathcal{A}}(\Sigma_{*,M}(\rho_{,M})), d^s_{\mathcal{Q},}(\rho_{,M}))$. We state for emphasis that M will be fixed throughout.

Recall that the starting point for the proof of Theorem 4.3.1 was to choose an order on the vertex set G^0, which in our case is $G^0_{0,M}$. The first step will be to replace this by $G^{2K_0}_{0,M}$, where K_0 satisfies the condition of Proposition 2.7.7. This has almost no effect, except for changing some t^k to t^{k-2K_0} and considering $G^k_{L,M}$, only for $k \geq 2K_0$.

The second step is to choose an order, instead of on $G^{2K_0}_{0,M}$, on the S_{M+1}-orbits in $G^{2K_0}_{0,M}$. One alternative is then to use this order and replace comparison of vertices with comparison of their orbits. Instead, we will simply extend this order to $G^{2K_0}_{0,M}$ so that the comparison of elements in $G^{2K_0}_{0,M}$ is done first by comparing their orbits and secondly, if they have the same orbit, by some pre-assigned order on each orbit.

With this special choice of an order on $G^{2K_0}_{L,M}$, we follow the proof exactly as it is given for Theorem 4.3.1, with some additional observations. The crucial one is the first: by part 3 of Proposition 2.7.7, if p is in $G^{2K_0}_{L,M}$ and two rows have the same $1 \times S_{M+1}$-orbits, then they are equal. So now, while it is *not* true that our order in $G^{2K_0}_{L,M}$ is $1 \times S_{M+1}$-invariant, it *is* true that if the rows of p in $G^{2K_0}_{L,M}$ are in strictly increasing order, then so are the rows of $p \cdot (1, \beta)$, for any β in S_{M+1}.

This means that both functions $Q^{\#}$ and $J^{\#}$ commute with the action of $1 \times S_{M+1}$. Also, the set $\mathcal{V}_I$ is globally invariant under $1 \times S_{M+1}$. Moreover, if v is in $\mathcal{V}_I$ and β in S_{M+1}, we have $\tilde{G}^k_{L+1,M}(v \cdot \beta) = \tilde{G}^k_{L+1,M}(v) \cdot (1, \beta)$ and $G^k_{L,M}(v \cdot \beta) = G^k_{L,M}(v) \cdot (1, \beta)$. For p in $G^k_{L,M}(v)$, we have $h_{v \cdot \beta}(p \cdot (1, \beta)) = h_v(p) \cdot (1, \beta)$. Finally, it follows from all of this that the function H_N is equivariant for the action of $1 \times G_{M+1}$. Therefore it maps $E_N \cap \oplus \mathcal{A}(G^k_{L,M}, 1 \times S_{M+1})$ to $E_N \cap \oplus \mathcal{A}(G^k_{L,M}, 1 \times S_{M+1})$. The remainder of the proof is exactly the same as in Chapter 4. This completes the proof of Theorem 5.3.1.

We now turn to the proof of Theorem 5.3.2. But we will make use of the results already established in the proof of Theorem 5.3.1. Consider the dynamical system (X, φ^{-1}). It is a trivial matter to see that this is also a Smale space whose local stable sets are the local unstable sets for (X, φ), etc. An s/u-bijective pair for it is $\bar{\pi} = (Z, \zeta^{-1}, \pi_u, Y, \psi^{-1}, \pi_s)$. For any $L \geq 0$, it is an easy matter to see that the s-bijective factor map for the $\bar{\pi}$ system, $\bar{\rho}_{,L}$, is equal to $\rho_{L,}$. (Of course, the reason the latter is u-bijective while the former is s-bijective is because their domains are the same space but with inverse maps.) The systems $\Sigma_{L,M}(\bar{\pi})$ coincide with $\Sigma_{M,L}(\pi)$, for all L, M, except that the inverse maps are used. Consider the complexes $(D^u(\Sigma_{*,L}(\bar{\rho}_{,L})), d^{u*}(\bar{\rho}_{,L}))$ and its subcomplex $(D^u_{\mathcal{A}}(\Sigma_{*,L}(\bar{\rho}_{,L})), d^{u*}_{\mathcal{A}}(\bar{\rho}_{,L}))$. By part 2 of Theorem 4.3.1, the inclusion map J from the latter to the former induces an isomorphism on homology. At this point, we must realize that we have made a simplification in out notation which is slightly misleading. We should, in fact, be considering the groups $D^u(\Sigma_{*,L}(\bar{\rho}_{,L}), \sigma^{-1})$. It is obvious from the definitions that

$$D^u(\Sigma_{*,L}(\bar{\rho}_{,L}), \sigma^{-1}) = D^s(\Sigma_{*,L}(\bar{\rho}_{,L}), \sigma) = D^s(\Sigma_{L,*}(\rho_{L,}), \sigma)$$

and that under this identification, the chain map $d^{u*}(\bar{\rho}_{,L}))$ equals $d^{s*}(\rho_{L,}))$.

Now we return to the proof we have given above for 5.3.1, as it applies to the s/u-bijective pair $\bar{\pi}$. We have noted that the map $J^{\#}$ is also equivariant for the action of S_{M+1} and hence the proof that J is an isomorphism is equally valid for the quotient complexes $D^u_{\mathcal{Q},}(\Sigma_{*,L}(\bar{\rho}_{,L}))$ and its subcomplex $D^u_{\mathcal{Q},\mathcal{A}}(\Sigma_{*,L}(\bar{\rho}_{,L}))$. This completes the proof.

5.4 Functorial properties

Our aim in this section is to show that our homology theories have exactly the same functorial properties as Krieger's original dimension groups.

Theorem 5.4.1. *Let* $\pi = (Y, \psi, \pi_s, Z, \zeta, \pi_u)$ *and* $\pi' = (Y', \psi', \pi'_s, Z', \zeta', \pi'_u)$ *be s/u-bijective pairs for the Smale spaces* (X, φ) *and* (X', φ')*, respectively. Let* $\eta = (\eta_X, \eta_Y, \eta_Z)$ *be a triple of factor maps*

$$\begin{aligned} \eta_X : (X, \varphi) &\to (X', \varphi') \\ \eta_Y : (Y, \psi) &\to (Y', \psi') \\ \eta_Z : (Z, \zeta) &\to (Z', \zeta') \end{aligned}$$

such that the diagrams

$$\begin{array}{ccc} (Y,\psi) & \xrightarrow{\pi_s} & (X,\varphi) \\ \downarrow{\scriptstyle \eta_Y} & & \downarrow{\scriptstyle \eta_X} \\ (Y',\psi') & \xrightarrow{\pi'_s} & (X',\varphi') \end{array}$$

and

$$\begin{array}{ccc} (Z,\zeta) & \xrightarrow{\pi_u} & (X,\varphi) \\ \downarrow{\scriptstyle \eta_Z} & & \downarrow{\scriptstyle \eta_X} \\ (Z',\zeta') & \xrightarrow{\pi'_u} & (X',\varphi') \end{array}$$

are both commutative and satisfy the conditions that $\eta_Y \times \pi_s$ and $\eta_Z \times \pi_u$ are surjective onto the respective fibred products.

1. *If η_X, η_Y and η_Z are s-bijective, then they induce chain maps and between the complexes $C^s_{\mathcal{Q},\mathcal{A}}(\pi)$ and $C^s_{\mathcal{Q},\mathcal{A}}(\pi')$ and hence group homomorphisms*
$$\eta^s : H^s_N(\pi) \to H^s_N(\pi'),$$
for every integer N.

2. *If η_X, η_Y and η_Z are u-bijective, then they induce chain maps and between the complexes $C^s_{\mathcal{Q},\mathcal{A}}(\pi')$ and $C^s_{\mathcal{Q},\mathcal{A}}(\pi)$ and hence group homomorphisms*
$$\eta^{s*} : H^s_N(\pi') \to H^s_N(\pi),$$
for every integer N.

These construction are functorial in the sense that if η_1 and η_2 are both triples of s-bijective factor maps and the ranges of θ_1 are the domains of θ_2, then

$$(\theta_2 \circ \theta_1)^s = \theta_2^s \circ \theta_1^s.$$

An analogous statement holds for the composition of u-bijective factor maps.

Proof. For each $L, M \geq 0$, we define $\eta : \Sigma_{L,M}(\pi) \to \Sigma_{L,M}(\pi')$ by setting

$$\eta(y_0, \ldots, y_L, z_0, \ldots, z_M) = (\eta_Y(y_0), \ldots, \eta_Y(y_L), \eta_Z(z_0), \ldots, \eta_Z(z_M)).$$

The fact that η is surjective is proved in much the same fashion is in the proof of Theorem 4.4.1. It is clear that these maps are s-bijective if η_Y and η_Z are and that they commute with the action of $S_{L+1} \times S_{M+1}$. The rest of the proof is routine and we omit the details. □

Theorem 5.4.2. *Let* (X, φ) *and* (X', φ') *be non-wandering Smale spaces and suppose* $\eta : (X, \varphi) \to (X', \varphi')$ *is a factor map.*

1. *If* η *is* s*-bijective, then there exist* s/u*-bijective pairs,* π *and* π'*, for* (X, φ) *and* (X', φ')*, respectively, and a triple of* s*-bijective factor maps* $\eta = (\eta, \eta_Y, \eta_Z)$ *which satisfy the hypotheses of Theorem 5.4.1.*

2. *If* η *is* u*-bijective, then there exist* s/u*-bijective pairs,* π *and* π'*, for* (X, φ) *and* (X', φ')*, respectively, and a triple of* u*-bijective factor maps* $\eta = (\eta, \eta_Y, \eta_Z)$ *which satisfy the hypotheses of Theorem 5.4.1.*

Proof. We prove the first part only. By Theorem 2.6.3, let Y, ψ, π_s be the first half of an s/u-bijective pair for (X, φ) and let Z', ζ', π'_u be the second half of an s/u-bijective pair for (X', φ'). Let (Z, ζ) be the fibred product of the maps $\eta : (X, \varphi) \to (X', \varphi')$ and $\pi'_u : (Z', \zeta') \to (X', \varphi')$. We denote by the two canonical factor maps by $\pi_u : (Z, \zeta) \to (X, \varphi)$ and $\eta_Z : (Z, \zeta) \to (Z', \zeta')$. By Theorem 2.5.13, the former is u-bijective since π'_u is, while the latter is s-bijective since θ is. Moreover, since the sets $Z'^s(z')$ are totally disconnected, for every z' in Z' and $\eta_Z : (Z, \zeta) \to (Z', \zeta')$ is s-bijective, it follows from Theorem 2.5.12 that $Z^s(z)$ is totally disconnected, for every z in Z. It then follows that $\pi = (Y, \psi, \pi_s, Z, \zeta, \pi_u)$ is an s/u-bijective pair for (X, φ).

On the other hand, since η is s-bijective, $\pi' = (Y, \psi, \eta \circ \pi_s, Z', \zeta', \pi'_u)$ is an s/u-bijective pair for (X', φ').

The proofs that these satisfy the hypotheses of Theorem 5.4.1 are immediate. □

5.5 Independence of s/u-bijective pair

In this section, we will prove that, for a given Smale space (X, φ) with an s/u-bijective pair π, the homology groups $H^s_N(\pi)$ and $H^u_N(\pi)$ are independent of the choice of π, and depend only on (X, φ). We then use this result and the results of the last section to deduce a final result on functoriality.

Theorem 5.5.1. *Let* (X, φ) *be a Smale space and suppose that* $\pi = (Y, \psi, \pi_s, Z, \zeta, \pi_u)$ *and* $\pi' = (Y', \psi', \pi'_s, Z', \zeta', \pi'_u)$ *are two* s/u*-bijective pairs for* (X, φ)*. Then there are canonical isomorphisms between* $H^s_N(\pi)$ *and* $H^s_N(\pi')$ *and between* $H^u_N(\pi)$ *and* $H^u_N(\pi')$*, for all integers* N*.*

Proof. It suffices to prove the result in the case that $Z' = Z, \zeta' = \zeta$ and $\pi'_u = \pi_u$, for a similar argument will deal with the case $Y = Y', \psi = \psi'$ and $\pi'_s = \pi_s$ and the result follows from these two special cases.

Assuming that $Z' = Z, \zeta' = \zeta$ and $\pi'_u = \pi_u$, let (Y'', ψ'') be the fibred product of $\pi_s : (Y, \psi) \to (X, \varphi)$ and $\pi'_s : (Y', \psi') \to (X, \varphi)$. Let $\eta : (Y'', \psi'') \to (Y, \psi)$ denote the usual map and let $\pi'' = (Y'', \psi'', \pi_s \circ \eta, Z, \zeta, \pi_u)$. We will show that η implements an isomorphism at the level of homology. This will complete the proof since the same argument also shows that $\eta' : (Y'', \psi'') \to (Y', \psi')$ (denoting the usual map) also induces an isomorphism on homology.

First, note that η is s-bijective by Theorem 2.5.13. Define $\theta : \Sigma(\pi'') \to \Sigma(\pi)$ by $\theta(y'', z) = (\eta(y''), z)$, for all y'' in Y and z in Z with $\pi''_s(y'') = \pi_u(z)$. It is a simple matter to see that θ is s-bijective. Also observe that $\rho''_s : \Sigma(\pi'') \to Z$ may be written as $\rho_s \circ \theta$.

We may find a graph G such that $(\Sigma(\pi), \sigma)$ is conjugate to (Σ_G, σ) and so that ρ_s and ρ_u are regular. Then we may find a graph H and left-covering graph homomorphism from H to G so that $(\Sigma(\pi''), \sigma)$ is conjugate to (Σ_H, σ) and, under these identifications, the map induced by the graph homomorphism agrees with θ. For convenience, we denote the graph homomorphism by θ also.

For each $M \geq 0$, θ also defines a graph homomorphism from $H_{0,M}$ to $G_{0,M}$. We claim that this is also left-covering. Fix a vertex in $H_{0,M}$. This consists of an $M+1$-tuple of vertices in H:

$$v = (v_0, \ldots, v_M) = (t(e^0(y'', z''_0)), \ldots, t(e^0(y'', z''_M)),$$

where $(y'', z''_0, \ldots, z''_M)$ is in $\Sigma_{0,M}(\pi'')$. The graph homomorphism θ is applied entry-wise and since it is injective on each set $t^{-1}\{v_m\}, 0 \leq m \leq M$, it is injective on $t^{-1}\{v\}$.

We now turn to the issue of surjectivity. With the vertex v in $H_{0,M}$ as above, consider an edge in $G_{0,M}$ whose terminus is $\theta(v)$. Such an edge is an $M+1$-tuple $(e^0(y, z_0), \ldots, e^0(y, z_M)$, where $(y, z_0, \ldots, z_M)$ is in $\Sigma_{0,M}(\pi)$, with $t(e^0(y, z_m)) = t(e^0(y'', z''_m))$, for all m. Since θ from H to G is left-covering, for each m we may find a point (y'_m, z'_m) in $\Sigma(\pi'')$ such that $\theta(e^k(y'_m, z'_m)) = e^k(y, z'_m)$ for all $k \leq 0$ and $t(e^0(y'_m, z'_m)) = t(e^0(y'', z''_m))$. The second condition and the regularity of θ imply that we may form $[(y'', z''_m), (y'_m, z'_m)] = ([y'', y'_m], [z''_m, z'_m)])$. Moreover, we have

$$\theta(e^k(([y'', y'_m], [z''_m, z'_m)])) = \theta(e^k((y'_m, z'_m))) = e^k(y, z'_m),$$

for $k \leq 0$. On the other hand, we have

$$\theta(e^k(([y'', y'_m], [z''_m, z'_m]))) = \theta(e^k((y'', z''_m))) = e^k(\theta(y''), z''_m),$$

for $k \geq 1$. It follows that

$$\begin{aligned} e^k(\theta[y'', y'_m], [z''_m, z'_m]) &= \theta(e^k(([y'', y'_m], [z''_m, z'_m]))) \\ &= e^k([(\theta(y''), z''_m), (y, z_m)]) \\ &= e^k([\theta(y''), y], [z''_m, z_m]), \end{aligned}$$

for all k. We conclude that $\theta[y'', y'_m] = [\theta(y''), y]$. For the various different values of m, the points $[y'', y'_m]$ are all stably equivalent to y'' and they have the same image under θ. Since θ is s-bijective, we conclude they are all equal.

From this it follows that the point $([y'', y'_0], [z''_0, z'_0], \ldots, [z''_M, z'_M])$ is in $\Sigma_{0,M}(\pi'')$. Moreover, for each m, we have

$$\begin{aligned} \theta(e^0([y'', y'_0], [z''_m, z'_m])) &= \theta(e^0([y'', y'_m], [z''_m, z'_m])) \\ &= \theta(e^0(y'_m, z'_m)) \\ &= e^0(y, z_m). \end{aligned}$$

Also, we have

$$\begin{aligned} t(e^0([y'', y'_0], [z''_m, z'_m])) &= i(e^1([y'', y'_0], [z''_m, z'_m])) \\ &= i(e^1(y'', z''_m)) \\ &= t(e^0(y'', z''_m)) \end{aligned}$$

as desired. This completes the proof that θ mapping $H_{0,M}$ to $G_{0,M}$ is left-covering.

Now, we may apply Theorem 4.5.1 to each $\theta : H_{0,M} \to G_{0,M}$ and $\rho_{,M} : \Sigma_{0,M}(\pi) \to (Z_M(\pi_u), \zeta)$. The conclusion is that, for each fixed M, the map θ will induce an isomorphism between the homology of the complex $(D^s(\Sigma_{*,M}(\rho_{,M} \circ \theta)), d^s(\rho_{,M} \circ \theta))$ that of $(D^s(\Sigma_{*,M}(\rho_{,M})), d^s(\rho_{,M}))$.

On the other hand, we know from part 1 of Theorem 4.3.1, that these complexes have the same homologies as $(D^s_{\mathcal{Q}}(\Sigma_{*,M}(\rho_{,M} \circ \theta)), d^s_{\mathcal{Q}}(\rho_{,M} \circ \theta))$ and $(D^s_{\mathcal{Q}}(\Sigma_{*,M}(\rho_{,M})), d^s_{\mathcal{Q}}(\rho_{,M}))$, respectively.

Let us return to consider our double complexes $C^s(\pi'')$ and $C^s(\pi)$. For each $L, M \geq 0$, the map

$$\theta(y_0, \ldots, y_L, z_0, \ldots, z_M) = (\eta(y_0), \ldots, \eta(y_L), z_0, \ldots, z_M)$$

is a factor map from $\Sigma_{L,M}(\pi'')$ to $\Sigma_{L,M}(\pi)$. It is easy to see that it is s-bijective and, after identifying the systems with their presentations by G, is induced by the graph homomorphism θ.

We use Theorem 5.3.2 to observe that the homologies of the complexes $C^s_{\mathcal{Q},\mathcal{A}}(\pi'')$ and $C^s_{\mathcal{Q},\mathcal{A}}(\pi)$ are isomorphic to those of $C^s_{\mathcal{Q},}(\pi'')$ and $C^s_{\mathcal{Q},}(\pi)$, so to establish our result, it suffices to show the map induced by θ on the latter pair induces an isomorphism on homology.

We consider the filtration

$$F^p C^s_{\mathcal{Q},}(\pi'') = \oplus_{L\geq 0, M\geq p} C^s_{\mathcal{Q},}(\pi'')_{L,M}$$

and the obvious analogue for π. These filtrations are decreasing and, while not bounded, they are exhaustive and since $\cap_{p\geq 0} F^p C^s_{\mathcal{Q},}(\pi'') = 0$, they are weakly convergent [25]. Finally, we claim that they are complete [25]. As noted in Theorem 5.1.10, the complex $C^s_{\mathcal{Q},}(\pi'')$ has non-zero entries only for $L < L_0$ and it follows that the entries of $F^p C^s_{\mathcal{Q},}(\pi'')$ are non-zero only for $L < L_0$ and $M \geq p$. (Here, L_0, M_0 are the constants for the maps $\pi_s \circ \eta$ and π_u.) But this means the homology of this double complex is non-zero only in dimensions $N = L - M < L_0 - p$. Combining Theorems 5.3.2 and 5.1.12, we see that the homology of $C^s_{\mathcal{Q},}(\pi'')$ is zero in dimensions $N < -M_0$. It follows that under the natural inclusion, the image of the homology of $F^p C^s_{\mathcal{Q},}(\pi'')$ in that of $C^s_{\mathcal{Q},}(\pi'')$ is zero, if $p > L_0 + M_0$. From this it follows at once that the filtration is complete. The same argument applies to the filtration of $C^s_{\mathcal{Q},}(\pi)$.

To compute the $E_1^{*,*}$ term for each of these filtrations is done by computing the homology of the rows of the complexes and we have already established above that we get the same answer for each. We may now apply Theorem 3.9 of [25] to complete the proof. □

With this result established, we set out the following notation.

Definition 5.5.2. *Let (X, φ) be a Smale space which has an s/u-bijective pair π. We define $H^s_N(X, \varphi) = H^s_N(\pi)$ and $H^u_N(X, \varphi) = H^u_N(\pi)$, for all integers N.*

The final result of this section is a re-statement of the functorial properties of Theorem 5.4.1 for the invariants $H^s_*(X, \varphi)$ and $H^u_*(X, \varphi)$.

Theorem 5.5.3. *1. The functor which associates the sequence of abelian groups $H^s_*(X, \varphi)$ to a (non-wandering) Smale space (X, φ) is covariant for s-bijective factor maps and contravariant for u-bijective factor maps.*

2. *The functor which associates the sequence of abelian groups $H^u_*(X,\varphi)$ to a (non-wandering) Smale space (X,φ) is contravariant for s-bijective factor maps and covariant for u-bijective factor maps.*

Proof. The existence of the desired induced maps on homology follows from Theorem 5.4.2. The main issue we must address is showing that the map induced by a composition is the same as the composition of the induced maps. Let $(X,\varphi),(X',\varphi')$ and (X'',φ'') be non-wandering Smale spaces. Let (Y,ψ,π_s) be the first half of an s/u-bijective pair for (X,φ) and let (Z'',ζ'',π''_u) be the second half of an s/u-bijective pair for (X'',φ''). Let (Z',ζ') be the fibred product of the maps $\eta'_{X'}$ and π''_u. Also, let π'_u be the canonical factor map from (Z',η') to (X',φ') and let $\eta'_{Z'}$ be the canonical factor map from (Z',η') to (Z'',ζ''). Repeat this procedure: let (Z,ζ) be the fibred product of the maps η_X and π'_u and define π_u and η_Z analogously. It follows that $\pi=(Y,\psi,\pi_s,Z,\zeta,\pi_u)$ is an s/u-bijective pair for (X,φ), $\pi'=(Y,\psi,\eta_X\circ\pi_s,Z',\zeta',\pi'_u)$ is an s/u-bijective pair for (X',φ') and $\pi''=(Y,\psi,\eta'_{X'}\circ\eta_X\circ\pi_s,Z'',\zeta'',\pi''_u)$ is an s/u-bijective pair for (X'',φ''). Finally, we define $\eta_Y,\eta'_{Y'}$ to be the identity maps. With these choices for s/u-bijective pairs, $\eta''=(\eta'_{X'}\circ\eta_X,\eta'_{Y'}\circ\eta_Y,\eta'_{Z'}\circ\eta_Z)$ and Theorem 5.5.1, the conclusion of the Theorem is immediate. □

Chapter 6

A Lefschetz formula

The goal of this chapter is to prove the following version of the Lefschetz formula for our homology theory. We present the statement in the first section and the remaining two sections are devoted to its proof; each deals with one side of the formula.

6.1 The statement

Let (X, φ) be a Smale space, which we assume has an s/u-bijective pair π. We may regard φ and φ^{-1} as s-bijective or u-bijective factor maps from (X, φ) to itself, then we consider φ^s and φ^{s*} to be the maps induced on $H^s_*(X, \varphi)$ and φ^u and φ^{u*} to be the maps induced on $H^u_*(X, \varphi)$. We denote by $\varphi^s_N \otimes 1_{\mathbb{Q}}$ and $\varphi^{s*} \otimes 1_{\mathbb{Q}}$ the associated maps on the rational vector space $H^s_N(X, \varphi) \otimes \mathbb{Q}$. We recall from Theorem 5.1.12 that this space is finite dimensional. There is analogous notation for the maps on $H^u_N(X, \varphi) \otimes \mathbb{Q}$.

Theorem 6.1.1. *Let (X, φ) be a Smale space which has an s/u-bijective* *pair. Then, for every $n \geq 1$, we have*

$$\begin{aligned}\#\{x \in X \mid \varphi^n(x) = x\} &= \sum_{N\in\mathbb{Z}} (-1)^N Tr(((\varphi^{-1})^s_N \otimes 1_{\mathbb{Q}})^n) \\ &= \sum_{N\in\mathbb{Z}} (-1)^N Tr((\varphi^u_N \otimes 1_{\mathbb{Q}})^n) \\ &= \sum_{N\in\mathbb{Z}} (-1)^N Tr((\varphi^{s*}_N \otimes 1_{\mathbb{Q}})^n) \\ &= \sum_{N\in\mathbb{Z}} (-1)^N Tr(((\varphi^{-1})^{u*}_N \otimes 1_{\mathbb{Q}})^n).\end{aligned}$$

In view of Theorem 5.5.1, we fix an s/u-bijective pair π for (X, φ), which we use to compute our homology theory. In fact, we will also compute the periodic point data for (X, φ) in terms of that of the systems $(\Sigma_{L,M}(\pi), \sigma)$. The computations involving the periodic point data are based on ideas of Anthony Manning and we will be done in the next section. The following section will deal with the homological data. The proof of the theorem follows from the fact that these two sections finally arrive at the same expression.

Before proceding to the proof, we note the following consequence. Recall that for a dymanical system (X, φ), its zeta function is

$$\zeta_\varphi(t) = \exp(\sum_{n=1}^{\infty} \frac{\#\{x \in X \mid \varphi^n(x) = x\}}{n} t^n)$$

at least formally, provided $\#\{x \in X \mid \varphi^n(x) = x\}$ is finite, for all $n \geq 1$. If (X, φ) is a Smale space, this series converges for small values of t an defines a rational function of t [23]. Our next result shows that our homology theory provides a *canonical* decomposition of the zeta function as a product of polynomials and their inverses.

Since our power series will actually converge uniformly and we are using a real variable t, it seems most convenient to use the real numbers rather than the rationals as our scalars for the vector space version of the homology. Theorem 6.1.1 remains true replacing $\mathbb{Q}$ by $\mathbb{R}$.

Corollary 6.1.2. *Let (X, φ) be an irreducible Smale space. For each integer N and real number t, define $p_N(t)$ to be the determinant of the linear transformation*

$$I - t(\varphi^{-1})^s_N \otimes 1_{\mathbb{R}} : H^s_N(X, \varphi) \otimes \mathbb{R} \to H^s_N(X, \varphi) \otimes \mathbb{R}.$$

Each of the vector spaces is finite dimensional and all but finitely many are trival so that $p_N(t)$ is well-defined and all but finitely many are identically one. Then we have

$$\begin{aligned}\zeta_\varphi(t) &= \Pi_{N\in\mathbb{Z}} p_N(t)^{(-1)^{N+1}} \\ &= \frac{\Pi_{N\ odd}\, p_N(t)}{\Pi_{N\ even}\, p_N(t)}.\end{aligned}$$

Proof. We give a proof under the assumption that Theorem 6.1.1 holds. It is fairly standard that if A is a square matrix with real entries, then

$$\exp\left(\sum_{n=1}^{\infty} \frac{Tr(A^n)}{n} t^n\right) = \det(I - tA)^{-1}.$$

(See 6.4.6 of [23].) In our case, let A_N be a matrix representing the linear transformation $(\varphi^{-1})^s_N \otimes 1_{\mathbb{R}}$. It follows from Theorem 6.1.1 that

$$\begin{aligned}\zeta_\varphi(t) &= \exp\left(\sum_{n=1}^{\infty} \frac{\sum_N (-1)^N Tr(A_N^n)}{n} t^n\right) \\ &= \exp\left(\sum_N (-1)^N \sum_{n=1}^{\infty} \frac{Tr(A_N^n)}{n} t^n\right) \\ &= \Pi_N \left(\exp\left(\sum_N (-1)^N \sum_{n=1}^{\infty} \frac{Tr(A_N^n)}{n} t^n\right)\right)^{(-1)^N} \\ &= \Pi_N \det(I - tA_N)^{(-1)^{N+1}}.\end{aligned}$$

□

6.2 The periodic point side

The heart of the proof (here presented in Theorem 6.2.1) is an argument due to A. Manning [24]. We will need some new notation for it. Suppose that $G = S_{N_1} \times S_{N_2} \times \cdots \times S_{N_I}$ is the product of permutation groups. We denote the identity element by 1. We extend the definition of the sign of a permutation to elements of G by setting

$$sgn(\alpha_1, \cdots, \alpha_I) = sgn(\alpha_1) \cdot sgn(\alpha_2) \cdots sgn(\alpha_I),$$

for any $(\alpha_1, \cdots, \alpha_I)$ in G.

If X is a set with an action of G (written on the right), we let X^1 denote the set of of those points of X with trivial isotropy; that is, x is in X^1 if the only element α of G with $x \cdot \alpha = x$ is the identity.

Suppose that X is a set with an action of G and φ is a bijection of X which commutes with the G action. For any integer $n \geq 1$, we define

$$Per^G(X, \varphi, n) = \{x \in X^1 \mid \varphi^n(x) = x \cdot \alpha, \text{ for some } \alpha \in G\}.$$

Notice that if x in $Per^G(X, \varphi, n)$, it is also in X^1 and hence, the α in the condition given is unique (for a fixed n). We define $sgn(x, n)$ to be $sgn(\alpha)$, where α in G is chosen so that $\varphi^n(x) = x \cdot \alpha$. We also notice that $Per^G(X, \varphi, n)$ is invariant under G and since

$$\varphi^n(x \cdot \beta) = \varphi^n(x) \cdot \beta = (x \cdot \alpha) \cdot \beta = (x \cdot \beta) \cdot (\beta^{-1}\alpha\beta),$$

the function $sgn(x, n)$ is constant on G orbits. Therefore, whenever $Per^G(X, \varphi, n)$ is finite, we define

$$per^G(X, \varphi, n) = \sum_{x \in Per^G(X, \varphi, n)/G} sgn(x, n).$$

If G is the trivial group with one element, we suppress the G in our notation and write

$$Per(X, \varphi, n) = \{x \in X \mid \varphi^n(x) = x\}$$

and

$$per(X, \varphi, n) = \#Per(X, \varphi, n),$$

provided it is finite.

Suppose that (Y, ψ) and (X, φ) each have actions of G, as above, and that

$$\pi : (Y, \psi) \to (X, \varphi)$$

is a finite-to-one factor map equivariant for the actions of G. That is, for all y in Y and α in G, we have $\pi(y \cdot \alpha) = \pi(y) \cdot \alpha$.

Theorem 6.2.1. *Let (Y, ψ) and (X, φ) have actions of G and let*

$$\pi : (Y, \psi) \to (X, \varphi)$$

be a finite-to-one factor map equivariant for the actions of G. If, for any $p \geq 1$, $Per^G(X, \varphi, n)$ is finite, then $Per^{G\times S_{N+1}}(Y_N(\pi), \psi, n)$ is also finite for all $N \geq 0$ and is empty for all but finitely many N. Moreover, we have

$$per^G(X, \varphi, n) = \sum_{N\geq 0} (-1)^N per^{G\times S_{N+1}}(Y_N(\pi), \psi, n).$$

Proof. First of all, we note that $Per^G(X, \varphi, n) = Per^G(X, \varphi^n, 1)$ and $per^G(X, \varphi, n) = per^G(X, \varphi^n, 1)$. It follows from this (and analogous statements for the systems $Y_N(\pi)$) that it suffices to prove the result for $n = 1$.

Next, it is clear that π maps $Per^{G\times S_{N+1}}(Y_N(\pi), \psi, 1)$ to $Per^G(X, \varphi, 1)$. Since the latter set is assumed to be finite and the map is finite-to-one, the former set must be finite also. In addition, as π is finite-to-one, there is a $N_0 \geq 1$ such that $\pi^{-1}\{x\}$ contains at most N_0 points, for any x in X. It follows then that for any $N \geq N_0$, any element of $Y_N(\pi)$ must contain a repeated entry. Hence, we see that $Y_N(\pi)^1$ is empty and hence so is $Per^{G\times S_{N+1}}(Y_N(\pi), \psi, 1)$, for all such N.

So if we fix x in $Per^G(X, \varphi, 1)$, we can consider $Per^{G\times S_{N+1}}(Y_N(\pi), \psi, 1) \cap \pi^{-1}\{x \cdot G\}$, which we denote by $Y_N(\pi, x)$, for convenience. We claim that

$$sgn(x, 1) = \sum_{N\geq 0} (-1)^N \sum_{y\in Y_N(\pi,x)/G\times S_{N+1}} sgn(y, 1)$$

Once this claim is established, summing both sides over x in $Per^G(X, \varphi, 1)$, we obtain the desired conclusion.

Let α be the unique element of G such that $\varphi(x) = x \cdot \alpha$. Next, using the fact that π is finite-to-one, write $\pi^{-1}\{x\} = \{y_0, y_1, \ldots, y_L\}$. Notice that, since we write this as a set, we assume the elements are all distinct. For each $0 \leq l \leq L$, we have

$$\pi(\psi(y_l \cdot \alpha^{-1})) = \varphi(\pi(y_l) \cdot \alpha^{-1}) = \varphi(x \cdot \alpha^{-1}) = x.$$

and it follows that there is a permutation σ in S_{L+1} such that

$$\psi(y_0, \ldots, y_L) = (y_0, \ldots, y_L) \cdot (\alpha, \sigma).$$

Moreover, as the y_l are all distinct, σ is unique. Write σ as the product of disjoint cycles;

$$\sigma = \sigma_1\sigma_2 \cdots \sigma_I.$$

(We include elements which are fixed as cycles of length one.) Let L_i be the length of cycle σ_i, $1 \leq i \leq I$. We have $\sum_i L_i = L + 1$ and also $sgn(\sigma_i) = (-1)^{L_i - 1}$. As the order on the y_l has been arbitrary up to this point, we may assume now that they are ordered so that we have

$$\begin{aligned}
\sigma_1 &= (0\,1\,\cdots\,L_1 - 1), \\
\sigma_2 &= (L_1\,L_1 + 1\,\cdots\,L_1 + L_2 - 1), \\
\sigma_i &= (L_1 + L_2 + \cdots L_{i-1}\,\cdots\,L_1 + L_2 + \cdots L_i - 1).
\end{aligned}$$

Now suppose that A is any non-empty subset of $\{1, 2, \ldots, I\}$. We define a point y_A as follows. First of all, let $N_A = \sum_{i \in A} L_i - 1$ and y_A will be in $Y_{N_A}(\pi)$. It consists of those y_l where l is part of the cycle σ_i, for some i in A. More precisely, for $L_1 + \cdots + L_{i-1} \leq l \leq L_1 + \cdots + L_{i-1} + L_i - 1$, for some i in A, we include y_l in Y_A. These are written in increasing order.

We observe the following properties of the points y_A. First, we have

$$\psi(y_A) = y_A \cdot (\alpha, \prod_{i \in A} \sigma_i).$$

In fact, the notation $\prod_{i \in A} \sigma_i$ is not good, as this is a permutation of $\{0, 1, \ldots, L\}$ and we have deleted a number of the entries of y in forming y_A. In fact, $\prod_{i \in A} \sigma_i$ is really meant to represent a permutation having exactly the same cyclic structure as $\prod_{i \in A} \sigma_i$, but with the fix-points removed. It follows that

$$\begin{aligned}
(-1)^{N_A} sgn(\alpha, \prod_{i \in A} \sigma_i) &= (-1)^{N_A} sgn(\alpha) \prod_{i \in A} (-1)^{L_i - 1} \\
&= (-1)^{N_A} sgn(\alpha) (-1)^{\sum_{i \in A} L_i} (-1)^{\# A} \\
&= (-1)^{N_A} sgn(\alpha) (-1)^{N_A + 1} (-1)^{\# A} \\
&= -sgn(\alpha)(-1)^{\# A}.
\end{aligned}$$

Next, we claim that y_A is in $Y_N(\pi)^1$; that is, the only element of $G \times S_{N_A+1}$ which fixes it is the identity. Suppose that $y_A \cdot (\alpha', \sigma') = y_A$, for some (α', σ') in $G \times S_{N_A+1}$. Applying π, we see that $x \cdot \alpha' = x$ and it follows from the fact that x is in X^1 that α' is the identity. It follows from the construction of y_A that all of its entries are distinct. This then implies σ' is also the identity.

We observe that the elements y_A are distinct for different sets A. This is because the y_l are all distinct. But more is true. No two lie in the same orbit of $G \times S_{N+1}$, for any N. Finally, we claim that if $N \geq 0$ and y' is any element of $Y_N(\pi, x)$, then y' is in the $G \times S_{N+1}$ orbit of some y_A. Suppose $\pi(y') = x \cdot \beta$, for some β in G. Then we have $\pi(y' \cdot \beta^{-1}) = x$ and so the entries of y' are of the form $y_l \cdot \beta$. Secondly, no $y_l \cdot \beta$ may be repeated. We know that $\psi(y') = y' \cdot (\alpha', \sigma')$, for some σ' in G and σ' in S_{N+1}. Applying π and arguing as before, we see that $\alpha' = \alpha$. Next, we note that if $y'_i = y_l \cdot \beta$, then $y'_{\sigma'(i)} = y_{\sigma(l)} \cdot \beta$. This means that the set of l such that $y_l \cdot \beta$ appears in y' is invariant under σ. After re-ordering the entries of y', we see that it is of the form $y_A \cdot \beta$, for some A, as desired. This completes the proof of our claim.

We may now compute

$$\begin{aligned}
\sum_N \sum_{y \in Y_N(\pi,x)/G \times S_{N+1}} (-1)^N sgn(y,1) &= \sum_{\emptyset \neq A \subset \{1,\ldots,I\}} (-1)^{N_A} sgn(y_A, 1) \\
&= \sum_{\emptyset \neq A \subset \{1,\ldots,I\}} (-1)^{N_A} sgn(\alpha, \prod_{i \in A} \sigma_i) \\
&= \sum_{\emptyset \neq A \subset \{1,\ldots,I\}} -sgn(\alpha)(-1)^{\#A} \\
&= sgn(x,1)(1 - \sum_{A \subset \{1,\ldots,I\}} (-1)^{\#A}).
\end{aligned}$$

For a fixed $0 \leq i \leq I$, there are exactly $\begin{pmatrix} I \\ i \end{pmatrix}$ sets A with i elements. This means we have

$$\begin{aligned}
\sum_{A \subset \{1,\ldots,I\}} (-1)^{\#A} &= \sum_{i=0}^{I} \begin{pmatrix} I \\ i \end{pmatrix} (-1)^i \\
&= (1-1)^I \\
&= 0.
\end{aligned}$$

□

We can apply the result above in the situation of an s/u-bijective pair π for a Smale space (X, φ). The proof is immediate from 6.2.1.

Theorem 6.2.2. *Let (X, φ) be a Smale space and suppose that $\pi = (Y, \psi, \pi_s, Z, \zeta, \pi_u)$ is an s/u-bijective pair for (X, φ). Then, for any*

$n \geq 1$ we have

$$\begin{aligned} per(X, \varphi, n) &= \sum_{L \geq 0} (-1)^L per^{S_{L+1}}(Y_L(\pi_s), \psi, n) \\ &= \sum_{M \geq 0} (-1)^M per^{S_{M+1}}(Z_M(\pi_u), \zeta, n) \\ &= \sum_{L,M \geq 0} (-1)^{L+M} per^{S_{L+1} \times S_{M+1}}(\Sigma_{L,M}(\pi), \sigma, n). \end{aligned}$$

We will conclude this section with a final result.

Lemma 6.2.3. *Let π be an s/u-bijective pair for a Smale space (X, φ). For every $L, M \geq 0$, $k \geq 2K_0$ as in Lemma 2.7.6, we suppose that $B^k_{L,M}$ is a subset of $G^k_{L,M}$ which meets each orbit of $S_{L+1} \times S_{M+1}$ with trivial isotropy in exactly one point and no other orbits. For $n \geq 1$, let*

$$A(n) = \{(q, \alpha, \beta) G^{k+n}_{L,M} \times S_{L+1} \times S_{M+1} \mid t^n(q) = i^n(q) \cdot (\alpha, \beta) \in B^k_{L,M}\}.$$

Then we have

$$per^{S_{L+1} \times S_{M+1}}(\Sigma_{L,M}(\pi), \sigma, n) = \sum_{(q,\alpha,\beta) \in A(n)} sgn(\alpha) sgn(\beta).$$

Proof. Define $\tilde{A}(n)$ to be the set of all (q, α, β) in $G^{k+n}_{L,M} \times S_{L+1} \times S_{M+1}$ such that $t^n(q) = i^n(q) \cdot (\alpha, \beta)$ and $t^n(q)$ has trivial isotropy for the action of $S_{L+1} \times S_{M+1}$.

For (q, α, β) in $\tilde{A}(n)$, we define $e(q, \alpha, \beta)$ in $\Sigma_{L,M}(\pi)$ by setting

$$e(q, \alpha, \beta)^{i+jn} = q^i \cdot (\alpha^j, \beta^j),$$

for $1 \leq i \leq n, j \in \mathbb{Z}$.

First, we show that $e(q, \alpha, \beta)$ is in $\Sigma_{L,M}(\pi)$. In the case $i < n$, we have

$$t(e(q, \alpha, \beta)^{i+jn}) = t(q^i) \cdot (\alpha^j, \beta^j) = i(q^{i+1}) \cdot (\alpha^j, \beta^j) = i(e(q, \alpha, \beta)^{i+jn+1}).$$

On the other hand, if $i = n$, then we have $t(e(q, \alpha, \beta)^{i+jn}) = t(q^n) \cdot (\alpha^j, \beta^j)$ while $i(e(q, \alpha, \beta)^{i+jn+1}) = i(q^1) \cdot (\alpha^{j+1}, \beta^{j+1})$. We use the fact that $t^n(q) = i^n(q) \cdot (\alpha, \beta)$; comparing their first entries, we have $q^{n+1} = q^1 \cdot (\alpha, \beta)$. From this it follows that

$$t(q^n) \cdot (\alpha^j, \beta^j) = i(q^{n+1}) \cdot (\alpha^j, \beta^j) = i(q^1) \cdot (\alpha^{j+1}, \beta^{j+1})$$

as desired.

Moreover, it is clear that $\sigma^n(e(q,\alpha,\beta)) = e(q,\alpha,\beta)\cdot(\alpha,\beta)$. Since (q,α,β) is in $\tilde{A}(n)$, there is no non-trivial element of $S_{L+1}\times S_{M+1}$ which fixes $t^n(q)$ and so $e(q,\alpha,\beta)$ is in $(\Sigma_{L,M}(\pi))^1$. Putting all this together, we see that (e,α,β) is in $Per^{S_{L+1}\times S_{M+1}}(\Sigma_{L,M},\sigma,n)$.

We now show the map $e(\cdot)$ is injective. If (q,α,β) and (q',α',β') are in A and $e(q,\alpha,\beta) = e(q',\alpha',\beta')$ then we know that

$$\begin{aligned} e(q,\alpha,\beta)\cdot(\alpha,\beta) &= \sigma^n(e(q,\alpha,\beta)) \\ &= \sigma^n(e(q',\alpha',\beta')) \\ &= e(q',\alpha',\beta')\cdot(\alpha',\beta') \\ &= e(q,\alpha,\beta)\cdot(\alpha',\beta'). \end{aligned}$$

But we have also shown that $e(q,\alpha,\beta)$ is in $(\Sigma_{L,M}(\pi))^1$ and so we conclude that $\alpha = \alpha'$ and $\beta = \beta'$. Then for $1 \le i \le n$, we have

$$q^i = e(q,\alpha,\beta)^i = e(q',\alpha',\beta')^i = (q')^i.$$

The condition $t^n(q) = i^n(q)\cdot(\alpha,\beta)$ means that $q^{i+jn} = q^i\cdot(\alpha^j,\beta^j)$, provided $1 \le i \le n$ and $i + jn \le k + n$ and a similar statement for q'. Hence, we have

$$q^{i+jn} = q^i\cdot(\alpha^j,\beta^j) = (q')^i\cdot(\alpha'^j,\beta'^j) = (q')^{i+jn}.$$

We have shown $(q,\alpha,\beta) = (q',\alpha',\beta')$ as desired.

We now show that $e(\cdot)$ is surjective. Suppose e is in $Per^{S_{L+1}\times S_{M+1}}(\Sigma_{L,M},\sigma,n)$, so that $\sigma^n(e) = e\cdot(\alpha,\beta)$, for some unique (α,β). We claim that $e^{[1,k]}$ has trivial isotropy. If not, then $e^{[1,k]}\cdot(\alpha',\beta') = e^{[1,k]}$, for some $(\alpha',\beta') \neq (1,1)$. We may apply the fourth part of Proposition 2.7.7 to conclude that $e^{[1,k]}\cdot(1,\beta') = e^{[1,k]}$. Then applying the sixth part of the same result i times, we see that $e^{[1,k+i]}\cdot(1,\beta') = e^{[1,k+i]}$, for any $k \ge 1$. But as e is periodic (in the usual sense), we find that $e\cdot(1,\beta') = e$. As e was assumed to be in $Per^{S_{L+1}\times S_{M+1}}(\Sigma_{L,M},\sigma,n)$, it follows that $\beta' = 1$. A similar argument proves that $\alpha' = 1$. This establishes our claim.

Define $q = e^{[1,k+n]}$. It follows immediately that (q,α,β) is in $\tilde{A}(n)$ and $e(q,\alpha,\beta) = e$, so $e(\cdot)$ is surjective.

The map sending (q,α,β) to $e(q,\alpha,\beta)$ is clearly covariant for the action of $S_{L+1}\times S_{M+1}$ and we have established above that it is a bijection from $\tilde{A}$ to $Per^{S_{L+1}\times S_{M+1}}(\Sigma_{L,M},\sigma,n)$. The conclusion now follows from the definition and the fact that $B^k_{L,M}$ meets each $S_{L+1}\times S_{M+1}$ orbit with trivial isotropy exactly once. □

6.3 The homological side

In this section, we turn to the right hand side of Theorem 6.1.1. We carry out the calculations for the first part only, then describe how the other three follow in a similar way.

First, we introduce some notation. If A is any finite set, we let $\mathbb{Q}A$ denote the rational vector space which is formal linear combinations of the elements of A. Obviously, A is a basis for this space. We suppress the obvious identification of $\mathbb{Z}A \otimes \mathbb{Q}$ with $\mathbb{Q}A$. We also use the same choices for B and B_0 as in the last section.

Let us fix $\pi = (Y, \psi, \pi_s, Z, \zeta, \pi_u)$, an s/u-bijective pair for (X, φ). We are considering $\eta_X = \varphi^{-1}$ as a factor map from (X, φ) to itself. This may be lifted to our s/u-bijective pair, as in Theorem 4.4.1, by setting $\eta_Y = \psi^{-1}$ and $\eta_Z = \zeta^{-1}$. The map induced on each $\Sigma_{L,M}(\pi)$ is then simply σ^{-1}.

To compute the homology $H^s_*(X, \varphi)$, we use the complex $C^s_{\mathcal{Q},\mathcal{A}}$. By Theorem 5.1.10, this has the advantage of being non-zero in only finitely many positions. We may therefore apply the Hopf trace formula [12] to conclude that

$$\sum_{N \in \mathbb{Z}} (-1)^N Tr(((\varphi^{-1})^s \otimes 1_{\mathbb{Q}})^n) = \sum_{L,M \geq 0} (-1)^{L+M} Tr(((\sigma^{-1})^s \otimes 1_{\mathbb{Q}})^n),$$

where σ is the usual shift map on $\Sigma_{L,M}(\pi)$. We may identify $D^s_{\mathcal{Q},\mathcal{A}}(\Sigma_{L,M}(\pi), \sigma)$ with $D^s_{\mathcal{Q},\mathcal{A}}(G^K_{L,M})$ by Theorem 5.2.3 Now we may assume that we have a graph G which presents π, that k is sufficiently large and that we have selected a subset $B^k_{L,M}$ of $G^k_{L,M}$ as in Theorem 5.2.9. This then, in turn, allows us to identify $D^s_{\mathcal{Q},\mathcal{A}}(G^K_{L,M})$ with the inductive limit in Theorem 5.2.9. These identifications also conjugate the automorphism $(\sigma^{-1})^s$ with the automorphism $\gamma^s_B[a, j] = [\gamma^s_B(a), j]$, for any a in $\mathbb{Z}B^k_{L,M}$.

The following general fact is probably well-known, but we state it here for completeness and provide a short sketch of the proof.

Proposition 6.3.1. *Let E be a finite-dimensional vector space and let $T : E \to E$ be a linear transformation. Let V be the inductive limit*

$$E \xrightarrow{T} E \xrightarrow{T} \cdots .$$

The automorphism of V sending $[v, k]$ to $[T^n v, k]$ has trace equal to the trace of T^n.

Proof. The sequence of subspaces $T^k E$ is decreasing and so their dimensions form a non-increasing sequence of integers bounded above by the dimension of E and below by zero. Therefore, there exists $K \geq 1$ such that $\dim(T^{K+1}E) = \dim(T^K E)$. Then T is a surjection from $T^K E$ to itself and hence is an isomorphism. We claim that $E = T^K E \oplus \ker(T^K)$. Let u be any element of E. Then $T^K u$ is in $T^K E$, on which T^K is an isomorphism. Hence, there exists v in $T^K E$ such that $T^K v = T^K u$ and we may write $u = v + (u - v)$, the former being in $T^K E$ and the latter in $\ker(T^K)$. As T^K is invertible on $T^K E$, its kernel there is trivial. This means that $T^K E \cap \ker(T^K) = 0$. Each of $T^K E$ and $\ker(T^K)$ are invariant under all powers of T and so the trace of T^p is the sum of its restrictions to the two subspaces. On the latter, it is nilpotent and so the trace there is zero.

Finally, it is an easy matter to check that the map from $T^K E$ to V sending u to $[u, 1]$ is an isomorphism of vector spaces and conjugates the restriction of T^n with the map induced by T^n on the inductive limit. □

Lemma 6.3.2. *Suppose that $B^k_{L,M} \subset G^k_{L,M}$ is as in Theorem 5.2.9 and $n \geq 1$. We have*

$$Tr((\gamma^s_B \otimes 1_{\mathbb{Q}})^n) = \sum_{(q,\alpha,\beta)\in A(n)} sgn(\alpha)sgn(\beta)$$

where

$$A(n) = \{(q, \alpha, \beta)G^{k+n}_{L,M} \times S_{L+1} \times S_{M+1} \mid t^n(q) = i^n(q) \cdot (\alpha, \beta) \in B^k_{L,M}\}.$$

is as in Lemma 6.2.3.

Proof. The set $B^k_{L,M}$ forms a basis for $\mathbb{Q}B^k_{L,M}$. Using Lemma 5.2.7, the coefficient of p in $(\gamma^s_B)^n(p)$ is the sum of $sgn(\alpha)sgn(\beta)$ over all (q, α, β) in $A(p, n)$ with $i^n(q) \cdot (\alpha, \beta) = p$. Now the trace of $(\gamma^s_B \otimes 1_{\mathbb{Q}})^n$ is obtained by summing this over all p in $B^k_{L,M}$ and this yields the desired result. □

The proof of the first equality in the Lefschetz formula of Theorem 6.1.1 is an immediate consequence of Theorem 6.2.2, Lemma 6.2.3, Proposition 6.3.1, Lemma 6.3.2 and the discussion above. The other three equalities are obtained in an analogous fashion. We omit the details.

Chapter 7

Examples

In this chapter, we present a number of simple examples to illustrate the theory. First, we consider a shift of finite type, (Σ, σ) and show that the the homology groups $H^s_N(\Sigma, \sigma)$ are all zero, except for the case $N = 0$, in which case, we recover Krieger's original invariant: $H^s_0(\Sigma, \sigma) \cong D^s(\Sigma, \sigma)$.

Secondly, we consider the case of an irreducible Smale space whose stable sets are totally disconnected. Here, we will show that the double complex used in the calculation of our homology from Chapter 5 essentially reduces to the complex appearing in Chapter 4.

Our third example is a special case of the second, where we consider the m^∞-solenoid (precise definitions will be given). The main ingredient in the s/u-bijective pair is the usual covering by the full m-shift. We will carry out the computation in full detail.

Our fourth example is a specific hyperbolic toral automorphism. We will not give a full treatment, but we summarize the computations which will appear elsewhere.

7.1 Shifts of finite type

Let (Σ, σ) be a shift of finite type. The first step in computing its homology is to find an s/u-bijective pair. It is easy to see that $(Y, \psi) = (\Sigma, \sigma) = (Z, \zeta)$ and $\pi_s = \pi_u = id_\Sigma$ will suffice. We denote it by id, for emphasis. (Of course, there are many other choices.) It follows at once from Theorem 5.1.10 that $C^s_{\mathcal{Q},\mathcal{A}}(id)_{L,M} = 0$ if either L or M are greater than zero. Moreover, we have $C^s_{\mathcal{Q},\mathcal{A}}(id)_{0,0} = D^s(\Sigma, \sigma)$. In consequence, we have the following.

Theorem 7.1.1. *Let (Σ, σ) be a shift of finite type.*

1. *We have*
$$H_N^s(\Sigma, \sigma) = \begin{cases} D^s(\Sigma, \sigma) & N = 0, \\ 0 & N \neq 0. \end{cases}$$

2. *We have*
$$H_N^u(\Sigma, \sigma) = \begin{cases} D^u(\Sigma, \sigma) & N = 0, \\ 0 & N \neq 0. \end{cases}$$

7.2 Totally disconnected stable sets

Let us begin by supposing that we have a Smale space, (X, φ), a shift of finite type, (Σ, σ) and an s-bijective factor map $\pi_s : (\Sigma, \sigma) \to (X, \varphi)$. It follows from Theorem 2.5.12 that $X^s(x)$ is totally disconnected, for every x in X. Conversely, if $X^s(x)$ is totally disconnected, for every x in X and if, in addition, we assume that (X, φ) is irreducible, then from Corollary 3 of [29], we may find an irreducible shift of finite type, (Σ, σ), and an s-resolving factor map $\pi_s : (\Sigma, \sigma) \to (X, \varphi)$. From Theorem 2.5.8, the map π_s is also s-bijective. In any case, it follows that $\pi = (\Sigma, \sigma, \pi_s, X, \varphi, id_X)$ is an s/u-bijective pair for (X, φ).

In this case, we have $M_0 = 1$ in Theorem 5.1.10 and so the complex $C^s_{\mathcal{Q},\mathcal{A}}(\pi)$ is non-zero only in row zero, where it is precisely $(D^s_{\mathcal{Q}}(\Sigma_*(\pi_s)), d^s(\pi_s))$.

Theorem 7.2.1. *Let (X, φ) be a Smale space and suppose that $\pi_s : (\Sigma, \sigma) \to (X, \varphi)$ is an s-bijective factor map. Then the homology $H^s_N(X, \varphi)$ is naturally isomorphic to the homology of the complex $(D^s_{\mathcal{Q}}(\Sigma_*(\pi_s)), d^s(\pi_s))$ of 4.1.7. Similarly, the homology $H^u_N(X, \varphi)$ is naturally isomorphic to the homology of the complex $(D^u_{\mathcal{A}}(\Sigma_*(\pi_s)), d^{u*}(\pi_s))$ of 4.1.7.*

Corollary 7.2.2. *Let (X, φ) be a non-wandering Smale space with $X^s(x)$ totally disconnected, for every x in X. Then we have $H^s_N(X, \varphi) \cong H^u_N(X, \varphi) \cong 0$, for all $N < 0$.*

Things are just slightly more complicated in the case of a u-bijective factor map $\pi_u : (\Sigma, \sigma) \to (X, \varphi)$. In this case, an s/u-bijective pair is $\pi = (X, \varphi, id_X, \Sigma, \sigma, \pi_u)$. The remaining argument as above is the same, reversing references to rows versus columns. In the end, the homology of the double complex, either H^s or H^u, is reduced to that of the first column. In the former case, this first column is the same as the complex $C^{s*}(\pi_u)$, except that the grading is reversed (N is replaced by $-N$).

Theorem 7.2.3. *Let (X, φ) be a Smale space and suppose that $\pi_u : (\Sigma, \sigma) \to (X, \varphi)$ is a u-bijective factor map. Then the homology $H^s_N(X, \varphi)$ is naturally isomorphic to the homology of the complex $(D^s_{\mathcal{A}}(\Sigma_*(\pi_s)), d^{s*}(\pi_u))$ of 4.1.7, with the reverse grading. Similarly, the homology $H^u_N(X, \varphi)$ is naturally isomorphic to the homology of the complex $(D^u_{\mathcal{Q}}(\Sigma_*(\pi_s)), d^u(\pi_u))$ of 4.1.7, with the reverse grading.*

Corollary 7.2.4. *Let (X, φ) be a non-wandering Smale space with $X^u(x)$ totally disconnected, for every x in X. Then we have $H^s_N(X, \varphi) \cong H^u_N(X, \varphi) \cong 0$, for all $N > 0$.*

7.3 Solenoids

In this section, we consider a specific example of the general type of the last section. We fix an integer $m \geq 2$ and consider the m^∞-solenoid. Let $\mathbb{T}$ denote the unit circle in the complex plane. Our space X is the inverse limit obtained from the system

$$\mathbb{T} \overset{z \to z^m}{\longleftarrow} \mathbb{T} \overset{z \to z^m}{\longleftarrow} \cdots$$

That is, we have

$$X = \{(z_0, z_1, \ldots) \mid z_n \in \mathbb{T}, z^m_{n+1} = z_n, \text{ for all } n \geq 0\}.$$

The map φ is given by $\varphi(z_0, z_1, \ldots) = (z^m_0, z^m_1, \ldots)$ with $\varphi^{-1}(z_0, z_1, \ldots) = (z_1, z_2, \ldots)$. This example was first done jointly with T. Bazett and will appear in [3], but we give most of the details here.

The metric on X is $d(w, z) = \sum_{n \geq 0} 2^{-n} \|w_n - z_n)\|$, where $\|w' - z'\|$ is the arclength distance on the circle. Our Smale space constant $\epsilon_X = \frac{\pi}{2}$. In particular, if $d(w, z) \leq \epsilon_X$, then $\|w_0 - z_0\| \leq \frac{1}{4}$ and so we may find a unique $-\frac{1}{4} \leq t \leq \frac{1}{4}$ such that $w_0 = z_0 e^{2\pi i t}$. In this case, the bracket is defined by

$$[w, z] = (z_0 e^{2\pi i t}, z_1 e^{\pi i t}, z_2 e^{\pi i t/2}, \ldots).$$

The shift of finite type which maps onto (X, φ) is the full m-shift. That is, the graph G consists of a single vertex v_0 and edges $\{0, 1, 2, \ldots, m-1\}$. We may write the factor map $\pi_s : (\Sigma_G, \sigma) \to (X, \varphi)$ quite explicitly as

$$\pi_s(e)_n = e^{2\pi i \sum_{k \geq 1} m^{-k} e^{k-n}}.$$

Consider e, f in Σ_G with $\pi_s(e) = \pi_s(f)$. There are three possibilities. The first is that $e = f$. The second is that $e^k = 0, f^k = m-1$, for all k (or vice verse). The third is that, for some K, $e^k = f^k$ for all $k < K$, while $e^k = 0, f^k = m-1$, for all $k \geq K$ (or vice verse). From this fact, we conclude that $L_0 = 2$.

By Theorem 5.1.10, the only two non-zero groups in our complex occur at positions $0, 0$ and $1, 0$. Here, the factor map is not regular, but if we replace G by G^2, then the induced map is regular. The vertex set for G^2 is just G^1 and there is exactly one edge between every pair of vertices. It is fairly easy to see that the map sending $[i, j], i \in G^{k-1}, j \in \mathbb{N}$ to m^{-j-k+1} is an isomorphism from $D^s(G^k)$ to $\mathbb{Z}[1/m]$, the subgroup of all rational numbers of the form l/m^j, where $l \in \mathbb{Z}$ and $j \geq 0$. To summarize, we have

$$D^s_{\mathcal{Q}}(\Sigma_0(\pi_s)) = D^s(\Sigma, \sigma) \cong \mathbb{Z}[\frac{1}{m}].$$

The vertex set for $G_1^2(\pi)$ is $\{(0,0), \ldots, (m-1, m-1), (0, m-1), (m-1, 0)\}$. The edge set consists of all pairs (p, p), where p is in G^2 and the edges $(0\,0, m-1\,m-1), (m-1\,m-1, 0\,0)$ and $(i\,0, i\,m-1), (i\,m-1, i\,0)$, where $0 \leq i < m$. To compute the homology, we first appeal to Theorem 7.2.1, which means that we will just consider the complex $(D^s_{\mathcal{Q}}(\Sigma_*(\pi_s)), d^s(\pi_s))$.

¿From this it is easy to see that $\mathcal{Q}(G_1^2, S_2)$ is an infinite cyclic group with generator $Q(0\,m-1)$. Moroever, $\gamma^s(0\,m-1) = (0\,m-1)$ and hence $D^s_{\mathcal{Q}}(G_1^2(\pi_s))$ is the integers and is generated by $[Q(0\,m-1), j]$, for any $j \geq 1$. We now need to compute

$$d^{s,K}_{\mathcal{Q}}(\pi_s)(Q(0\,m-1)) = \delta_0^{s,K}(0\,m-1) - \delta_1^{s,K}(0\,m-1).$$

We compute $\delta_0^{s,K}(0\,m-1)$ using the definition given in 3.4.3: we must consider all paths q in G_1^{1+K} with $t^K(q) = (0\,m-1)$. We do need to list them all since we are going to apply δ_0 and then take the sum (without repetition). It is clear from the graph that we get every p in G^{1+K} with $t^K(p) = 0$. Using the isomorphism above with $\mathbb{Z}[1/m]$, we see that the image of our generator is $1/m$. A similar computation shows the same result for δ_1 and we conclude that $d^s(\pi_s)_1 = 0$.

We summarize the conclusion as follows:

$$H^s_N(X, \varphi) = \begin{cases} \mathbb{Z}[1/m] & N = 0, \\ \mathbb{Z} & N = 1, \\ 0 & N \neq 0, 1. \end{cases}$$

We also observe that the groups $H^u_N(X, \varphi)$ are exactly the same. In this case, the groups in the complex are exactly the same as above. Curiously, the computation of the boundary map is made easier by the fact that the zero map is the *only* group homomorphism from $\mathbb{Z}[1/m]$ to $\mathbb{Z}$.

7.4 A hyperbolic toral automorphism

We consider the matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. In the usual way, it induces an automorphism of the two-torus $\mathbb{T}^2$, which we denote by φ. We also let $q : \mathbb{R}^2 \to \mathbb{T}^2$ denote the usual quotient map. The local stable and unstable sets are given by the eigenvectors of the matrix, which are $(\gamma, 1)$ and $(1, -\gamma)$, where γ denotes the golden mean. The associated eigenvalues are γ and $-\gamma^{-1}$, respectively.

The detailed computation was carried out jointly with T. Bazett and the details will appear elsewhere [3]. Here, we present a summary. The (Y, ψ) and (Z, ζ) we choose are both DA systems [30]. The space Y is obtained from $\mathbb{T}^2$ by removing the stable set of $q(0, 0)$, which is the line $\{q(s, -\gamma s) \mid s \in \mathbb{R}\}$ and replacing it with two copies of itself, separated by a gap. The space Z is obtained from a similar alteration to the unstable set at $q(0, 0)$. The system $(\Sigma(\pi), \sigma)$ is associated to the Markov partition of $\mathbb{T}^2$ having three rectangles which may be found in many texts on dynamical systems. For example, see page 207-212 of [23]. A description of the graph presenting (Σ, σ) can be found in [23]. It has three vertices and five edges. Here, the map γ^s is an isomorphism and we have $D^s(\Sigma, \sigma) \cong \mathbb{Z}^3$.

The constants L_0 and M_0 of 5.1.10 are both two. The only non-zero groups in our double complex (besides the one above) are

$$\begin{aligned} C^s_{\mathcal{Q},\mathcal{A}}(\pi)_{1,0} &\cong \mathbb{Z}^2, \\ C^s_{\mathcal{Q},\mathcal{A}}(\pi)_{0,1} &\cong \mathbb{Z}^2, \\ C^s_{\mathcal{Q},\mathcal{A}}(\pi)_{1,1} &\cong \mathbb{Z}. \end{aligned}$$

The boundary maps may also be computed explicitly and the resulting homology is as follows:

$$\begin{aligned} H^s_1(\mathbb{T}^2, \varphi) &\cong \mathbb{Z}, \\ H^s_0(\mathbb{T}^2, \varphi) &\cong \mathbb{Z}^2, \\ H^s_{-1}(\mathbb{T}^2, \varphi) &= \mathbb{Z}. \end{aligned}$$

It is also worth noting that the automorphism of these groups induced by φ is the identity on $H^s_{-1}(\mathbb{T}^2, \varphi)$, minus the identity on $H^s_1(\mathbb{T}^2, \varphi)$ and given by the matrix A^{-1} on $H^s_0(\mathbb{T}^2, \varphi)$.

Chapter 8

Questions

8.1 Order on homology groups

One the great strengths of Krieger's invariant which has not appeared at all in our presentation is that it is not simply an abelian group, but an *ordered* abelian group. That is, given a shift of finite type (Σ, σ), there are natural positive cones, which we denote by $D^s(\Sigma, \sigma)^+$ and $D^u(\Sigma, \sigma)^+$, in $D^s(\Sigma, \sigma)$ and $D^u(\Sigma, \sigma)$, respectively. In the former case, this is simply the sub-semigroup generated by the equivalence classes of the compact open sets in $CO^s(\Sigma, \sigma)$. These provide a great deal more information than the groups on their own. It is natural to ask whether this has any generalization to our homology theory. Probably the right formulation is the following.

Question 8.1.1. *For a Smale space (X, φ), is there a natural order structure on on the groups $H^s_0(X, \varphi)$ and $H^u_0(X, \varphi)$? That is, are there natural subsemigroups $H^s_0(X, \varphi)^+ \subset H^s_0(X, \varphi)$ and $H^u_0(X, \varphi)^+ \subset H^u_0(X, \varphi)$ which generate the groups in question? This should be functorial in the sense that, if $\pi : (Y, \psi) \to (X, \varphi)$ is an s-bijective factor map, then the induced map $\pi^s : H^s_0(Y, \psi) \to H^s_0(X, \varphi)$ should be positive; i.e. $\pi^s(H^s_0(Y, \psi)^+) \subset H^s_0(X, \varphi)^+$. Analogous statements should hold for a u-bijective factor map and for H^u as well.*

Concentrating just on the case of H^s for the moment, one could proceed as follows. Let π be an s/u-bijective pair for (X, φ). Observe that $D^s_{\mathcal{Q},\mathcal{A}}(\Sigma_{0,0}(\pi)) = D^s(\Sigma_{0,0}(\pi))$ and so this group carries a natural order. Next define an order on the group $\oplus_{L-M=0} D^s_{\mathcal{Q},\mathcal{A}}(\Sigma_{L,M}(\pi))$ by setting the positive cone to be those elements whose entry in the summand $L = M = 0$

are strictly positive, together with the zero element. That is, entries for $L = M > 0$ do not effect positivity. Now we can (attempt to) define a positive cone in $H_0^s(\pi)$ as those elements which are represented by a positive cocycle in $\oplus_{L-M=0} D^s_{\mathcal{Q},\mathcal{A}}(\Sigma_{L,M}(\pi))$.

In short, three things need to be verified. The first that this positive cone generates the group. The second is that there are no elements which are positive and their inverses are positive as well. (Strictly speaking, the definition of an ordered group does not prohibit this. For example, letting the positive cone be the entire group is a valid order. But it seems desirable in our situation that we avoid this.) Finally, one needs to see that the positive cone is independent of the s/u-bijective pair which is used.

For the first point, observe that the kernel of

$$\delta^{s*}_{,0} - \delta^{s*}_{,1} : D^s_{\mathcal{Q},\mathcal{A}}(\Sigma_{0,0}(\pi)) \to D^s_{\mathcal{Q},\mathcal{A}}(\Sigma_{0,1}(\pi))$$

also lies in the kernel of $\oplus_{L-M=0} d^s(\pi)_{L,M}$ and hence determine homology classes. It suffices to prove that

$$\ker(\delta^{s*}_{,0} - \delta^{s*}_{,1}) \cap D^s(\Sigma_{0,0}(\pi))^+$$

generates $\ker(\delta^{s*}_{,0} - \delta^{s*}_{,1})$.

For the second point, it suffices to verify that the coboundaries, that is, the range of $\oplus_{L-M=1} d^s(\pi)_{L,M}$ lies in the part of the group which does not effect positivity. Immediately from our definition, the maps $d^s(\pi)_{L,M}$ satisfy this condition for $L > 1$. So it suffices to verify that this condition holds for

$$\delta^s_{0,} - \delta^s_{1,} : D^s_{\mathcal{Q},\mathcal{A}}(\Sigma_{1,0}(\pi)) \to D^s_{\mathcal{Q},\mathcal{A}}(\Sigma_{0,0}(\pi)).$$

Since the range is the direct sum of simple acyclic dimension groups, positivity is determined by the states on this group [13] and so this should provide a method for verifying the desired property.

As a final comment, it is well-known that Krieger's invariant is a *dimension group* [13]. In our generalization, it seems unclear whether one should expect $H^0(X, \varphi)$ to be torsion free.

8.2 Machinery

Of course, one would like to be able to actually compute the homology theory in specific examples. In the few examples we have given, this has been done mostly by resorting to the definition. In algebraic topology, the computations of homology and cohomology are done by appealing to very powerful machinery such as long exact sequences, excision, homotopy invariance, and so on. It is natural to look for such techniques here as well, but exact analogues of these are not so clear. One possibility would be something along the following lines.

Question 8.2.1. *If $(X, \varphi), (Y, \psi)$ and (Z, ζ) are Smale spaces and*

$$\pi : (Y, \psi) \to (X, \varphi), \rho : (Z, \zeta) \to (X, \varphi)$$

are factor maps, are there some hypotheses on these maps which will yield a long exact sequence in H^s involving the three Smale spaces and the fibred product of the maps?

In particular, if (X, φ) is a Smale space with s/u-bijective pair $\pi = (Y, \psi, \pi_s, Z, \zeta, \pi_u)$, can the homology of (X, φ) be computed from that of (Y, ψ), (Z, ζ) and $\Sigma_{0,0}(\pi)$?

One particularly difficult aspect in the computation of our homology is that it is done via a double complex. It may be possible to eliminate some of this. As an example, here is a fairly simple result. The hypothesis is slightly odd, but the conclusion is quite nice.

Theorem 8.2.2. *Suppose that (X, φ) is a Smale space with s/u-bijective pair $\pi = (Y, \psi, \pi_s, Z, \zeta, \pi_u)$ which satisfies*

$$\min\{\#\pi_s^{-1}\{x\}, \#\pi_u^{-1}\{x\}\} = 1,$$

for all periodic points x in X. Then $C^s_{\mathcal{Q},\mathcal{A}}(\pi)_{L,M} = 0$, if either $L \geq 1$ or $M \geq 1$.

We will not give a proof of this result, but it is a fairly simple consequence of the results we have already.

Let us observe that it could be used for the computation done for the hyperbolic toral automorphism in Section 7.4 if we use a different s/u-bijective pair. In fact, Y, ψ, π_s would be exactly the same. But instead of obtaining Z by replacing the line which is the stable set $(0, 0)$, we would do exactly the same with the three lines which are the stable sets of the periodic orbit consisting of $(1/2, 0), (1/2.1/2)$ and $(0, 1/2)$. We will not complete the computation except to observe that this s/u bijective pair satisfies the hypothesis of the theorem. In fact, the same technique can be used for any hyperbolic map of the 2-torus.

Question 8.2.3. *Does any irreducible Smale space have an s/u-bijective pair satisfying the hypotheses of Theorem 8.2.2?*

On the positive side, since the maps in an s/u-bijective pair can be chosen so that π_s and π_u are both one-to-one on a dense G_δ, there is some genericity sense that this might be possible or even likely. Nevertheless, it seems too good to be true in general.

8.3 Relation with Cech (co)homology

It seems natural ask if there is some relation between our homology for the Smale space (X, φ) and the usual Cech (co)homology of X. There are old questions about which manifolds admit an Anosov or even Axiom A diffeomorphism. It seems conceivable that an explicit relation between our homology and the Cech cohomology might have applications for such questions.

Let us formulate a couple of concrete questions.

Question 8.3.1. *Let (X, φ) be an irreducible Smale space. Under which conditions is there a group homomorphism from $H^*(X)$ to $H^s(X, \varphi) \otimes H^u(X, \varphi)$?*

There are several problems here, since it is not really clear in what sense the range group is a functor.

However, let us observe that the result holds for a shift of finite type, (Σ, σ). In this case, the Cech cohomology lives in dimension zero and can be seen as being generated by the clopen subsets of Σ. In fact, it is generated by clopen sets which are rectangles. That is, clopen sets of the form $[U, V]$, where U is a clopen subset of $\Sigma^u(e, \epsilon/2)$ and V is a clopen subset of $\Sigma^s(e, \epsilon/2)$. The map that sends $[U, V]$ to $[V] \otimes [U]$ can be shown to give a well-defined homomorphism of the type desired.

The Cech cohomology of the product of two spaces is closely related to the tensor product of their cohomology (see the Künneth Theorem [34]). The question above would provide a homological remnant of the local product structure.

Another question which may be more reasonable, but perhaps more difficult, would be to know cases where our homology and the Cech cohomology agree, up to a shift in degree. This is mainly based on examples.

Question 8.3.2. *If the stable (or unstable) sets in a Smale space (X, φ) are contractible in the topology of 2.1.12, is there an integer k such that*

$$H^n(X) \cong H^s_{n-k}(X, \varphi),$$

for all $n \geq 0$?

8.4 C^*-algebras

As we mentioned, Krieger showed how C^*-algebras could be constructed from a shift of finite type. Moreover, these have a relatively nice structure: they are inductive limits of finite-dimensional C^*-algebras. This fact makes their K-theory readily computable, and it is exactly Krieger's invariant.

The construction of C^*-algebras was extended to the general setting of Smale spaces by David Ruelle [32, 28]. To an irreducible Smale space (X, φ), we associate C^*-algebras to stable and unstable equivalence, denoted by $S(X, \varphi)$ and $U(X, \varphi)$, respectively. However, unlike the situation for shifts of finite type, the result has no obvious inductive limit structure and so the computation of the K-theory of these C^*-algebras is not immediately clear. The work of this paper began as an attempt to compute these K-groups and in future work we plan to show how this may be done from the homology presented here.

Question 8.4.1. *For an irreducible Smale space (X, φ), can the K-theory of $S(X, \varphi)$ and $U(X, \varphi)$ may be computed from $H^s_*(X, \varphi)$ and $H^u_*(X, \varphi)$?*

The answer is presumably some type of spectral sequence.

Due to some regrettable choices for terminology, the K-theory of $S(X, \varphi)$ should be computed from $H^u_*(X, \varphi)$.

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