ORDER ON THE HOMOLOGY GROUPS OF SMALE SPACES

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Abstract. Smale spaces were defined by D. Ruelle to describe the properties of the basic sets of an Axiom A system for topological dynamics. One motivation for this was that the basic sets of an Axiom A system are merely topological spaces and not submanifolds. One of the most important classes of Smale spaces is shifts of finite type. For such systems, W. Krieger introduced a pair of invariants, the past and future dimension groups. These are abelian groups, but are also with an order which is an important part of their structure. The second author showed that Krieger's invariants could be extended to a homology theory for Smale spaces. In this paper, we show that the homology groups on Smale spaces (in degree zero) have a canonical order structure. This extends that of Krieger's groups for shifts of finite type.

1. Introduction

The original notion of a Smale space is due to David Ruelle, based on the observation that the basic sets of Smale's Axiom A systems do not form submanifolds of the ambient manifold [16, 17, 2, 9, 8]. In fact, Smale spaces are the topological dynamical systems that admit a hyperbolic structure in terms of canonical coordinates of contracting and expanding (or stable and unstable) directions. Hyperbolic toral automorphisms, one-dimensional generalized solenoids as described by R.F. Williams and shifts of finite type are all examples of Smale spaces. In fact, any totally disconnected (irreducible) Smale space is conjugate to a shift of finite type. In [11], W. Krieger defined two abelian groups for shift of finite type called the past and future dimension groups in terms of clopen sets of the stable and unstable sets. One of their most important features is a natural order structure.

In [14], the second author defined a homology for Smale spaces which extends the dimension groups for shifts of finite type. However, the

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homology groups as defined in [14] are not given any order structure. In this paper, we prove that the homology groups of Smale spaces in degree zero have a canonical order structure.

The paper is organized as follows. In the second section, we introduce the basic concepts and notations (based on [14]) and state the main results of this paper, which are proved in Sections 4 and 5. The shifts of finite type which play an important role in the homology of Smale spaces are reviewed in the third section and their dimension groups are discussed as ordered groups.

2. Preliminaries

2.1. **Smale spaces.** A pair (X, φ) is called a dynamical system if X is a topological space and φ is a homeomorphism of X. A dynamical system (X, φ) is called irreducible if for every ordered pair of non-empty open sets, U, V in X, there is a non-negative integer n such that $\varphi^n(U) \cap V$ is non-empty. It is called mixing if for every ordered pair of non-empty open sets, U, V in X, there is a non-negative integer N such that $\varphi^n(U) \cap V$ is non-empty for any $n \geq N$ [2, 14].

Definition 2.1. ([16], 2.1.6 of [14]) For a compact metric space X, the dynamical system (X, φ) is called a Smale space if there exist constants ε_X and $0 < \lambda < 1$ and a continuous map from

$$\triangle_{\varepsilon_X} = \{(x, y) \in X \times X \mid d(x, y) \leq \varepsilon_X\}$$

to X (denoted by $[\cdot, \cdot]$) such that, for every $x, y, z \in X$,

- $B\,1$ $[x, x] = x$,
- $B\,2$ $[x, [y, z]] = [x, z]$,
- $B\,3$ $[[x, y], z] = [x, z]$,
- $B\,4$ $[\varphi(x), \varphi(y)] = \varphi([x, y])$,

whenever both sides of the above equations are defined, and

- $C\,1$ $d(\varphi(x), \varphi(y)) \leq \lambda\, d(x, y)$, whenever $[x, y] = y$,
- $C\,2$ $d(\varphi^{-1}(x), \varphi^{-1}(y) \leq \lambda\, d(x, y)$, whenever $[x, y] = x$.

In a Smale space (X, φ), the local stable and unstable sets are defined, for x in X and $\epsilon_X \geq \epsilon > 0$, by

$$\begin{aligned} X^s(x, \epsilon) &= \{y \in X \mid d(x, y) \leq \epsilon, [x, y] = y\}, \\ X^u(x, \epsilon) &= \{y \in X \mid d(x, y) \leq \epsilon, [y, x] = y\}. \end{aligned}$$

It is a simple matter to show that, for any ϵ sufficiently small, $[\cdot, \cdot] : X^u(x, \epsilon) \times X^s(x, \epsilon) \to X$ is a homeomorphism to its image, which is a neighbourhood of x in X. The inverse is obtained by mapping y to $([x, y], [y, x])$.

Let (X, φ) be a Smale space. Two points $x, y \in X$ are stable (resp, unstable) equivalent if

$$\lim_{n \to +\infty} d(\varphi^n(x), \varphi^n(y)) = 0$$

(resp, $\lim_{n \to +\infty} d(\varphi^{-n}(x), \varphi^{-n}(y)) = 0$). We denote the stable (unstable) equivalence class of x by $X^s(x)$ (resp, $X^u(x)$)) [14].

Examples of Smale spaces include Anosov diffeomorphisms, the basic sets from Smale's Axiom A systems, various solenoids and certain substitution tiling spaces [10, 17, 4, 20, 21, 22]. Key examples are the shifts of finite type, namely the doubly infinite path space of a finite directed graph. We provide a more complete description in the next section. In this case, the underlying space is totally disconnected [12, 14]. Conversely, any irreducible Smale space which is totally disconnected is topologically conjugate to a shift of finite type.

A factor map π between dynamical systems (Y, ψ) and (X, φ) is a continuous, surjective map $\pi : Y \to X$ satisfying $\varphi \circ \pi = \pi \circ \psi$. A factor map π is finite-to-one if there is an upper bound on the cardinality of the sets $\pi^{-1}\{x\}$, as x runs over X [14]. It is almost one-to-one if $\#\pi^{-1}\{x\} = 1$, for each x in some dense G_δ subset of X.

A map $\pi : (Y, \psi) \to (X, \varphi)$ between Smale spaces is called s-bijective (resp, u-bijective) if the restriction of π to $Y^s(y)$ (resp, $Y^u(y)$) is a bijection to $X^s(\pi(y))$ (resp. $X^u(\pi(y))$), for any $y \in Y$. Every s-bijective (or u-bijective) map is finite-to-one [14].

Definition 2.2. (2.6.2 of [14]) Let (X, φ) be a Smale space. Then

$$\pi = (Y, \psi, \pi_s, Z, \zeta, \pi_u)$$

is an s/u-bijective pair for (X, φ) if

- (Y, ψ) and (Z, ζ) are Smale spaces,
- $\pi_s : (Y, \psi) \to (X, \varphi)$ is s-bijective and $X^u(y)$ is totally disconnected, for every $y \in Y$,
- $\pi_u : (Z, \zeta) \to (X, \varphi)$ is u-bijective and $X^s(y)$ is totally disconnected, for every $z \in Z$.

Theorem 2.3. *(2.6.3 of [14]) Every irreducible Smale space (X, φ) admits an s/u-bijective pair.*

This result plays a crucial role in [14]. The homology is defined and computed from such an object. While there may be many such s/u-bijective pairs for a given (X, φ), it is shown in 5.5.1 of [14] that the homology is independent of the choice.

Our first contribution here is to improve this situation by proving the existence of s/u-bijective pairs with certain advantageous extra features. These will be important for our proofs later, but presumably, will have many other applications.

Theorem 2.4. *Every irreducible Smale space* (X, φ) *admits an* s/u-*bijective pair* $\pi = (Y, \psi, \pi_s, Z, \zeta, \pi_u)$ *such that the Smale spaces* (Y, ψ), (Z, ζ) *are irreducible and both maps* π_s *and* π_u *are almost one-to-one.*

The proof is based on [15] and will be given in the subsection 4.1.

Definition 2.5. For any Smale space (X, φ), we say that an s/u-bijective pair $(Y, \psi, \pi_s, Z, \zeta, \pi_u)$ is *irreducible* if both (Y, ψ) and (Z, ζ) are irreducible and both maps π_s and π_u are almost one-to-one.

For a Smale space (X, φ) and s/u-bijective pair $\pi = (Y, \psi, \pi_s, Z, \zeta, \pi_u)$, for each $L, M \geq 0$, we define

$$\begin{aligned} \Sigma_{L,M}(\pi) &= \{(y_0, y_1, \ldots, y_L, z_0, z_1, \ldots, z_M) \in Y^{L+1} \times Z^{M+1} \mid \\ &\quad \pi_s(y_l) = \pi_u(z_m), \text{ for all } 0 \leq l \leq L, 0 \leq m \leq M\}. \end{aligned}$$

If we let σ be the obvious map on $\Sigma_{L,M}(\pi)$ induced by ψ and ζ, $(\Sigma_{L,M}(\pi), \sigma)$ is a dynamical system. Indeed, it is also a Smale space with totally disconnected stable and unstable sets, and so is a shift of finite type. In the special case that $L = M = 0$, this is usually called the fibred product of (Y, ψ) and (Z, ζ). On the other hand, $(\Sigma_{L,M}(\pi), \sigma)$ has an obvious action of the group $S_{L+1} \times S_{M+1}$, where S_{N+1} denotes the permutation group of $\{0, 1, \ldots, N\}$ [14].

If the s/u-bijective pair is irreducible in the sense above, then the fibred product is irreducible. By this we mean the shift of finite type $\Sigma_{0,0}(\pi), \sigma)$ is irreducible. The other $\Sigma_{L,M}(\pi), \sigma)$ will not be, in general. The proof of this result is long and will be given in subsection 4.2.

Theorem 2.6. *Suppose* $(X, \varphi), (Y, \psi)$ *and* (Z, ζ) *are irreducible Smale spaces,*

$$\pi_s : (Y, \psi) \to (X, \varphi)$$

is an almost one-to-one, s-bijective factor map and

$$\pi_u : (Z, \zeta) \to (X, \varphi)$$

is an almost one-to-one, u-bijective factor map. Then the fibred product

$$Y \times_X Z = \{(y, z) \in Y \times Z \mid \pi_s(y) = \pi_u(z)\}$$

of maps π_s *and* π_u *with* $\psi \times \zeta$ *is an irreducible Smale space.*

In addition, if (X, φ) *is mixing then so is* $(Y \times_X Z, \psi \times \zeta)$*.*

Of course, one application of the theorem is to an irreducible s/u-bijective pair for (X, φ), but the result is more general, since it makes no assumptions that the local stable sets of Z or the local unstable sets of Y are totally disconnected.

For a shift of finite type, (Σ, σ), Krieger introduced the dimension group invariants, denoted here by $D^s(\Sigma, \sigma)$ and $D^u(\Sigma, \sigma)$. These are countable abelian groups and, if the shift of finite type is presented as the edge shift of a finite directed graph G, they may be computed directly as inductive limits from the adjacency matrix of G. We discuss this more thoroughly in the next section.

The second author developed a homology theory for Smale spaces in [14]. Let us briefly review the construction here. First, one considers the dimension groups $D^s(\Sigma_{L,M}(\pi), \sigma)$ of the system, over all $L, M \geq 0$. At each index, a quotient of a certain subgroup is taken, denoted by $D^s_{\mathcal{Q},\mathcal{A}}(\Sigma_{L,M}(\pi), \sigma)$, which takes into account the action of the permutation groups (section 5.1 of [14]). These groups are assembled into a double complex, $\mathcal{C}^s_{\mathcal{Q},\mathcal{A}}(\pi)_{L,M} = D^s_{\mathcal{Q},\mathcal{A}}(\Sigma_{L,M}(\pi), \sigma), L, M \geq 0$, whose homology is denoted by $H^s_*(\pi)$. There is an analogous construction of $H^u_*(\pi)$, using the unstable dimension groups D^u. In [14], it is shown that the result is independent of the choice of π, and so is written as $H^s_*(X, \varphi)$ or $H^u_*(X, \varphi)$ (Theorem 5.5.1 of [14]).

For the remainder of this paper, we will concentrate on $H^s(X, \varphi)$. Analogous results hold for $H^u(X, \varphi)$.

The above construction is analogous to computing the Čech cohomology of a compact manifold by considering a 'nice', finite, open cover and the homology of its nerve. Here, the s/u-bijective pair replaces the open cover. The shifts $(\Sigma_{L,M}(\pi), \sigma)$ evidently play the role of the nerve of the cover, keeping track of the multiplicities of the cover. Finally, Krieger's dimension group invariant replaces the homology of the open balls in the 'nice' cover.

One of the most important features of Krieger's invariant for a shift of finite type is that it also carries a natural order structure. Moreover, this is also easily computed from the corresponding directed graph. The aim of this paper is to define a natural and canonical order structure on the homology groups $H^s_0(X, \varphi)$ and $H^u_0(X, \varphi)$.

Let us begin with the definition of an ordered abelian group.

Definition 2.7. [3] A pair (G, G^+) is called an ordered abelian group if G is an abelian group with a positive cone G^+, which is a subset of G satisfying the following conditions:

(1) $G^+ + G^+ \subseteq G^+$,
(2) $G^+ - G^+ = G$,
(3) $G^+ \cap -G^+ = \{0\}$.

The elements of G^+ are called positive elements of G and, for g_1, g_2 in G, we write $g_1 \geq g_2$ (or $g_2 \leq g_1$) when $g_1 - g_2 \in G^+$.

A homomorphism $\Gamma : G \to H$ of ordered groups is called positive if $\Gamma(G^+) \subseteq H^+$. An isomorphism $\Gamma : G \to H$ of ordered groups, is an order isomorphism if both Γ and Γ^{-1} are positive homomorphisms (equivalently, if $\Gamma(G^+) = H^+$). We remark that the inverse of a positive isomorphism is not positive in general. For example, consider the ordered group $\mathbb{Z}^2$ with the positive cone $\{(m,n) | m,n \geq 0\}$. The map $\alpha(m,n) = (m+n,n)$ is a positive automorphism of $\mathbb{Z}^2$ whose inverse is not positive.

The groups $D^s(\Sigma_{L,M}(\pi)), L, M \geq 0$, all carry canonical orders. Unfortunately, these do not induce orders on the groups $D^s_{\mathcal{Q},\mathcal{A}}(\Sigma_{L,M}(\pi))$ in our double complex, except in the special case when $L = M = 0$, where $D^s_{\mathcal{Q},\mathcal{A}}(\Sigma_{0,0}(\pi))$ and $D^s(\Sigma_{0,0}(\pi))$ are equal. We intend to lift this order to the degree zero group in our double complex, namely on $\oplus_{L-M=0}\mathcal{C}^s_{\mathcal{Q},\mathcal{A}}(\pi)_{L,M}$, by setting the positive cone to be those elements whose entries in the summand $L = M = 0$ are strictly positive, together with the zero element. In particular, the entries in the position $L = M > 0$ do not effect positivity. The positive cone $H^s_0(\pi)^+$ in $H^s_0(\pi)$ is then defined as those elements which are represented by a positive cocycle in $\oplus_{L-M=0}\mathcal{C}^s_{\mathcal{Q},\mathcal{A}}(\pi)_{L,M}$. The difficulty is to show that this gives a well-defined and well-behaved order on the homology.

Definition 2.8. Let $\pi = (Y, \psi, \pi_s, Z, \zeta, \pi_u)$ be an s/u-bijective pair for the Smale space (X, φ). Let $(\mathcal{C}^s_{\mathcal{Q},\mathcal{A}}(\pi), d^s_{\mathcal{Q},\mathcal{A}}(\pi))$ be the double complex associated with π and $H^s_*(\pi)$ be the homology of this double complex. We define the corresponding cones as follows:

$$\Big(\bigoplus_{L-M=0} \mathcal{C}^s_{\mathcal{Q},\mathcal{A}}(\pi)_{L,M}\Big)^+ = \{0\} \cup \{a \mid 0 \neq a_{0,0} \in \mathcal{C}^s_{\mathcal{Q},\mathcal{A}}(\pi)^+_{0,0}\},$$

and

$$\begin{aligned} H^s(\pi)^+ \;=\; & \{a + \mathrm{Im}\Big(\bigoplus_{L-M=1} d^s_{\mathcal{Q},\mathcal{A}}(\pi)_{L,M}\Big) \mid \\ & a \in \mathrm{Ker}\Big(\bigoplus_{L-M=0} d^s_{\mathcal{Q},\mathcal{A}}(\pi)_{L,M}\Big) \cap \Big(\bigoplus_{L-M=0} \mathcal{C}^s_{\mathcal{Q},\mathcal{A}}(\pi)_{L,M}\Big)^+\}. \end{aligned}$$

Of course, both definitions are the obvious ones. The issue is now to show that this provides good order structures, at least for irreducible Smale spaces. The strategy is a simple one: we first assume that our s/u-bijective pair is irreducible. We reduce to the case that the shift of finite type, $(\Sigma_{0,0}(\pi), \sigma)$, is mixing. It follows that the order structure on its dimension group is completely determined by the state which arises from its unique measure of maximal entropy, or the Parry measure - see Theorem 3.4.

To take homology, we first pass to a subgroup (the cocycles) and then take a quotient (by the coboundaries). The following rather elementary result summarizes our task.

Theorem 2.9. [3] *Let* (G, G^+) *be an ordered abelian group and let* $H \subseteq G$ *be a subgroup.*

(i) *If* $G^+ \cap H = \{0\}$*, then with*

$$(G/H)^+ = \{a + H \mid a \in G^+\},$$

$(G/H, (G/H)^+)$ *is an ordered abelian group.*

(ii) *If* $G^+ \cap H$ *generates* H*, that is,* $(G^+ \cap H) - (G^+ \cap H) = H$*, then* $(H, G^+ \cap H)$ *is an ordered abelian group.*

The conditions for the subgroup and quotient in the above theorem are complementary and could not hold at the same time (except for trivial cases), but one should note that these conditions are going to be applied to two separate cases with distinct subgroups (the subgroup condition is applied to a "kernel" in the complex, whereas the quotient condition is used for the preceding "image").

Our first task is to show that $G = \mathrm{Ker}(\bigoplus_{L-M=0} d^s_{\mathcal{Q},\mathcal{A}}(\pi)_{L,M})$ and $H = \mathrm{Im}(\bigoplus_{L-M=1} d^s_{\mathcal{Q},\mathcal{A}}(\pi)_{L,M})$ satisfy the hypotheses of the first part of 2.9.

Theorem 2.10. *Let* $\pi = (Y, \psi, \pi_s, Z, \zeta, \pi_u)$ *be an irreducible s/u-bijective pair for the irreducible Smale space* (X, φ)*. We have*

$$(\bigoplus_{L-M=0} \mathcal{C}^s_{\mathcal{Q},\mathcal{A}}(\pi)_{L,M})^+ \cap \mathrm{Im}(\bigoplus_{L-M=1} d^s_{\mathcal{Q},\mathcal{A}}(\pi)_{L,M}) = \{0\}.$$

Our second task is to show that $G = \bigoplus_{L-M=0} \mathcal{C}^s_{\mathcal{Q},\mathcal{A}}(\pi)_{L,M}$ and $H = \mathrm{Ker}(\bigoplus_{L-M=0} d^s_{\mathcal{Q},\mathcal{A}}(\pi)_{L,M})$ satisfy the hypotheses of the second part of 2.9.

Theorem 2.11. *Let* $\pi = (Y, \psi, \pi_s, Z, \zeta, \pi_u)$ *be an irreducible s/u-bijective pair for the irreducible Smale space* (X, φ)*. The subgroup generated by*

$$(\bigoplus_{L-M=0} \mathcal{C}^s_{\mathcal{Q},\mathcal{A}}(\pi)_{L,M}))^+ \cap \mathrm{Ker}(\bigoplus_{L-M=0} d^s_{\mathcal{Q},\mathcal{A}}(\pi)_{L,M})$$

is $\mathrm{Ker}(\bigoplus_{L-M=0} d^s_{\mathcal{Q},\mathcal{A}}(\pi)_{L,M})$.

As an immediate consequence of Theorems 2.9, 2.11 and 2.10, we get our main result as follows.

Theorem 2.12. *Let (X,φ) be an irreducible Smale space and $\pi = (Y,\psi,\pi_s,Z,\zeta,\pi_u)$ be an irreducible s/u-bijective pair for (X,φ). Then $H^s_0(\pi)$ is an ordered abelian group with the positive cone defined in 2.8.*

The next issue is to see that the resulting order is independent of the choice of π, in a suitable sense.

Theorem 2.13. *Suppose (X,φ) is an irreducible Smale space, and*

$$\pi = (Y,\psi,\pi_s,Z,\zeta,\pi_u), \quad \tilde{\pi} = (\tilde{Y},\tilde{\psi},\tilde{\pi}_s,\tilde{Z},\tilde{\zeta},\tilde{\pi}^u)$$

are s/u-bijective pairs for (X,φ). Assume that π is irreducible. Then

1. *$H^s_0(\tilde{\pi})$ is an ordered abelian group with the positive cone defined in 2.8,*
2. *There is an order isomorphism $\mathcal{H}$ from $H^s_0(\pi)$ to $H^s_0(\tilde{\pi})$.*

We also want to show that our order structure behaves well as a functor. Already in [14], the functoriality for the groups alone is somewhat subtle; H^s is covariant for s-bijective factor maps and contravariant for u-bijective factor maps. We will show that the maps induced at the level of groups from s-bijective factor maps and u-bijective factor maps between the dynamical systems are positive group homomorphisms.

Theorem 2.14. *Suppose (X,φ) and (X',φ') are irreducible Smale spaces.*

1. *If $\rho : (X,\varphi) \to (X',\varphi')$ is an s-bijective factor map. Then the group homomorphism $\rho^s_0 : H^s_0(X,\varphi) \to H^s_0(X',\varphi')$ of [14] is positive; that is, we have*
$$\rho^s_0(H^s_0(X,\varphi)^+) \subseteq H^s_0(X',\varphi')^+.$$
2. *If $\rho : (X,\varphi) \to (X',\varphi')$ is a u-bijective factor map. Then the group homomorphism $\rho^{s*}_0 : H^s_0(X',\varphi') \to H^s_0(X,\varphi)$ of [14] is positive; that is, we have*
$$\rho^{s*}_0(H^s_0(X',\varphi')^+) \subseteq H^s_0(X,\varphi)^+.$$

A couple of remarks are in order. All of our results are stated for irreducible Smale spaces. They extend easily to Smale spaces in which every point is non-wandering, since any such Smale space is the disjoint union of a finite number of irreducible subsystems.

The ordered groups introduced by Krieger have a number of special features. They are unperforated: if, for any element a, na is positive, for some $n \geq 1$, then a itself is positive. They also satisfy the Riesz interploation property (see [7] for details). At this point, it is not clear exactly which nice properties our ordered groups $H^s_0(X,\varphi)$ may have. However, one may observe, using [1], that they may have elements of finite order, which means that they are not unperforated in general. It may be reasonable to expect them to be weakly unperforated: if $na > 0$, for some $n \geq 1$, then $a > 0$.

3. Dimension Groups and the Perron-Frobenius Theorem

3.1. **Shifts of finite type.** Shifts of finite type are usually defined in terms of the alphabets and (forbidden) words, but here we use an equivalent formulation in terms of graphs, which is more suitable for our purposes.

A graph G consists of finite sets G^0 and G^1, consisting of vertices and edges, respectively, and maps $i, t : G^1 \to G^0$, marking the initial and terminal points. The graph is drawn by depicting each vertex as a dot and each edge e as an arrow from $i(e)$ to $t(e)$.

A path of length k in G is a sequence $(e_1, ..., e_k)$, with $e_i \in G^1$, for $1 \leq i \leq k$, such that $t(e_i) = i(e_{i+1})$, for $1 \leq i < k$. Let G^k denote the set of all paths of the length k. For each k, G^k is a graph with vertices G^{k-1} and edges G^k, and the initial and terminal maps

$$i(e_1, ..., e_k) = (e_1, ..., e_{k-1}),\ t(e_1, ..., e_k) = (e_2, ..., e_k),$$

for $(e_1, \ldots, e_k)$ in G^k. To any graph G, a pair (Σ_G, σ) is associated, where

$$\begin{aligned} \Sigma_G \quad &= \quad \{\ (e_n)_{n\in\mathbb{Z}} \mid e_n \in G^1, t(e_n) = i(e_{n+1}), n \in \mathbb{Z}\}, \\ \sigma : \Sigma_G \to \Sigma_G \quad &; \quad \sigma(e)_n = e_{n+1}. \end{aligned}$$

This is a dynamical system with the following metric

$$d(e, f) = \inf\{1, 2^{-K-1} \mid K \geq 0,\ e_{[1-K,K]} = f_{[1-K,K]}\}$$

on the Σ_G, where $e_{[K,L]} = (e_K, e_{K+1}, \ldots, e_L)$, for $K \leq L$, and $e_{[K+1,K]} = t(e_K) = i(e_{K+1})$.

It is easy to see that (Σ_G, σ) is an Smale space with constants $\varepsilon_X = \lambda = \frac{1}{2}$ and

$$[e, f]_k = \begin{cases} f_k & k \leq 0 \\ e_k & k \geq 1. \end{cases}$$

The system (Σ_G, σ) is called the shift of finite type associated to the graph G.

3.2. **Dimension groups.** In [11], Krieger defines two ordered groups in terms of the clopen sets, for the shift of finite type, called the past and future dimension groups.

Suppose (Σ, σ) is a shift of finite type and $\Sigma^s(e)$ is the stable equivalence class of $e \in \Sigma$. By Proposition 2.1.12 in [14], the set $\Sigma^s(e)$ admits a topology that is second countable and locally compact. This may be different from the relative topology of Σ. Let $CO(\Sigma,\sigma)$ be the set of non-empty, open and compact subsets of $\Sigma^s(e)$, over all e in Σ, and $\sim$ be the smallest equivalence relation on $CO(\Sigma,\sigma)$ such that $E \sim F$ if $[E,F] = E$ and $[F,E] = F$ and $E \sim F$ if and only if $\sigma(E) \sim \sigma(F)$, and let $[E]$ denote the equivalence class of E.

Let $\mathcal{D}^s(\Sigma,\sigma)$ be the free abelian group on $\sim$-equivalence classes of $CO^s(\Sigma,\sigma)$ and and H be the subgroup generated by $[E\cup F]-[E]-[F]$, where E, F and $E\cup F$ are in $CO^s(\Sigma,\sigma)$ and E and F are disjoint. The group $D^s(\Sigma,\sigma)$ is defined to be $\mathcal{D}^s(\Sigma,\sigma)/H$.

The order is obtained by defining

$$\mathcal{D}^s(\Sigma,\sigma)^+ = \{[E] \mid E \in CO^s(\Sigma,\sigma)\},$$

and then

$$D^s(\Sigma,\sigma)^+ = \{a + H \mid a \in \mathcal{D}^s(\Sigma,\sigma)^+\}.$$

The ordered abelian group $D^u(\Sigma,\sigma)$ is defined in a similar way, by replacing the unstable equivalence classes $\Sigma^u(e)$ by $\Sigma^s(e)$.

Krieger showed how this ordered group could be computed from the underlying graph of the shift of finite type.

Before going into more details, we need some notation. Let A is a finite set, then the free abelian group generated by A, $\mathbb{Z}A$, is an ordered abelian group with the positive cone $\{z_1a_1 + \ldots + z_na_n \mid z_1,\ldots,z_n \in \mathbb{Z}^+\cup\{0\}, a_1,\ldots,a_n \in A, n \in \mathbb{N}\}$. In our notation, A is considered as a subset of $\mathbb{Z}A$.

If A , B are finite sets and $\tau : A \to B$ is any function, then there is a unique positive homomorphism $\Gamma : \mathbb{Z}A \to \mathbb{Z}B$ extending τ.

For the finite set A, the integer-valued bilinear form $\langle,\rangle$ is defined on $\mathbb{Z}A \times \mathbb{Z}A$ which is additive in each variable, and for each $a,b \in A$,

$$\langle a,b\rangle = \begin{cases} 1 & if \quad a = b, \\ 0 & if \quad a \neq b \end{cases}$$

For two finite sets A, B, and homomorphism $h : \mathbb{Z}A \to \mathbb{Z}B$, there is a unique homomorphism $h^* : \mathbb{Z}B \to \mathbb{Z}A$ such that

$$\langle h(a), b\rangle = \langle a, h^*(b)\rangle,$$

for all a in $\mathbb{Z}A$ and b in $\mathbb{Z}B$.

Let $A = \{a_1,...,a_m\}$ and $B = \{b_1,...,b_n\}$. We associate a matrix $[h_{ij}]_{n\times m}$ to the homomorphism h such that the entry h_{ij} is equal with the coefficient b_j in $h(a_i)$ when $h(a_i)$ is written in terms of the generating set B. We have

$$\langle h(a), b\rangle = \langle a, h^*(b)\rangle,$$

for a in $\mathbb{Z}A$ and b in $\mathbb{Z}B$, that is, $[h^*_{ij}]_{m\times n} = ([h_{ij}]_{n\times m})^T$, where M^T denotes the transpose of a matrix M.

Now we compute the dimension group in terms of the underlying graph of the shift of finite type.

Let (G^0, G^1, i, t) be a graph and (Σ_G, σ) be the associated shift of finite type. Suppose $\mathbb{Z}G^0$ is the free abelian group on the generating set G^0 and consider the homomorphism

$$\gamma_G^s : \mathbb{Z}G^0 \to \mathbb{Z}G^0; \quad \gamma_G^s(v) = \sum_{t(e)=v} i(v) \quad (v \in G^0).$$

The past dimension group $D^s(G)$ is defined as the inductive limit of the system

$$\mathbb{Z}G^0 \xrightarrow{\gamma_G^s} \mathbb{Z}G^0 \xrightarrow{\gamma_G^s} \dots .$$

Since $\mathbb{Z}G^0$ is an ordered group and γ_G^s is a positive homomorphism, $D^s(G)$ inherits an order structure in a natural way. Let us give a brief and simple description of the elements of $D^s(G)$. Two points (a, m) and (b, n) in $\mathbb{Z}G^0 \times \mathbb{N}$ are equivalent, denoted $(a, m) \sim_s (b, n)$, if there exists $l \in \mathbb{N} \cup \{0\}$ such that $(\gamma_G^s)^{n+l}(a) = (\gamma_G^s)^{m+l}(b)$. The equivalence class of (a, m) is denoted by $[a, m]_s$ and $D^s(G)$ is the set of all equivalence classes. The positive cone in this group consists of those elements $[a, m]_s$ with $(\gamma_G^s)^l(a) \in (\mathbb{Z}G^0)^+$, for some l *in* $\mathbb{N}$.

The future dimension group for the graph (G^0, G^1, i, t) is defined in a similar way, by replacing the homomorphism $\gamma_G^u : \mathbb{Z}G^0 \to \mathbb{Z}G^0$ by γ_G^s, where

$$\gamma_G^u(v) = \Sigma_{i(e)=v} t(v),$$

for all v in G^0. Note that $\gamma_G^u = (\gamma_G^s)^*$.

It is worth noting that in some places in the computation of the homology, it is necessary to use the graph G^k instead of G, which does not effect the answer. This can be viewed as a consequence of the next theorem. The next two result appear as Theorems 3.3.3 and 3.5.5 in [14], but without the order structure.

Theorem 3.1. *Suppose G is a graph, (Σ_G, σ) is the associated shift of finite type and $k \geq 1$. The homomorphism Ψ from $D^s(\Sigma_G, \sigma)$ to $D^s(G^k)$, defined on the generating elements by $\Psi([\Sigma_G^s(e, 2^{-j})]) = [e^{[1-j, k-j-1]}, j-k+1]$, $e \in \Sigma_G$, $j \geq k$, is an order isomorphism.*

We recall some notation from section 3.1 of [14], that if B is any subset of A, $Sum(B) = \sum_{b \in B} b \in \mathbb{Z}A$.

Theorem 3.2. *Let G and H be graphs with a graph homomorphism $\pi : H \to G$ and suppose that the associated map $\pi : (\Sigma_H, \sigma) \to (\Sigma_G, \sigma)$ is s-bijective, $k \geq 1$, and K satisfies the conclusion of Lemma 2.7.1 in [14] for π. The induced map $\pi^s[a, j] = [\pi^{s,K}(a), j]$ from $D^s(H^k)$ to*

$D^s(G^{k+K})$ is a positive homomorphism, where $a \in \mathbb{Z}H^{k-1}$, $j \geq 1$ and $\pi^{s,K}(q) = Sum\{\pi(q') \mid q' \in H^{k+K}, t^K(q') = q\}$.

3.3. **The Perron-Frobenius Theorem.** Let G be a finite directed graph. The adjacency matrix, A_G, is $\#G^0 \times \#G^0$ whose entries are the number of edges between the different vertices of G. The shift of finite type (Σ_G, σ) is irreducible if and only if the graph G is irreducible, in the sense that, for each ordered pair of vertices u and v in G, there exists a path p in G starting at u and terminating at v. This is also equivalent to the adjacency matrix being irreducible, in the sense that, for each ordered pair of indices i, j, there is some non-negative integer n such that $(A_G)^n_{i,j} > 0$.

The shift of finite type (Σ_G, σ) is mixing if and only if there is a positive integer n such that, for every ordered pair of vertices u and v in G, there exists a path of length n in G starting at u and terminating at v. This is also equivalent to the adjacency matrix being primitive; that is, there is some positive integer n such that $(A_G)^n_{i,j} > 0$, for all $1 \leq i, j \leq m$. If this holds for some fixed n, it also holds for all higher values of n [12].

Let us recall the Perron-Frobenius Theorem (Theorem 4.2.3 of [12]). If A is a non-negative irreducible square matrix, then it has a positive eigenvalue λ_A and a right positive eigenvector $\mathbf{v}_A$ associated to λ_A, called the Perron eigenvalue and the Perron eigenvector, respectively, such that $|\mu| \leq \lambda_A$, for every eigenvalue μ of A, and the corresponding eigenspace of λ_A is both geometrically and algebraically simple.

Given our presentation using homomorphisms rather than matrices, we state this in the following fashion. We apply this to both the adjacency matrix for the graph and its transpose, but these share the same Perron eigenvalue. Assuming that the graph G is irreducible, there is $\lambda_G > 0$ and vectors v^s_G, v^u_G in $\mathbb{R}^+G^0$ such that

$$\gamma^s_G(v^s_G) = \lambda_G v^s_G, \gamma^u_G(v^u_G) = \lambda_G v^u_G.$$

We have extended the definition of γ^s_G, γ^u_G in the obvious way. We remark that if we replace G by G^k, for some $k \geq 1$, we obtain a higher block presentation of the shift (see Definition 1.4.1 of [12]). The Perron eigenvectors are changed, but not the eigenvalue: $\lambda_{G^k} = \lambda_G$.

The Perron eigenvalue in the above result is related to the notion of entropy as the below result shows. This could be defined for a general dynamical system, but here we only deal with the shifts of finite type. Let G be a graph and (Σ_G, σ) be the corresponding shift of finite type.

The entropy of (Σ_G, σ) is defined (4.4.1 of [12]) by

$$h(\Sigma_G, \sigma) = \lim_{n\to\infty} \frac{1}{n} \log \#G^n$$

where $\#G^n$ is the number of paths of length n in G.

Theorem 3.3 (Theorem 4.4.4 [12])**.** *If G is a graph, then we have $h(\Sigma_G, \sigma) = log(\lambda_G)$.*

The Perron-Frobenius Theorem also has a nice application for the computation of the order structure of $D^s(\Sigma_G, \sigma)$, particularly in the mixing case. This follows from Corollary 4.2 and Theorem 6.1 of [7].

Theorem 3.4. *Let G be a finite directed graph whose associated shift of finite type is mixing. For any $n \geq 1$ and a in $\mathbb{Z}G^0$, the element $[a, n]$ is in $D^s(G)^+ - \{0\}$ if and only if $< a, v_G^u >$ is positive.*

We end this section with a result which gives a sufficient condition for the surjectivity of maps between shifts of finite type.

Theorem 3.5 (Corollary 4.4.9 [12])**.** *Suppose G and H are graphs and $\pi : (\Sigma_G, \sigma) \to (\Sigma_H, \sigma)$ is a finite-to-one map. If the graph H is irreducible and $h(\Sigma_G, \sigma) = h(\Sigma_H, \sigma)$, then π is onto.*

4. Irreducible s/u-bijective pairs and fibred products

4.1. **Irreducible bijective pairs.** The proof of the existence of s/u-bijective pairs comes from [15]. Our proof of the existence of irreducible ones must go back to the same starting point to see how the results of [15] can be improved.

Suppose (X, φ) and (Y, ψ) are irreducible Smale spaces and $\pi : (X, \varphi) \to (Y, \psi)$ is an almost one-to-one map. In [15], it is shown that that there exist irreducible Smale spaces $(\tilde{X}, \tilde{\varphi})$, $(\tilde{Y}, \tilde{\psi})$ and factor maps α, β, $\tilde{\pi}$ such that the following diagram is commutative.

$$\begin{array}{ccc} (\tilde{X}, \tilde{\varphi}) & \xrightarrow{\tilde{\pi}} & (\tilde{Y}, \tilde{\psi}) \\ \alpha \downarrow & & \downarrow \beta \\ (X, \varphi) & \xrightarrow{\pi} & (Y, \psi) \end{array} \tag{4.1}$$

Moreover, the maps α, β are u-bijective and the map $\tilde{\pi}$ is s-bijective. Regretably, it is not shown that $\alpha, \beta, \tilde{\pi}$ are almost one-to-one, which is what we undertake now. In fact, it will be enough to consider β. (The space (X, φ) is appearing in a somewhat unfortunate position as the domain, but we follow [15] for the moment.)

The proof involves finding a periodic point y_0 in Y with $\pi^{-1}\{y_0\} = \{x_0\}$, a single point in X. Then W is the unstable set of the orbit of x_0 and it is shown that $\pi(W)$ is the unstable set of the orbit of

y_0. Let d_X, d_Y be the metrics on X and Y, respectively. We view X and Y as the completions of the space (W, d_X) an d $(\pi(W), d_Y)$. The proof of [15] involves introduing new metrics on W and $\pi(W)$, δ_X and δ_Y, respectively so that $\tilde{X}$ and $\tilde{Y}$ are their completions. As these new metrics are greater than or equal to the old ones, the factor maps α, β appear automatically.

Here, we claim that $\beta^{-1}\{y_0\} = \{x_0\}$. (The references here will all be to [15].) To see this, it suffices to consider a sequence y_n in $\pi(W)$ which is Cauchy is δ_Y and converges to y_0 in d_Y and prove that it converges to y_0 in δ_Y. For n sufficiently large, $[y_0, y_n]$ is defined and using part 4 of 2.18, we have

$$\begin{aligned}\delta_Y(y_0, y_n) &\leq \delta_Y(y_0, [y_0, y_n]) + \delta_Y([y_0, y_n], y_n) \\ &\leq \delta_Y(y_0, [y_0, y_n]) + (1 - r\lambda)^{-1} d_Y([y_0, y_n], y_n).\end{aligned}$$

It suffices for us to show that $[y_0, y_n]$ converges to y_0 in δ_Y. By replacing y_n by $[y_0, y_n]$, we may assume that y_n is in $V^s(y_0, \epsilon_Y)$. By part 2 of 2.12, we may assume that y_n and y_0 are ρ-compatible and then by 2.10, for all $k \geq 0$, there is $N_k \geq 1$, such that $g^{-k}(y_0)$ and $g^{-k}(y_n)$ are ρ-compatible for $n \geq N_k$.

Let $\epsilon > 0$ be given. From the definition of δ_Y^0 in 2.14, it is bounded by D. We may find $K \geq 1$ such that $\sum_{k>K} r^k D < \epsilon/2$. Find $N \geq \max\{N_k \mid 1 \leq k \leq K\}$ so that, for $n \geq N$ and $0 \leq k \leq K$, we have

$$d_Y(g^{-k}(y_0), g^{-k}(y_n) < \frac{\epsilon}{2(K+1)}.$$

It follows from 2.17 and part 4 of 2.15 that for such n,

$$\begin{aligned}\delta_Y(y_0, y_n) &= \sum_{k=0}^{\infty} r^k \delta_Y^0(g^{-k}(y_0), g^{-k}(y_n)) \\ &\leq \sum_{k=0}^{K} d_Y(g^{-k}(y_0), g^{-k}(y_n)) + \sum_{k=K+1}^{\infty} r^k D \\ &< \sum_{k=0}^{K} \frac{\epsilon}{2(K+1)} + \frac{\epsilon}{2} \\ &= \epsilon.\end{aligned}$$

Exactly as in [15], we apply this result as follows. We begin with our irreducible Smale space (X, φ) and find an irreducible shift of finite type (Σ, σ) and an almost one-to-one factor map $\pi : (\Sigma, \sigma) \to (X, \varphi)$. The system which is called $(\tilde{Y}, \tilde{g})$ above, we denote by (Z, ζ) and the map β by π_u. The fact that Z has totally disconnected stable sets follows from the facts that $\tilde{\Sigma}$ is also a shift of finite type and $\tilde{\pi}$ is s-bijective. Now, we also know that there is x_0 in X with $\#\pi_u^{-1}\{x_0\} = 1$.

We next want to show that if there is a single point x with $\#\pi^{-1}\{x\} = 1$, this will also hold for all points with dense forward or backward orbit if we also assume that π is s-bijective or u-bijective. Recall that the forward orbit of a point x is $\{\varphi^n(x) \mid n \geq 0\}$, while the backward orbit is $\{\varphi^n(x) \mid n \leq 0\}$.

Lemma 4.1. *Let (Y, ψ) and (X, φ) be Smale spaces and $\pi : (Y, \psi) \to (X, \varphi)$ be an s-bijective (or u-bijective) factor map. Assume there is x_0 in X such that $\pi^{-1}\{x_0\} = 1$. Then for any point x in X with a dense forward (backward) orbit, we have $\#\pi^{-1}\{x\} = 1$.*

Proof. We prove result in the case that π is s-bijective. List $\pi^{-1}\{x\} = \{y_1, ..., y_I\}$. Since the orbit of x is dense, we may find an increasing sequence of positive integers n_k such that $\varphi^{n_k}(x)$ converges to x_0. Passing to a subsequence, we may assume that for each $1 \leq i \leq I$, the sequence $\psi^{n_k}(y_i)$ converges to some point of Y, and by continuity, these points must all lie in $\pi^{-1}(x_0)$. It remains to show that no such two sequences can have the same limit. If there is $1 \leq i \neq j \leq I$, then $d(\psi^{n_k}(y_i), \psi^{n_k}(y_j))$ tends to zero, as k goes to infinity. Then we have

$$\pi(\psi^{n_k}(y_i)) = \varphi^{n_k}(\pi(y_i)) = \varphi^{n_k}(x) = \varphi^{n_k}(\pi(y_j)) = \pi(\psi^{n_k}(y_i)).$$

Using the fact that π is s-bijective, Proposition 2.5.2 in [14] implies that, for k sufficiently large,

$$\psi^{n_k}(y_i) \in Y^u(\psi^{n_k}(y_j), \epsilon_\pi),$$

which implies that

$$y_i \in Y^u(y_j, \lambda^{n_k}\epsilon_\pi).$$

Since this is true for all k, we conclude $y_i = y_j$, and we are done. □

The set of points with a dense forward orbit is rather large in an irreducible system. The following result is standard; see, for example, Theorem 5.9 of [18].

Proposition 4.2. *Let (X, φ) be a dynamical system, with X a compact metric space. If (X, φ) is irreducible, then the set of all points x with dense forward orbit is a dense G_δ subset of X.*

It is probabaly worth noting that Lemma 4.1 and Proposition 4.2 together prove the following.

Corollary 4.3. *Let (Y, ψ) and (X, φ) be Smale spaces and $\pi : (Y, \psi) \to (X, \varphi)$ be an s-bijective (or u-bijective) factor map. Then π is almost one-to-one if and only if there is a point x_0 in X such that $\#\pi^{-1}\{x_0\} = 1$.*

We have also now proved Theorem 2.3, that every irreducible Smale space has an irreducible s/u-bijective pair.

4.2. **The fibred product of maps.** Let $\pi_1 : (Y, \psi) \to (X, \varphi)$ and $\pi_2 : (Z, \zeta) \to (X, \varphi)$ be maps between Smale spaces and

$$Y \times_X Z = \{(y, z) \in Y \times Z \mid \pi_1(y) = \pi_2(z)\}$$

be the fibred product of π_1, π_2, with the relative topology of $Y \times Z$. By Theorem 2.4.2 in [14], $Y \times_X Z$ is an Smale space, with $\psi \times \zeta(y, z) = (\psi(y), \zeta(z))$, for $(y, z) \in Y \times_X Z$. We also note that there are natural maps $\rho_2 : (Y \times_X Z, \psi \times \zeta) \to (Z, \zeta)$ defined by $\rho_2(y, z) = z$ and $\rho_1 : (Y \times_X Z, \psi \times \zeta) \to (Y, \psi)$ defined by $\rho_1(y, z) = y$. We note that if π_1 is s-bijective (or u-bijective), then so is ρ_2.

The drawback is that the fibred product of maps on irreducible Smale spaces is not irreducible in general. In this section, we prove the irreducibility of the fibred product $(Y \times_X Z, \psi \times \zeta)$ under certain natural conditions.

Proposition 4.4. *Let $\pi_1 : (Y, \psi) \to (X, \varphi)$ and $\pi_2 : (Z, \zeta) \to (X, \varphi)$ be either s-bijective or u-bijective, almost one-to-one factor maps between irreducible Smale spaces. Then the natural maps ρ_1 and ρ_2 be from the fibred product to Y and Z, respectively, are also almost one-to-one.*

Proof. The set of x in X with $\#\pi_1^{-1}\{x\} = 1$ is a dense G_δ, as is the set of x with $\#\pi_2^{-1}\{x\} = 1$. It follows that their intersection is non-empty. If x is in this intersection and $\pi_1(y) = x, \pi_2(z) = x$, it is a simple matter to see that

$$\rho_2^{-1}\{z\} = \{(y, z)\} = \rho_1^{-1}\{y\}.$$

Noting that ρ_1 and ρ_2 are also either s-bijective or u-bijective and Corollary 4.3, this completes the proof. □

We will need two technical results for the proof of Theorem 2.6. The first is a characterization of irreducibility.

Lemma 4.5. *Let (X, φ) be a Smale space. If there exists a point x in X whose forward orbit clusters on every periodic point of X, then (X, φ) is irreducible.*

Proof. Let y be an accumulation point of the backward orbit of x. It is clearly non-wandering and so it is in the closure of the periodic points. It follows that y is also a limit point of the forward orbit of x. By patching the forward orbit of x that gets close to y with part of the backward orbit of x that begins close to y, we can form pseudo-orbits from x to itself and conclude that x is in the non-wandering set. The orbit of x will remain in the same irreducible component of the non-wandering set. Hence all periodic points are in the same irreducible component. This implies that there is only one irreducible component. If X contained a wandering point, its forward orbit and backward orbits would limit on two distinct irreducible components. As this is not possible, X has no wandering points. □

Lemma 4.6. *Let $\pi_s : (Y, \psi) \to (X, \varphi)$ be an s-bijective almost one-to-one factor map between irreducible Smale spaces and (X, φ) and (Y, ψ). Let x_0 be a periodic point of X with $\pi^{-1}\{x_0\} = \{y_1, y_2, ..., y_I\}$. For $\delta, \epsilon > 0$, put $U = [X^u(x_0, \delta), X^s(x_0, \delta)]$ and, for $1 \leq i \leq I$, let*

$$V_i = \{x \in U \mid \pi^{-1}\{[x, x_0]\} \subseteq Y(y_i, \epsilon)\},$$

where $Y(y_i, \epsilon)$ denotes the open ball at y_i of radius ϵ. Then there exist arbitrarily small positive pairs δ, ϵ such that

(i) *V_i is open,*
(ii) *V_i is non-empty,*
(iii) *$[V_i, U] \subset V_i$.*

Proof. First choose ϵ to be smaller than ϵ_π and also smaller than half of the distance between y_i and y_j, over all $1 \leq i \neq j \leq I$. Then choose $\delta > 0$ so that Lemma 2.5.11 of [14] holds. It follows easily from the continuity of the bracket and Lemma 2.5.9 of [14] that V_i is open for all i. Let us next fix i and prove that V_i is non-empty. By hypothesis, there exists a point x' with dense forward orbit and $\#\pi^{-1}\{x'\} = 1$. Notice that any point in the orbit of x' also has these properties, as does any point stably equivalent to a point in the orbit of x'. Let y' in Y be the unique point with $\pi(y') = x'$. Since (Y, ψ) is irreducible, the stable equivalence class of the orbit of y' is dense. So there exists y'' stably equivalent to some point in the orbit of y' in $Y^u(y_i, \epsilon)$. Let us check that $\pi(y'')$ is in V_i. As $\epsilon < \epsilon_\pi$, we know $[\pi(y''), x_0] = \pi([y'', y_i]) = \pi(y'')$ and hence $\pi^{-1}\{[\pi(y''), x_0] = \{y''\}\}$ is in $Y(y_i, \epsilon)$. Finally, we verify that last condition. Suppose that x is in V_i and x_1 is in $X^u(x_0, \delta)$ and x_2 in $X^u(x_0, \delta)$. Since $[x, [x_1, x_2]] = [x, x_2]$ is in V_i, $[[x, x_2], x_0] = [x, x_0]$ and the conclusion follows. □

We have most of the ingredients for the proof of Theorem 2.6, but for the last statement, we need some convenient characterizations of mixing.

Lemma 4.7. *Suppose (X, φ) is an irreducible Smale space. The following are equivalent.*

(1) *(X, φ) is mixing*

(2) *For any periodic point x in X, we have $X^s(x) \cap X^u(\varphi(x)) \neq \emptyset$ and $X^u(x) \cap X^s(\varphi(x)) \neq \emptyset$.*
(3) *For some periodic point x in X, we have $X^s(x) \cap X^u(\varphi(x)) \neq \emptyset$ and $X^u(x) \cap X^s(\varphi(x)) \neq \emptyset$.*

Proof. This is a consequence of the Smale's spectral decomposition. Let $\sim$ be the equivalence relation on the periodic points of (X, φ) in the Smale's spectral decomposition, that is, for two periodic points $x, y \in X$, $x \sim y$, if and only if $X^s(x) \cap X^u(y) \neq \emptyset$ and $X^u(x) \cap X^s(y) \neq \emptyset$. Then there are pairwise disjoint clopen sets $X_1, \ldots, X_N$ whose union is X, $\varphi(X_i) = \varphi(X_{i+1})$ for $1 \leq i \leq N-1$, $\varphi(X_N) = X_1$ and (X_i, φ^N) is a mixing Smale space, for every $1 \leq i \leq N$. Moreover each X_i is the closure of an equivalence class of $\sim$ and these sets are unique up to relabeling.

If we assume that (X, φ) is mixing, then N above must equal 1 and the second condition holds. The second part obviously implies the third. Finally, if x is periodic point, so is $\varphi(x)$. Suppose $x \in X_i$, for some $1 \leq i \leq N-1$, $X^s(x) \cap X^u(\varphi(x)) \neq \emptyset$ and $X^u(x) \cap X^s(\varphi(x)) \neq \emptyset$. Then $x \sim \varphi(x)$, thus $\varphi(x) \in X_i \cap X_{i+1}$. Since X_i's are pairwise disjoint, $X_i = X_{i+1}$. The same argument shows that $X_i = X_{i+1} = \ldots = X_N$. Similarly, if $x \in X_N$, then $X_N = X_1 = \ldots = X_{N-1}$. Therefore, $N = 1$, hence $X = X_i$ and (X, φ) is a mixing space. □

Proposition 4.8. *If $\pi : (Y, \psi) \to (X, \varphi)$ is an almost one-to-one factor map between Smale space, (Y, ψ) is irreducible and (X, φ) is mixing, then (Y, ψ) is mixing also.*

Proof. We will verify the condition of the last Lemma. Suppose y is in Y and x is in X such that $\pi^{-1}\{x\} = \{y\}$. Since (X, φ) is mixing, it is irreducible and hence by Proposition 2.3 in [15], we can find a periodic point $x_0 \in X$ with $\#\pi^{-1}\{x_0\} = 1$. Let $y_0 \in Y$ with $\pi(y_0) = x_0$. Since x_0 is periodic and π is finite-to-one, y_0 is a periodic point. By the argument of the proof of Lemma 2.4 in [15], $\pi^{-1}(X^s(x_0)) = Y^s(y_0)$, $\pi^{-1}(X^s(\varphi(x_0))) = Y^s(\psi(y_0))$, $\pi^{-1}(X^u(x_0)) = Y^u(y_0)$ and $\pi^{-1}(X^u(\varphi(x_0))) = Y^u(\psi(y_0))$. Since (X, φ) is a mixing Smale space, we have

$$X^s(x_0) \cap X^u(\varphi(x_0)) \neq \emptyset, \qquad X^u(x_0) \cap X^s(\varphi(x_0)) \neq \emptyset$$

that implies

$$\begin{aligned}Y^s(y_0) \cap Y^u(\psi(y_0)) &= \pi^{-1}(X^s(x_0)) \cap \pi^{-1}(X^u(\varphi(x_0))) \\ &= \pi^{-1}(X^s(x_0) \cap X^u(\varphi(x_0))) \neq \emptyset, \\ Y^u(y_0) \cap Y^s(\psi(y_0)) &= \pi^{-1}(X^u(x_0)) \cap \pi^{-1}(X^s(\varphi(x_0))) \\ &= \pi^{-1}(X^u(x_0) \cap X^s(\varphi(x_0))) \neq \emptyset.\end{aligned}$$

Therefore, by the Lemma 4.7, (Y, ψ) is mixing. □

Proof. (Theorem 2.6) The sets of points of X with dense forward and backward orbits are both dense G_δ's and so their intersection is non-empty. Let x be a point in X with a dense forward orbit and a dense backward orbit. Let y and z be its unique pre-images under π_s and π_u, respectively. By Lemma 4.5, it suffices to prove that the forward orbit of (y, z) clusters on every periodic point. Let (y_1, z_1) be a periodic point in the fibred product. Let $x_1 = \pi_s(y_1) = \pi_u(z_1)$. Enumerate $\pi_s^{-1}\{x_1\} = \{y_1, ..., y_I\}$ and $\pi_u^{-1}\{x_1\} = \{z_1, ..., z_J\}$.

For small δ, ϵ, let V_i, $1 \leq i \leq I$ and W_j, $1 \leq j \leq J$ be the result of applying Lemma 4.6 to the maps π_s and π_u, respectively. Observe that since π_u is u-bijective, the last condition on W_j is $[U, W_j] \subseteq W_j$. We have

$$V_1 \cap W_1 \supseteq [V_1, U] \cap [U, W_1] \supseteq [V_1, W_1],$$

which is clearly non-empty. Also $V_1 \cap W_1$ is open. It follows that there is $n \geq 1$ with $\varphi^n(x) \in V_1 \cap W_1$. This implies that $\psi^n(y) \in Y(y_1, \epsilon)$ and $\zeta^n(z) \in Z(z_1, \epsilon)$. Since ϵ was arbitrary, this completes the proof of the first part. The mixing case follows from two applications of Proposition 4.8. □

5. Homology

In this section, we prove the main results on the homology of Smale spaces, stated in the first section. Suppose (X, φ) is a Smale space, then so is (X, φ^n), for any positive integer n, and if $\pi = (Y, \psi, \pi_s, Z, \zeta, \pi_u)$ is an s/u-bijective pair for (X, φ), then $\pi_n = (Y, \psi^n, \pi_s, Z, \zeta^n, \pi_u)$ is an s/u-bijective pair for (X, φ^n). The results in chapters 4 and 5 of [14] show that $(C_{\mathcal{Q},\mathcal{A}}(\Sigma(\pi)), d_{\mathcal{Q},\mathcal{A}}(\Sigma(\pi)))$ and $(C_{\mathcal{Q},\mathcal{A}}(\Sigma(\pi_n)), d_{\mathcal{Q},\mathcal{A}}(\Sigma(\pi_n)))$ admit the same cocycle and coboundaries. On the other hand, by Smale's spectral decomposition, for every irreducible Smale space, (X, φ), X can be written as a union of pairwise disjoint clopen subsets $X_1, ..., X_L$ such that $\varphi^L(X_i) = X_i$, for each $1 \leq i \leq L$, and (X_i, φ^L)'s are mixing Smale spaces [17]. Hence

$$H^s_N(X,\varphi) \cong H^s_N(X,\varphi^L) \cong \bigoplus_{i=1}^{L} H^s_N(X_i,\varphi^L),$$

for any positive integer N, and this along with Theorem 2.13 allows us to replace an irreducible Smale space by a mixing one.

Under the assumption that (X,φ) is mixing, we find π, an irreducible s/u-bijective pair for (X,φ). It follows at once from Propositions 4.8, 4.4 and Theorem 2.6, that $(\Sigma_{0,0}(\pi),\sigma) = (Y \times_X Z, \psi \times \zeta)$ is mixing and ρ_s and ρ_u are almost one-to-one.

We start with two lemmas that are simpler versions of Theorems 2.11, 2.10. Both of these consider the following situation: a shift of finite type (Σ,σ), a Smale space (Y,ψ) and a factor map $\rho : (\Sigma,\sigma) \to (Y,\psi)$ which is either s-bijective or u-bijective. In Chapter 4 of [14], a complex is formed from such a map. It is a simpler object than the double complex associated to an s/u-bijective pair, but its importance lies in the fact that the individual rows and columns of the double complex all arise in this fashion. Applying this to our map $\rho_s : (\Sigma_{0,0}(\pi),\sigma) \to (Y,\psi)$ yields the bottom row of our double complex. Similarly, applying this to our map $\rho_u : (\Sigma_{0,0}(\pi),\sigma) \to (Z,\zeta)$ yields the left column of our double complex.

To a factor map ρ as above, we let

$$\Sigma_N(\rho) = \{(x_0,\ldots,x_N) \in \Sigma^{N+1} \mid \rho((x_0) = \ldots = \rho(x_N)\},$$

for all $N \geq 0$. There are obvious maps $\delta_n : \Sigma_N(\rho) \to \Sigma_{N-1}(\rho)$, for $0 \leq n \leq N$ and $N \geq 1$.

Lemma 5.1. *Let (Σ,σ) be a mixing shift of finite type, (Y,ψ) be a mixing Smale space and $\rho_s : (\Sigma,\sigma) \to (Y,\psi)$ be an s-bijective, almost one-to-one factor map. Then* $\mathrm{Im}(\delta^s_0 - \delta^s_1) \cap D^s(\Sigma_0(\rho_s))^+ = \{0\}$.

Proof. We begin by finding a graph G whose associated shift (Σ_G,σ) is conjugate to (Σ,σ). (We suppress the conjugacy in our notation.) From Theorem 4.2.8 in [14], this G may be chosen so that the map ρ_s is regular. (The definition of regular is given in 2.3.3 of [14]. We will not really need it here, but we will indicate where it is used shortly.)

If (x_0,x_1) is in $\Sigma_1(\rho_s)$, then x_0 and x_1 are bi-infinite paths in G and if we take their 0-th entries we obtain a pair in $G^1 \times G^1$. We let G^1_1 be the set of all such pairs over all (x_0,x_1) in $\Sigma_1(\rho_s)$ and G^0_1 be the image of this set under $t \times t$. Then G_1 is a graph with obvious i,t maps. The significance of our choice that ρ_s is regular is that

$$\Sigma_{G_1} = \Sigma_1(\rho_s).$$

(The elements of the set on the left are infinite sequences of pairs of edges of G, while those on the right are pairs of infinite sequences of edges of G, but we feel no confusion will arise from equating the two.)

It is clear from letting $x_0 = x_1$ that G_1^i contains all pairs (a, a) where a is in G^i. We denote this subgraph by G_1^Δ. As ρ_s is s-bijective, any edge in G_1 which terminates in G_1^Δ must actually be in G_1^Δ.

Let G_1' consist of those vertices not in G_1^Δ and all edges whose initial vertex is not in G_1^Δ. This is a graph and its infinite path space $\Sigma_{G_1'}$ maps to Σ_G by δ_0. If this map is surjective, then every point of Y has at least two distinct pre-images under ρ_s, contrary to our hypothesis. Using Theorem 3.5, we conclude that

$$\log(\lambda_{G_1'}) = h(\Sigma_{G_1'}, \sigma)) = h(\delta_0(\Sigma_{G_1'}, \sigma)) < h(\Sigma_G, \sigma) = \log(\lambda_G).$$

It follows that there is a constant C such that $\#(G_1')^j \leq C(\lambda_{G_1'})^j$, for all $j \geq 1$.

Following the discussion prior to Theorem 4.2.13 of [14], for $k \geq 0$, we choose B_1^k to be a subset of of G_1^k which contains no paths of the form (p_0, p_1) if $p_0 = p_1$ and for $p_0 \neq p_1$, it contains exactly one of (p_0, p_1) and (p_1, p_0). Following Theorem 4.2.13 of [14], for any $k \geq 0$, $j \geq 1$, p in B_1^k, we let

$$t^*_{\mathcal{A}}(p, j) = \{(q, \alpha) \in G_1^{k+j} \times S_2 \mid t^j(q) = p, i^j(q) \cdot \alpha \in B_1^k\}.$$

The point here is that any path q with $i^j(q) = p \in B_1^k \subseteq (G_1')^k$ must lie entirely in G_1'. It is then clear that $\#t^*_{\mathcal{A}}(p, j) \leq C(\lambda_{G_1'})^{j+k}$. The map $\gamma^s_{B_1^k} : \mathbb{Z}B_1^k \to \mathbb{Z}B_1^k$ is defined just before 4.2.13 of [14]. We conclude from the first part of 4.2.13 of [14] that if $\eta : \mathbb{Z}B_1^{k-1} \to \mathbb{R}$ is any group homomorphism and a is in $\mathbb{Z}B_1^{k-1}$, then there is a constant D (depending on a) such that
$\eta((\gamma^s_{B_1})^j(a)) < D(\lambda_{G_1'})^j$, for all $j \geq 1$.

Consider the following diagram

$$\begin{array}{ccc}
\mathbb{Z}B_1^k & \xrightarrow{\gamma^s_{B_1^k}} & \mathbb{Z}B_1^k \\
{\scriptstyle Q}\downarrow & & \downarrow{\scriptstyle Q} \\
\mathcal{Q}(G_1^k, S_2) & \xrightarrow{\gamma^s_{G_1^k}} & \mathcal{Q}(G_1^k, S_2) \\
{\scriptstyle \delta_0^{s,K} - \delta_1^{s,K}}\downarrow & & \downarrow{\scriptstyle \delta_0^{s,K} - \delta_1^{s,K}} \\
\mathbb{Z}G^{k+K} & \xrightarrow{\gamma^s_{G^{k+K}}} & \mathbb{Z}G^{k+K}.
\end{array}$$

The second part of 4.2.13 of [14] tells us that the top square commutes and that the vertical maps are isomorphisms. The bottom square commutes by Theorem 4.2.3, Definition 4.2.4 and Theorem 4.2.5 of [14].

We consider $\eta(\cdot) =< (\delta_0^{s,K} - \delta_1^{s,K}) \circ Q(\cdot), v^u_{G^{k+K}} >$, where $v^u_{G^{k+K}}$ is the Perron eigenvector for $\gamma^u_{G^{k+K}}$. It follows that, for any a in $\mathbb{Z}B_1^k$, there is D such that

$$\begin{aligned} D(\lambda_{G_1'})^j &\geq \eta((\gamma^s_{B_1^k})^j(a)) \\ &= < (\delta_0^{s,K} - \delta_1^{s,K}) \circ Q((\gamma^s_{B_1^k})^j(a)), v^u_{G^{k+K}} > \\ &= < (\gamma^s_{G^{k+K}})^j(\delta_0^{s,K} - \delta_1^{s,K}) \circ Q(a)), v^u_{G^{k+K}} > \\ &= < (\delta_0^{s,K} - \delta_1^{s,K}) \circ Q(a)), (\gamma^u_{G^{k+K}})^j(v^u_{G^{k+K}}) > \\ &= \lambda_G^j < (\delta_0^{s,K} - \delta_1^{s,K}) \circ Q(a)), v^u_{G^{k+K}} > . \end{aligned}$$

As $0 < \lambda_{G_1'} < \lambda_G$, the conclusion we draw from this is that $< (\delta_0^{s,K} - \delta_1^{s,K}) \circ Q(a)), v^u_{G^{k+K}} >$ is not positive. This implies that $(\delta_0^{s,K} - \delta_1^{s,K}) \circ Q(a)$ is not in $D^s(G^k)^+ - \{0\}$. This holds for every a in $\mathbb{Z}B_1^k$, but as Q is an isomorphism, we also see that $\mathrm{Im}(\delta_0^{s,K} - \delta_1^{s,K}) \cap D^s(G^k)^+ = \{0\}$. The conclusion follows. □

Lemma 5.2. *Let (Σ, σ) be a mixing shift of finite type, (Z, ζ) be a mixing Smale space and $\rho_u : (\Sigma, \sigma) \to (Z, \zeta)$ be a u-bijective, almost one-to-one factor map. Then the subgroup generated by $\mathrm{Ker}(\delta_0^{s*} - \delta_1^{s*}) \cap D^s(\Sigma_0(\rho_u))^+$ is $\mathrm{Ker}(\delta_0^{s*} - \delta_1^{s*})$.*

Proof. First, suppose that we have a strictly positive element a in $(\mathbb{Z}G^{k+K})^+$ such that $(\delta_0^{s*,K} - \delta_1^{s*,K})(a) = 0$. Then $[a, j] \in Ker(\delta_0^{s*} - \delta_1^{s*})$, for every j in $\mathbb{N}$. It follows that every $[b, j]$ in $\mathrm{Ker}(\delta_0^{s*} - \delta_1^{s*})$ can be written as the difference elements

$$[b, j] = [b + na, j] - [na, j],$$

in which $n \in \mathbb{N}$. It is a simple consequence of Theorem 3.4 that we may chose n large enough such that $b + na \in (\mathbb{Z}G_{0,0}^{k+K})^+$. This means $\mathrm{Ker}(\delta_0^{s*} - \delta_1^{s*}) \cap D^s(\Sigma_0(\rho_u))^+$ generates $Ker(\delta_0^{s*} - \delta_1^{s*})$.

In order to obtain the element a as above, let us return to the proof of Lemma 5.1, using ρ_u and replacing s with u throughout. We now consider the diagram

$$\begin{array}{ccc} \mathbb{Z}B_1^k & \xrightarrow{\gamma^u_{B_1^k}} & \mathbb{Z}B_1^k \\ \downarrow J & & \downarrow J \\ \mathcal{A}(G_1^k, S_2) & \xrightarrow{\gamma^u_{G_1^k}} & \mathcal{A}(G_1^k, S_2) \\ \downarrow \delta_0^{u,K} - \delta_1^{u,K} & & \downarrow \delta_0^{u,K} - \delta_1^{u,K} \\ \mathbb{Z}G^{k+K} & \xrightarrow{\gamma^u_{G^{k+K}}} & \mathbb{Z}G^{k+K} \end{array}$$

The third part of 4.2.13 of [14] tells us that the top square commutes and that the vertical maps are isomorphisms. The bottom square commutes by Theorem 4.2.3, Definition 4.2.4 and Theorem 4.2.5 of [14].

The same argument as given earlier shows that $< (\delta_0^{u,K} - \delta_1^{u,K})(a), v^s_{G^{k+K}} >$ is not positive, for every a in $\mathcal{A}(G_1^k, S_2)$. But this also applies to $-a$ and it follows that

$$0 =< (\delta_0^{u,K} - \delta_1^{u,K})(a), v^s_{G^{k+K}} >$$

for every a. Then by Lemma 3.5.6 of [14] (where there is a typo, switching $s*$ and $u*$), we get

$$0 =< a, (\delta_0^{s*,K} - \delta_1^{s*,K})(v^s_{G^{k+K}}) >$$

for every a. It follows that $(\delta_0^{s*,K} - \delta_1^{s*,K})(v^s_{G^{k+K}}) = 0$. If $v^s_{G^{k+K}}$ had integer entries, we would be done.

If we view $(\delta_0^{s*,K} - \delta_1^{s*,K})$ as a linear map, the condition above means that it has a non-trivial kernel. That kernel has a basis and since the transformation has matrix with integer entries, we can obtain a basis for the kernel consisting of rational vectors. We know that $v^s_{G^{k+K}}$ is a positive vector and it also must be a linear combination of the rational basis for the kernel. If we carefully choose rational scalars, we may find a rational vector, also in the kernel, and sufficiently close to $v^s_{G^{k+K}}$ so that all its entries are positive. If we then multiply by a suitable integer, we find a positive integer vector $a \in \mathbb{Z}G^{k+K}$ in the kernel of $(\delta_0^{s*,K} - \delta_1^{s*,K})$. This completes the proof. □

Proof. (Theorems 2.11 and 2.10) Consider the fibred product $\Sigma_{0,0}(\pi)$ of maps π_s and π_u, and let G be a presentation of π. Since (X, φ) is mixing, so is $\Sigma_{0,0}(\pi)$, by 2.6. From Theorem 5.1.4 of [14], the bottom row in our double complex is the same as the complex for the map ρ_s while the first column is the same as the complex for the map ρ_u. Now the two theorems follow from Lemmas 5.1 and 5.2, respectively. □

Suppose π and $\tilde{\pi}$ are the s/u-bijective pairs given in Theorem 2.13. It is shown in [14] that the homology of Smale spaces is independent of the corresponding s/u-bijective pair. This is done in Section 4.5 of [14], where an isomorphism is found between the homology of the rows of the complexes

$$(\oplus_{L-M=N}(C^s_{\mathcal{Q},\mathcal{A}}(\pi)_{L,M}, \oplus_{L-M=N}(d^s_{\mathcal{Q},\mathcal{A}}(\pi)_{L,M})$$

and

$$(\oplus_{L-M=N}(C^s_{\mathcal{Q},\mathcal{A}}(\tilde{\pi})_{L,M}, \oplus_{L-M=N}(d^s_{\mathcal{Q},\mathcal{A}}(\tilde{\pi})_{L,M}),$$

and then using Theorem 3.9 of [13], it is extended to an isomorphism between the homologies of the complexes

$$(\oplus_{L-M=N}(C^s_{\mathcal{Q},\mathcal{A}}(\pi)_{L,M}, \oplus_{L-M=N}(d^s_{\mathcal{Q},\mathcal{A}}(\pi)_{L,M})$$

and

$$(\oplus_{L-M=N}(C^s_{\mathcal{Q},\mathcal{A}}(\tilde{\pi})_{L,M}, \oplus_{L-M=N}(d^s_{\mathcal{Q},\mathcal{A}}(\tilde{\pi})_{L,M}).$$

We use these isomorphisms to show that $H^s_0(\tilde{\pi})$ is an ordered group with the positive cone defined in 2.8, and that, these are indeed ordered isomorphisms.

Let us first remind the reader that there is a minor mistake in the statement of Theorem 3.5.11 in [14] used to prove the independence and functorial properties of the homology for Smale spaces (see Sections 5.4 and 5.5 in [14]). Deeley and co-authors proved that the surjectivity condition in this theorem must be replaced by the conjugacy condition [6]. It follows that we also need the conjugacy condition in Theorem 5.4.1 in [14]. Here we state the correct versions of these results from [6].

Theorem 5.3. *Suppose that*

$$\begin{array}{ccc} (\Sigma,\sigma) & \xrightarrow{\eta_1} & (\Sigma_1,\sigma) \\ \eta_2\downarrow & & \downarrow\pi_1 \\ (\Sigma_2,\sigma) & \xrightarrow{\pi_2} & (\Sigma_0,\sigma) \end{array}$$

is a commutative diagram of non-wandering shifts of finite type, in which η_1 and π_2 are s-bijective factor maps, and η_2 and π_1 are u-bijective factor maps. If $\eta_2\times\eta_1 : (\Sigma,\sigma)\to(\Sigma_2,\sigma)\ {}_{\pi_2}\times_{\pi_1}(\Sigma_1,\sigma)$ is a conjugacy, then

$$\eta^s_1\circ\eta^{s^*}_2 = \pi^{s^*}_1\circ\pi^s_2 : D^s(\Sigma_2,\sigma)\to D^s(\Sigma_1,\sigma). \tag{5.1}$$

Theorem 5.4. *Let $\pi = (Y,\psi,\pi_s,Z,\zeta,\pi_u)$ and $\pi' = (Y',\psi',\pi'_s,Z',\zeta',\pi'_u)$ be s/u-bijective pairs for the Smale spaces (X,φ) and (X',φ'), respectively, and $\eta = (\eta_Y,\eta_X,\eta_Z)$ be a triple of factor maps such that the following diagram commutes:*

$$\begin{array}{ccccc} (Y,\psi) & \xrightarrow{\pi_s} & (X,\varphi) & \xleftarrow{\pi_u} & (Z,\zeta) \\ \eta_Y\Big\downarrow & & \eta_X\Big\downarrow & & \eta_Z\Big\downarrow \\ (Y',\psi') & \xrightarrow{\pi'_s} & (X',\varphi') & \xleftarrow{\pi'_u} & (Z',\zeta'). \end{array}$$

(i) *If η is a triple of s-bijective maps and*

$$\pi_u\times\eta_Z : (Z,\zeta)\to(X,\varphi)\ {}_{\eta_X}\times_{\pi'_u}(Z',\zeta')$$

is a conjugacy, then for $L \geq 0, M \geq 1$,

$$\begin{array}{ccc} (\Sigma_{L,M}(\pi)) & \stackrel{\eta_{L,M}}{\longrightarrow} & (\Sigma_{L,M}(\pi')) \\ \delta_{,m}\downarrow & & \delta'_{,m}\downarrow \\ (\Sigma_{L,M-1}(\pi)) & \stackrel{\eta_{L,M-1}}{\longrightarrow} & (\Sigma_{L,M-1}(\pi')), \end{array}$$

and for $L \geq 1, M \geq 0$,

$$\begin{array}{ccc} (\Sigma_{L,M}(\pi)) & \stackrel{\eta_{L,M}}{\longrightarrow} & (\Sigma_{L,M}(\pi')) \\ \delta_{l,}\downarrow & & \delta'_{l,}\downarrow \\ (\Sigma_{L-1,M}(\pi)) & \stackrel{\eta_{L-1,M}}{\longrightarrow} & (\Sigma_{L-1,M}(\pi')) \end{array}$$

are commutative diagrams and

$$\eta_{L,M} \times \delta_{,m} : (\Sigma_{L,M}(\pi)) \to (\Sigma_{L,M}(\pi')) \ {}_{\delta'_{,m}} \times_{\eta_{L,M-1}} (\Sigma_{L,M-1}(\pi))$$

is a conjugacy. Moreover, η *induces chain maps between the complexes* $C^s_{\mathcal{Q},\mathcal{A}}(\pi)$ *and* $C^s_{\mathcal{Q},\mathcal{A}}(\pi')$, *and hence group homomorphisms* $\eta^{s*} : H^s_N(\pi) \to H^s_N(\pi')$, *for every integer* N.

(ii) *If* η *is a triple of u-bijective maps and*

$$\pi_s \times \eta_Y : (Y, \psi) \to (X, \varphi) \ {}_{\eta_X} \times_{\pi'_s} (Y', \psi')$$

is a conjugacy, then for $L \geq 0, M \geq 1$,

$$\begin{array}{ccc} (\Sigma_{L,M}(\pi)) & \stackrel{\eta_{L,M}}{\longrightarrow} & (\Sigma_{L,M}(\pi')) \\ \delta_{,m}\downarrow & & \delta'_{,m}\downarrow \\ (\Sigma_{L,M-1}(\pi)) & \stackrel{\eta_{L,M-1}}{\longrightarrow} & (\Sigma_{L,M-1}(\pi')), \end{array}$$

and for $L \geq 1, M \geq 0$,

$$\begin{array}{ccc} (\Sigma_{L,M}(\pi)) & \stackrel{\eta_{L,M}}{\longrightarrow} & (\Sigma_{L,M}(\pi')) \\ \delta_{l,}\downarrow & & \delta'_{l,}\downarrow \\ (\Sigma_{L-1,M}(\pi)) & \stackrel{\eta_{L-1,M}}{\longrightarrow} & (\Sigma_{L-1,M}(\pi')) \end{array}$$

are commutative diagrams and

$$\eta_{L,M} \times \delta_{l,} : (\Sigma_{L,M}(\pi)) \to (\Sigma_{L,M}(\pi')) \ {}_{\delta'_{l,}} \times_{\eta_{L-1,M}} (\Sigma_{L-1,M}(\pi))$$

is a conjugacy. Moreover, η *induces chain maps between the complexes* $C^s_{\mathcal{Q},\mathcal{A}}(\pi')$ *and* $C^s_{\mathcal{Q},\mathcal{A}}(\pi)$, *and hence group homomorphisms* $\eta^{s*} : H^s_N(\pi') \to H^s_N(\pi)$, *for every integer* N.

We remark that the results obtained in [14] (the independence and functorial properties) are all correct, because the diagrams constructed there satisfy the conjugacy condition.

By Theorem 3.2, both maps $\eta_1^s \circ \eta_2^{s*}$ and $\pi_1^{s*} \circ \pi_2^s$ in 5.1 are positive homomorphisms.

Theorem 5.5. *For graphs G, H, suppose $\theta : H \to G$ is a left-covering graph homomorphism, (X, φ) is a Smale space and $\rho : (\Sigma_G, \sigma) \to (X, \varphi)$ is a regular s-bijective factor map. The map θ induces an isomorphism between the homologies of the chain complexes $(D^s(\Sigma_*(\rho\circ\theta)), d^s(\rho\circ\theta))$ and $(D^s(\Sigma_*(\rho), d^s(\rho))$.*

In fact, the map θ induces homomorphisms θ_*^s at all levels of the complexes with the following commutative diagram:

$$\begin{array}{ccc} D^s(H_N^k) & \overset{d^s(\rho\circ\theta)_N}{\longrightarrow} & D^s(H_{N-1}^{k+K}) \\ \theta_N^s \downarrow & & \downarrow \theta_{N-1}^s \\ D^s(G_N^k) & \overset{d^s(\rho)_N}{\longrightarrow} & D^s(G_{N-1}^{k+K}), \end{array}$$

for each $N \geq 1$ and $k \geq 0$, where $K \geq 1$ satisfies the conclusion of Lemma 2.7.2 in [14], for the map ρ.

To show that the induced homomorphism on the homology of the above complexes by θ_*^s, is an isomorphism, one could choose a lifting map $\lambda : G^0 \to H^0$ with $\theta \circ \lambda = Id_{G_0}$. Then Lemma 4.5.4 in [14] shows that, for each $N \geq 0$,

$$d^{s,K}(\rho\circ\theta)_N \circ \lambda = \lambda \circ d^{s,K}(\rho)_N. \tag{5.2}$$

We claim that

$$\theta^s(\mathrm{Ker}(d^s(\theta\circ\rho)_N) \cap (D^s(H_N^k))^+) = \mathrm{Ker}(d^s(\rho)_N) \cap (D^s(G_N^k))^+,$$

$$\theta^s(\mathrm{Im}(d^s(\theta\circ\rho)_{N+1}) \cap (D^s(H_N^{k+K}))^+) = \mathrm{Im}(d^s(\rho)_{N+1}) \cap (D^s(G_N^{k+K}))^+.$$

By Theorem 3.2,

$$\theta^s(\mathrm{Ker}(d^s(\theta\circ\rho)_N) \cap (D^s(H_N^k))^+) \subseteq \mathrm{Ker}(d^s(\rho)_N) \cap (D^s(G_N^k))^+$$

and

$$\theta^s(\mathrm{Im}(d^s(\theta\circ\rho)_{N+1}) \cap (D^s(H_N^{k+K}))^+) \subseteq \mathrm{Im}(d^s(\rho)_{N+1}) \cap (D^s(G_N^{k+K}))^+.$$

Suppose that $b \in \mathbb{Z}G_{N+1}^k$ and $j \geq 0$, with $d^s(\rho)_{N+1}([b,j])$ in $D^s(G_N^{k+K})^+$. By Theorem 4.2.3 in [14],

$$d^s(\rho)_{N+1}([b,j]) = [d^{s,K}(\rho)_{N+1}(b), j] \in D^s(G_N^{k+K})^+,$$

which implies that, for some $j' \geq 0$,

$$(\gamma_{G_N}^s)^{j'}(d^{s,K}(\rho)_{N+1}(b)) \in (\mathbb{Z}G_N^{k+K})^+.$$

By Theorem 4.2.3 in [14],

$$(\gamma^s_{G_N})^{j'}(d^{s,K}(\rho)_{N+1}(b)) = d^{s,K}(\rho)_{N+1}((\gamma^s_{G_N})^{j'}(b)) \in (\mathbb{Z}G_N^{k+K})^+.$$

Let $b_1 = (\gamma^s_{G_N})^{j'}(b)$ and $j_1 = j' + j$. Then

$$[d^{s,K}(\rho)_{N+1}(b), j] = [d^{s,K}(\rho)_{N+1}(b_1), j_1],$$

and since $\lambda((\mathbb{Z}G^k_{N+1})^+) \subseteq (\mathbb{Z}H^k_{N+1})^+$, it follows from 5.2 that,

$$d^{s,K}(\rho \circ \theta)_{N+1} \circ \lambda(b_1) = \lambda \circ d^{s,K}(\rho)_{N+1}(b_1) \in (\mathbb{Z}H_N^{k+K})^+.$$

Let $a_1 = \lambda(b_1)$. Applying $\theta^{s,0} = \theta$ to both sides of the above equality,

$$\theta^{s,0}(d^{s,K}(\rho \circ \theta)_{N+1}(a_1)) = \theta^{s,0}(\lambda \circ d^{s,K}(\rho)_{N+1}(b_1)) = d^{s,K}(\rho)_{N+1}(b_1),$$

hence

$$[\theta^{s,0}(d^{s,K}(\rho \circ \theta)_{N+1}(a_1)), j_1] = [d^{s,K}(\rho)_{N+1}(b_1), j_1] = [d^{s,K}(\rho)_{N+1}(b), j],$$

and so

$$\theta^s(d^s(\rho)_{N+1}[a_1, j_1]) = d^s(\rho)_{N+1}([b, j]).$$

Since b is an arbitrary element in $\mathbb{Z}G^k_{N+1}$ with

$$d^s(\rho)_{N+1}([b, j]) \in D^s(G_N^{k+K})^+,$$

the last equality implies

$$\mathrm{Im}(d^s(\rho)_{N+1}) \cap (D^s(G^{k+K}))^+ \subseteq \theta^s(\mathrm{Im}(d^s(\theta \circ \rho)_{N+1}) \cap (D^s(H^{k+K}))^+).$$

A similar argument shows that

$$\mathrm{Ker}(d^s(\rho)_N) \cap (D^s(G^k))^+ \subseteq \theta^s(\mathrm{Ker}(d^s(\theta) \circ \rho)_N \cap (D^s(H^k))^+).$$

Combining Theorems 3.1, 5.5, with Theorem 4.5.3 in [14], we get the following result.

Theorem 5.6. *Suppose (X, φ) is a Smale space and (Σ, σ), (Σ', σ) are shifts of finite type with s-bijective maps $\rho : (\Sigma, \sigma) \to (X, \varphi)$ and $\rho' : (\Sigma', \sigma) \to (X, \varphi)$. Let (Y'', ψ'') be the fibred product of maps $\rho : (\Sigma, \sigma) \to (X, \varphi)$ and $\rho' : (\Sigma', \sigma) \to (X, \varphi)$, and η, η' be the natural s-bijective maps from (Y'', ψ'') to (Y, ψ) and (Y', ψ'), respectively. Then*

(i) *There is a chain map η^s from $(D^s(\Sigma_N(\rho \circ \eta)), d^s(\rho \circ \eta)_N)$ to $(D^s(\Sigma_N(\rho)), d^s(\rho)_N)$ such that*

$$\eta^s(\mathrm{Ker}(d^s(\rho \circ \eta)_N) \cap D^s(\Sigma_N(\rho \circ \eta))^+) = \mathrm{Ker}(d^s(\rho)_N \cap D^s(\Sigma_N(\rho))^+),$$

$$\eta^s(\mathrm{Im}(d^s(\rho \circ \eta)_N) \cap D^s(\Sigma_N(\rho \circ \eta))^+) = \mathrm{Im}(d^s(\rho)_N \cap D^s(\Sigma_N(\rho))^+).$$

(ii) *There is a chain map $\boldsymbol{C}'$ from $(D^s(\Sigma_N(\rho' \circ \eta')), d^s(\rho' \circ \eta')_N)$ to $(D^s(\Sigma_N(\rho')), d^s(\rho')_N)$ such that*

$$\eta'^s(\mathrm{Ker}(d^s(\rho' \circ \eta')_N) \cap D^s(\Sigma_N(\rho' \circ \eta'))^+) = \mathrm{Ker}(d^s(\rho')_N \cap D^s(\Sigma_N(\rho'))^+),$$

$$\eta'^s(\mathrm{Im}(d^s(\rho' \circ \eta')_N) \cap D^s(\Sigma_N(\rho' \circ \eta'))^+) = \mathrm{Im}(d^s(\rho')_N \cap D^s(\Sigma_N(\rho'))^+).$$

(iii) *η^s and η'^s induce isomorphisms at the level of the associated homologies of the chain complexes.*

As in Section 5.5 of [14], we prove Theorem 2.13 in the case $Z = \tilde{Z}$, $\zeta = \tilde{\zeta}$ and $\pi_u = \tilde{\pi_u}$. The cases $Y = \tilde{Y}$, $\psi = \tilde{\psi}$, and $\pi_s = \tilde{\pi_s}$ is proved in a similar way, and the general result follows from these two special cases.

Let (Y', ψ') denote the fibred product of the maps $\pi_s : (Y, \psi) \to (X, \varphi)$, $\tilde{\pi_s} : (\tilde{Y}, \tilde{\psi}) \to (X, \varphi)$, and η', $\tilde{\eta}'$ denote the the natural s-bijective maps from (Y', ψ') to (Y, ψ) and $(\tilde{Y}, \tilde{\psi})$, respectively. Then $\pi' = (Y', \psi', \pi_s \circ \eta', Z, \zeta, \pi_u)$ is an s/u-bijective pair for the Smale space (X, φ), and the following diagram is commutative:

$$\begin{array}{ccccc} (Y', \psi') & \xrightarrow{\pi_s \circ \eta} & (X, \varphi) & \xleftarrow{\pi_u} & (Z, \zeta) \\ \eta' \big\downarrow & & Id_X \big\downarrow & & \big\downarrow Id_Z \\ (Y, \psi) & \xrightarrow{\pi_s} & (X, \varphi) & \xleftarrow{\pi_u} & (Z, \zeta) \end{array} \tag{5.3}$$

This diagram satisfies the conditions of the first part of Theorem 5.4, and the triple of s-bijective $\eta = (\eta', Id_X, Id_Z)$ induces a chain map on the double complexes used to define $H^s_N(\pi')$ and $H^s_N(\pi)$, $N \in \mathbb{Z}$. Since $D^s_{\mathcal{Q},\mathcal{A}}(\Sigma(\pi), \sigma) = D^s(\Sigma(\pi), \sigma)$, $D^s_{\mathcal{Q},\mathcal{A}}(\Sigma(\pi'), \sigma) = D^s(\Sigma(\pi'), \sigma)$, and $\eta^s_{L,M}$, $\eta^{u*}_{L,M}$ are positive homomorphisms, by Theorem 3.2, we have

$$\begin{aligned} \eta^s_{0,0}(\mathrm{Ker}(d^s(\pi)_{0,0}) \cap D^s(\Sigma(\pi), \sigma)^+) &\subseteq \mathrm{Ker}(d^s(\pi')_{0,0}) \cap D^s(\Sigma(\pi'), \sigma)^+ \\ \eta^s_{0,0}(\mathrm{Im}(d^s(\pi)_{1,0}) \cap D^s(\Sigma(\pi))^+) &\subseteq \mathrm{Im}(d^s(\pi')_{1,0}) \cap D^s(\Sigma(\pi'), \sigma)^+ \end{aligned} \tag{5.4}$$

Let $H_N(\eta)$ be the induced homomorphism by the chain map $\eta^s_{*,*}$ at the level of homologies from $H_N(\pi')$ to $H^s_N(\pi)$. This is known to be an isomorphism. We claim that this is ordered isomorphism after proving that $H_N(\pi')$ is an ordered group. To prove that $H_N(\pi')$ is an ordered group, it suffice to show that inclusions in 5.4 are indeed equalities.

To prove that $H_N(\eta)$ is an isomorphism, for $N \in \mathbb{Z}$, one need to consider the filterations $F^p C^s_{\mathcal{Q},}(\pi')$ and $F^p C^s_{\mathcal{Q},}(\pi)$ for the differential graded abelian groups $(H^s(\pi'), d^s_{\mathcal{Q},\mathcal{A}}(\pi'))$ and $(H^s(\pi), d^s_{\mathcal{Q},\mathcal{A}}(\pi))$, respectively, as in Section 5.5 of [14]. These filterations satisfy the conditions of Theorem 3.9 in [13]. According to this theorem, every isomorphism Φ_1 between $E^{*,*}_1$ terms of the associated spectral sequences (of these filtration differential graded modules) induces an isomorphism Φ_∞ between E_∞ terms of the associated spectral sequences (roughly, $\Phi_\infty(a) = \Phi_1(a)$, when we regard a as an element of the associated $E^{*,*}_1$ term). The isomorphisms $H_N(\eta)$ is then constructed using the isomorphisms between the E_∞ terms, for $N \in \mathbb{N}$. The $E^{*,*}_1$ terms, for each of these filterations, are the homologies of the rows of the corresponding complexes, that is,

$$E_1^{p,q}(\pi) = \mathrm{Ker}(\tilde{d}^s(\rho_{,M})_L)/\mathrm{Im}(\hat{d}^s(\rho_{,M})_{L+1}),$$

and the same for π', where

$$\tilde{d}^s(\rho_{,M})_L = d^s(\rho_{,M})_L \mid_{\bigoplus_{L\geq 2p+q, M=p} C^s_{\mathcal{Q},}(\pi)}$$

and

$$\hat{d}^s(\rho_{,M})_{L+1} = d^s(\rho_{,M})_{L+1} \mid_{\bigoplus_{L\geq 2p+q+1, M=p} C^s_{\mathcal{Q},}(\pi)} .$$

Since the maps θ and ρ_u in $\Sigma(\pi') \xrightarrow{\theta} \Sigma(\pi) \xrightarrow{\rho_u} (Z,\zeta)$ are s-bijective, where $\theta((y,\tilde{y}),z) = (y,z)$, by Theorem 5.6, we have a chain map θ^s from $(C^s(\pi')_{*,M}, d^s((\rho_u\circ\theta)_{,M})_*)$ to $(C^s(\pi)_{*,M}, d^s((\rho_u)_{,M})_*)$, that induces an isomorphism H_θ at the level of homologies of the complexes, for fixed $M\geq 0$, so that

$$\theta^s(\mathrm{Ker}(d^s(\rho_{,M})_L)\cap(C^s(\pi')_{L,M})^+) = \mathrm{Ker}(d^s(\rho_{,M})_L)\cap(C^s(\pi)_{L,M})^+,$$

and

$$\theta^s(\mathrm{Im}(d^s(\rho_{,M})_{L+1})\cap(C^s(\pi')_{L,M})^+) = \mathrm{Im}(d^s(\rho_{,M})_{L+1})\cap(C^s(\pi)_{L,M})^+,$$

for each $L\geq 0$, and fixed $M\geq 0$.

If one lifts H_θ at the level of homologies of the complexes

$$(C^s_{\mathcal{Q},}(\pi')_{*,M}, d^s_{\mathcal{Q},}((\rho_u\circ\theta)_{,M})_*),\quad (C^s_{\mathcal{Q},}(\pi)_{*,M}, d^s_{\mathcal{Q},}((\rho_u)_{,M})_*),$$

by the first part of Theorem 4.3.1 in [14], for fixed $M\geq 0$, since

$$C^s_{\mathcal{Q},}(\pi')_{0,0} = C^s(\pi')_{0,0} = D^s(\Sigma_{0,0}(\pi'),\sigma),$$

and

$$C^s_{\mathcal{Q},}(\pi_0)_{0,0} = C^s(\pi)_{0,0} = D^s(\Sigma_{0,0}(\pi),\sigma),$$

for

$$K^s(\pi) := \mathrm{Ker}(d^s(\rho_{,0})_0)\cap(D^s(\Sigma_{0,0}(\pi'),\sigma))^+),$$

and

$$I^s(\pi) := \mathrm{Im}(d^s(\rho_{,0})_0)\cap(D^s(\Sigma_{0,0}(\pi'),\sigma))^+),$$

we have

$$C_\theta(K^s(\pi')) = K^s(\pi),\quad C_\theta(I^s(\pi')) = I^s(\pi').$$

In fact, H_θ is an isomorphism between the terms $E_1^{*,*}(\pi')$ and $E_1^{*,*}(\pi)$. Therefore, Theorem 3.9 in [13] implies that there is an isomorphism $\mathcal{H}_\theta$ at the level of homologies of the complexes $(C^s_{\mathcal{Q},}(\pi'), d^s_{\mathcal{Q},}(\pi'))$ and $(C^s_{\mathcal{Q},}(\pi), d^s_{\mathcal{Q},}(\pi))$, that is constructed by the induced isomorphism H_∞ on $E_\infty^{*,*}$ terms with H_θ (roughly, $H_\infty(a) = H_\theta(a)$ when we regard $a\in E_\infty^{*,*}$ as an element of $E_1^{*,*}$). Since the isomorphism $\mathcal{H}_\theta$ is directly defined by H_∞ (or H_θ), it is the same as the induced homomorphism by the chain map $\eta^s_{\mathcal{Q},}$, where $\eta = (\eta', Id_X, Id_Z)$ is the triple s-bijective in the diagram 5.3 and $\eta^s_{\mathcal{Q},}$ exactly behaves like θ^s, when θ^s is considered as a map on the domain of $\eta^s_{\mathcal{Q},}$. On the other hand, since the maps u and $\bar{u}$ in the proof of Theorem 3.9 in [13], are natural and $D^s_{\mathcal{Q},}(\Sigma_{0,0},\sigma) = D^s(\Sigma_{0,0},\sigma)$, for

$$K^s_{\mathcal{Q},}(\pi) := \mathrm{Ker}(d^s_{\mathcal{Q},}(\pi')_{0,0}) \cap (D^s(\Sigma_{0,0}(\pi'),\sigma))^+,$$

and

$$I^s_{\mathcal{Q},}(\pi) := \mathrm{Im}(d^s_{\mathcal{Q},}(\pi')_{1,0}) \cap (D^s(\Sigma_{0,0}(\pi'),\sigma))^+$$

we have

$$\eta^s_{\mathcal{Q},}(K^s_{\mathcal{Q},}(\pi')) = K^s_{\mathcal{Q},}(\pi), \quad \eta^s_{\mathcal{Q},}(I^s_{\mathcal{Q},}(\pi')) = I^s_{\mathcal{Q},}(\pi). \tag{5.5}$$

Let $\mathcal{J}(\pi')$ and $\mathcal{J}(\pi)$ be the isomorphisms induced by the chain maps $J_{\mathcal{Q}}(\pi')$ and $J_{\mathcal{Q}}(\pi)$, as in Theorem 5.3.2 in [14], respectively. Then $H_0(\eta) = \mathcal{J}(\pi) \circ \mathcal{H}_\theta \circ \mathcal{J}(\pi')^{-1}$ and it is an isomorphism from $H^s_N(\pi')$ to $H^s_N(\pi)$, and since $D^s_{\mathcal{Q},\mathcal{A}}(\Sigma_{0,0},\sigma) = D^s(\Sigma_{0,0},\sigma)$, for

$$K^s_{\mathcal{Q},\mathcal{A}}(\pi) := \mathrm{Ker}(d^s_{\mathcal{Q},\mathcal{A}}(\pi)_{0,0}) \cap (D^s(\Sigma_{0,0}(\pi),\sigma))^+,$$

and

$$I^s_{\mathcal{Q},\mathcal{A}}(\pi) := \mathrm{Im}(d^s_{\mathcal{Q},\mathcal{A}}(\pi)_{1,0}) \cap (D^s(\Sigma_{0,0}(\pi),\sigma))^+,$$

we have

$$J_{\mathcal{Q}}(\pi)(K^s_{\mathcal{Q},\mathcal{A}}(\pi')) = K^s_{\mathcal{Q},}(\pi), \quad J_{\mathcal{Q}}(\pi)(I^s_{\mathcal{Q},\mathcal{A}}(\pi')) = I^s_{\mathcal{Q},}(\pi). \tag{5.6}$$

The equalities 5.5, 5.6 show that $\mathrm{Ker}(d^s_{\mathcal{Q},\mathcal{A}}(\pi')_{0,0})$ contains positive elements if and only if $\mathrm{Ker}(d^s_{\mathcal{Q},\mathcal{A}}(\pi)_{0,0})$ does so (and the same holds for $\mathrm{Im}(d^s_{\mathcal{Q},\mathcal{A}}(\pi')_{1,0})$ and $\mathrm{Im}(d^s_{\mathcal{Q},\mathcal{A}}(\pi)_{1,0})$). Since $\mathrm{Im}(d^s_{\mathcal{Q},\mathcal{A}}(\pi)_{1,0})$ does not contain any positive element, and $\mathrm{Ker}(d^s_{\mathcal{Q},\mathcal{A}}(\pi)_{0,0})$ contains at least one positive element, $\mathrm{Im}(d^s_{\mathcal{Q},\mathcal{A}}(\pi')_{1,0})$ could not contain any positive element and $\mathrm{Ker}(d^s_{\mathcal{Q},\mathcal{A}}(\pi')_{0,0}0$ contains at least one positive element, and these imply that $H^s_0(\pi')$ is an ordered group with the positive cone defined as above. Also by 5.5 and 5.6, $H(\eta)$ is an order isomorphism. Replacing $(\tilde{Y},\tilde{\psi})$ by (Y,ψ) in 5.3, we get that $H^s_0(\tilde{\pi})$ is an ordered group with the positive cone defined as in Definition 2.8 and $H(\tilde{\eta}) = H^s_N(\pi') \to H^s_N(\tilde{\pi})$ is an order isomorphism. Finally, $H_0(\tilde{\eta}) \circ H_0(\eta)^{-1}$ is an order isomorphism from $H^s_0(\pi)$ to $H^s_0(\tilde{\pi})$.

Proof. (Theorem 2.14) We only prove the first part. The other is proved in a similar way. By Theorem 4.2 in [6], we can find s/u-bijective pairs $\pi = (Y,\psi,\pi_s,Z,\zeta,\pi_u)$ and $\pi' = (Y',\psi',\pi'_s,Z',\zeta',\pi'_u)$ for Smale spaces (X,φ) and (X',φ'), respectively, and s-bijective maps η_Y and η_Z, such that the following diagram commutes:

$$\begin{array}{ccccc}
(Y,\psi) & \xrightarrow{\pi_s} & (X,\varphi) & \xleftarrow{\pi_u} & (Z,\zeta) \\
\eta_Y \big\downarrow & & \rho \big\downarrow & & \eta_Z \big\downarrow \\
(Y',\psi') & \xrightarrow{\pi'_s} & (X',\varphi') & \xleftarrow{\pi'_u} & (Z',\zeta'),
\end{array}$$

and $\pi_u \times \eta_Z : (Z,\zeta) \to (X,\varphi)\ {}_{\rho}\times_{\pi'_u} (Z',\zeta')$ is a conjugacy. Therefore, ρ induces a positive homomorphism $\rho_0^s : H_0^s(X,\varphi) \to H_0^s(X',\varphi')$, by Theorems 5.4, 3.2 and 2.13 and the order structure is independent of the s/u-bijective pair. □

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