Some classifiable groupoid C^*-algebras with prescribed K-theory

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Abstract

Given a simple, acyclic dimension group G_0 and countable, torsion-free, abelian group G_1, we construct a minimal, amenable, étale equivalence relation, R, on a Cantor set whose associated groupoid C^*-algebra, $C^*(R)$, is tracially AF, and hence classifiable in the Elliott classification scheme for simple, amenable, separable C^*-algebras, and with $K_*(C^*(R)) \cong (G_0, G_1)$.

1 Introduction

Over the past twenty-five years, there has been an enormous amount of truly remarkable work devoted to classifying simple, amenable, separable C^*-algebras by simple invariants which can be roughly described as K-theoretic. This is the so-called Elliott program.

Let us discuss a specific important result in this program which is of interest to us here. In [9], Huaxin Lin introduced the notion for a C^*-algebra to be tracially AF (or tracially approximately finite-dimensional). We recall the definition, in the case of a unital C^*-algebra, A: for any finite set $\mathcal{F}$ in A containing a non-zero element x_1, $\epsilon > 0$, $n \geq 1$ and a, a full element of A^+, there is a finite-dimensional C^*-subalgebra $F \subseteq A$ with unit p satisfying

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1. $\|px - xp\| < \epsilon$, for all x in $\mathcal{F}$,
2. $pxp \in_\epsilon F$, for all x in $\mathcal{F}$, and $\|px_1p\| > \|x_1\| - \epsilon$,
3. $n[1-p] \leq [p]$ in the sense of Murray and von Neumann and $1-p$ is equivalent to a projection in the hereditary subalgebra generated by a.

In [10], Lin proves that two unital, separable, simple, amenable, tracially AF C^* algebras satisfying the Universal Coefficient Theorem (see [2]) are isomorphic if and only if their K-zero groups are isomorphic as ordered abelian groups with order unit and their K-one groups are isomorphic.

One important aspect of the Elliott program is to understand the range of the invariant. In practical terms, this means specifying some K-theoretic data and constructing a C^*-algebra within the class having this data as its K-theory. The typical approach is to consider inductive limits of simpler C^*-algebras. These are usually formed from continuous functions on compact spaces, finite-dimensional C^*-algebras, tensor products of these and, finally, C^*-subalgebras of those.

This is quite natural in the sense that many of the approximation techniques are very well-suited to dealing with these simpler algebras and also with inductive limits. On the other hand, many important C^*-algebras arise from geometric, topological, number theoretic or dynamical situations. In particular, the construction of C^*-algebras from groupoids provides a very general source for C^*-algebras. Moreover, this has been the basis of many very fruitful interactions between these other fields and operator algebras. The classification program itself has devoted a lot of attention to the case of a crossed product of a discrete group acting on a commutative C^*-algebra. In most of these cases, inductive limit structures are not obviously available.

In this paper, we address the following question: which C^*-algebras that are classifiable in Elliott's sense, may be constructed from an étale groupoid? Or, put in a better way, which possible Elliott invariants may be realized as coming from the C^*-algebra of an étale groupoid? If, in addition, it is shown that the C^*-algebra is classifiable, then the classification results will provide isomorphisms with C^*-algebras constructed by other means having the same invariant.

The class of possible invariants we consider in our main result is rather restricted. On the other hand, the groupoids which we produce are, in fact, equivalence relations and the underlying space is a Cantor set. By a Cantor set, we mean a compact, totally disconnected, metrizable space with no isolated points.

We recall the notion of an étale equivalence relation. Let X be a set and let R be an equivalence relation on X. It is a groupoid with the set of composable pairs $R^2 = \{((x,y),(y,z)) \in R \times R\}$, product $(x,y)\cdot(y,z) = (x,z)$, for $((x,y),(y,z))$ in R^2 and inverse $(x,y)^{-1} = (y,x)$. For convenience, we will identify the diagonal in R with X in the usual way. This means that the range and source maps $r, s : R \to X$ are simply $r(x,y) = y, s(x,y) = x$.

If X, Y are topological spaces, a function $f : X \to Y$ is a local homeomorphism, if, for every x in X, there is a neighbourhood U of x such that $f(U)$ is open and $f : U \to f(U)$ is a homeomorphism.

We say that a topology on an equivalence relation R on a topological space X is *étale* if the two maps r, s are local homeomorphisms. We also say that R is an étale equivalence relation. We say that R is minimal if every equivalence class is dense in X.

If R is an étale equivalence relation on a space X, the linear space of continuous complex functions of compact support on R becomes a $*$-algebra with the operations

$$\begin{aligned} f \cdot g(x,y) &= \sum_{(x,z)\in R} f(x,z)g(z,y), \\ f^*(x,y) &= \overline{f(y,x)} \end{aligned}$$

for f, g continuous and compact supported and (x,y) in R. The completion of this algebra in a suitable norm is a C^*-algebra which we denote by $C^*(R)$.

One particularly nice class of examples are the AF-equivalence relations. In this case, the associated C^*-algebra is an AF-algebra and its ordered K-zero group can be computed rather easily. It is a dimension group (see [4] and [14]). We say this ordered abelian group is simple if it has no order ideals. This is equivalent to the C^*-algebra being simple which, in turn, is equivalent to the equivalence relation being minimal. We describe this in detail in the next section.

There are a number of different equivalent conditions for a groupoid to be amenable. We will use condition (ii) of Proposition 2.2.13 of [1] which follows. There exists a sequence of functions $g_l, l \geq 1$, on R which are non-negative, continuous and compactly supported and satisfy the following:

1. for every x in X,
$$\sum_{(x,y)\in R} g_l(x,y) = 1,$$
and

2. the function
$$(x,y) \to \sum_{(x,z)\in R} |g_l(x,z) - g_l(y,z)|,$$
converges uniformly to zero on compact subsets of R.

If R is an amenable equivalence relation, then $C^*(R)$ is an amenable C^*-algebra [1]. Moreover, in this case, $C^*(R)$ is simple if and only if R is minimal [14].

We state our main result.

Theorem 1.1. *Let G_0 be a simple, acyclic dimension group with order unit and let G_1 be a countable, torsion-free, abelian group. There exists an étale equivalence relation, R, on a Cantor set, X, such that*

1. *R is minimal,*
2. *R is amenable,*
3. *$K_0(C^*(R)) \cong G_0$, as ordered abelian groups with order unit,*
4. *$K_1(C^*(R)) \cong G_1$, as groups, and*
5. *$C^*(R)$ is tracially AF.*

In view of the classification theorem of [10], it is important to note that Lemma 3.5 and Proposition 10.7 of [16] show that the C^*-algebra of an amenable, étale equivalence relation satisfies the Universal Coefficient Theorem. Hence, our $C^*(R)$ falls within the realm of Lin's result above and the Elliott classification program generally.

Let us briefly discuss the main ingredients in the proof. One begins with the dimension group G_0. This can be realized as the K_0-group of a simple AF-algebra and we begin with a Bratteli diagram, (V, E), for it. This diagram has an infinite path space which will be our Cantor set X. This space has an étale equivalence relation R_E whose C^*-algebra is the AF-algebra. Next, we find another Bratteli diagram, (W, F), whose associated dimension group, without the order structure, is isomorphic to G_1 and we find two disjoint embeddings of this diagram into (V, E). The equivalence relation R is then generated by the original R_E and all equivalences between the two embedded copies of the paths of (W, F). This description as an equivalence relation is rather easy; what is more subtle is endowing R with an étale topology.

To verify the conditions of the theorem, the first, second and fifth parts are relatively straightforward. The main difficulty lies in the computation of the K-theory. The main tool here is the results of [12] and [13]. We have R_E as an open subequivalence relation of R and hence we have an inclusion $C^*(R_E) \subseteq C^*(R)$. The results of [13] essentially allow us to say that, since the difference between the groupoids R_E and R is described by the two embeddings of the diagram (W, F), the relative K-theory for the inclusion of their C^*-algebras can also be described from (W, F). This is then used to show that the inclusion of $C^*(R_E)$ in $C^*(R)$ induces an isomorphism on K-zero groups and that the K-one group of $C^*(R)$ is G_1.

Let us make some comments on some special cases of the theorem and other potential results along these lines.

Of course, the case $G_1 = 0$ goes back to the seminal work of Elliott [5] on AF-algebras coupled with Renault's construction of them from groupoids [14]. The case $G_1 \cong \mathbb{Z}$ is closely linked with the work of the author with Giordano, Herman and Skau [7].

The author, along with Deeley and Strung [3], showed that the Jiang-Su algebra [8, 15] could be realized via an étale equivalence relation. This is a case when $G_0 \cong \mathbb{Z}$, $G_1 = 0$, but the lack of projections in the C^*-algebra means that we cannot use a Cantor set for the space X. The main idea is to begin with a minimal, uniquely ergodic homeomorphism of a sphere of odd dimension at least 3. There are a couple of ways in which this construction might be extended. The first, already discussed in [3], is to begin with a minimal homeomorphism of the sphere with a more complex set of invariant measures. This, of course, produces more traces on the C^*-algebras. If one takes the product of one of these equivalence relations with one of the equivalence relations of Theorem 1.1, one would obtain the K-theory from 1.1, but with more traces. This produces examples where the C^*-algebras are not real rank zero nor tracially AF.

Another extension of the results in [3] is as follows. The key idea in [3] is to alter the sphere and the dynamics so as to insert a 'tube' which is invariant under the homeomorphism. If one instead inserted k of these tubes, one obtains a space Z with a minimal homeomorphism ζ such that

$K_0(C(Z) \times_\zeta \mathbb{Z}) \cong \mathbb{Z}^k \cong K_1(C(Z) \times_\zeta \mathbb{Z})$.

Finally, it appears that the statement of 1.1 can be extended to include the case G_1 is finite. But this work is still in progress and requires more general excision results than those available in [12] and [13].

The paper is organized as follows. In the second section, we provide background information on étale equivalence relations. We also define our étale equivalence relation R and prove some basic properties, including amenability and minimality. A key part of the structure of R is that it contains an open subequivalence relation, R_E, which is AF.

The third section deals with the computation of the K-theory of $C^*(R)$. The key point here is the presence of the AF-equivalence relation R_E, which provides us with a C^*-subalgebra, $C^*(R_E) \subseteq C^*(R)$ and results from [12] and [13] which allow us to compute the relative K-theory of this pair.

The fourth section is devoted to showing that $C^*(R)$ is tracially AF.

2 Groupoids

We begin with a general disucssion of étale equivalence relations, particularly those on a Cantor set, and AF-equivalence relations.

Returning to our undergraduate days, we recall that a function is defined as a set of ordered pairs. The first thing that one usually does with this definition is to forget it and treat functions as a kind of black box. Here, however, it is useful to keep this definition. The reason is simply that we are used to thinking of equivalence relations as sets of ordered pairs and our equivalence relations will be expressed as the union of a collection of functions. This is particularly useful as these functions provide a basis for the étale topology which we require. We will use following theorem, which is designed specifically for use on the Cantor set. Note that if $f, g \subseteq X \times X$ are local homeomorphisms, then their composition is

$$f \circ g = \{(x, y) \mid (x, z) \in f, (z, y) \in g\}.$$

For any clopen set U, we denote the identity function on U by $id_U = \{(x, x) \mid x \in U\}$.

Theorem 2.1. *Let X be a Cantor set and let Γ be a collection of subsets of $X \times X$ satisfying the following:*

1. *each element γ of Γ is a local homeomorphism of X with $r(\gamma)$ and $s(\gamma)$ clopen,*

2. *$\{U \subseteq X \mid U$ clopen $, id_U \in \Gamma\}$ is a basis for the topology of X,*

3. *for all γ in Γ, γ^{-1} is in Γ,*

4. *if γ_1, γ_2 are in Γ, then so is $\gamma_1 \circ \gamma_2$,*

5. *if γ_1, γ_2 are in Γ, then so is $\gamma_1 \cap \gamma_2$.*

Let R_Γ be the union of the elements of Γ. Then Γ is a basis for a topology on R_Γ in which it is an étale equivalence relation.

The easiest example of such a collection is to begin with φ, a free action of a discrete group, G, on a Cantor set, X. That is, for every g in G, φ^g is a homeomorphism of X and $\varphi^g \circ \varphi^h = \varphi^{gh}$, for all g, h in G. The action is free if $\varphi^g(x) = x$ occurs only for $g = e$. Here, the collection $\varphi^g|_U$, where g varies over G and $U \subseteq X$ is clopen, satisfies the hypotheses of 2.1 and the associated equivalence relation is the orbit relation.

The next class of examples are the AF-equivalence relations. These may be described rather abstractly as those étale equivalence relations on a Cantor set which can be written as the countable union of an increasing sequence of compact, open subequivalence relations [6]. They can also be described more concretely as being constructed from a Bratteli diagram. By a Bratteli diagram, (V, E), we mean a vertex set, V, which is the union of finite non-empty subsets $V_n, n \geq 0$, with $V_0 = \{v_0\}$ and an edge set, E, which is the union of finite non-empty subsets $E_n, n \geq 1$ along with initial and terminal maps $i : E_n \to V_{n-1}, t : E_n \to V_n$. Associated to such a diagram, we have its infinite path space

$$X_E = \{(x_1, x_2, \ldots) \mid x_n \in E_n, t(x_n) = i(x_{n+1}), n \geq 1\}.$$

For $m < n$, we let $E_{m,n}$ denote the finite paths from vertices in V_m to vertices in V_n with the obvious maps $i : E_{m,n} \to V_m$, $t : E_{m,n} \to V_n$. For each p in $E_{m,n}$, we let

$$U(p) = \{x \in X_E \mid x_i = p_i, m < i \leq n\}.$$

We observe that, for any such p,

$$U(p) = \cup_{i(e)=t(p)} U(pe),$$

and that the sets on the right are pairwise disjoint. It follows that sets $U(p), p \in E_{0,n}, n \geq 1$, form the basis for a topology on X_E. In this topology, the sets $U(p)$ are closed as well as open and X_E is totally disconnected. It is also metrizable.

We now introduce an étale equivalence relation on X_E. It will be convenient to let I_n denote the set of all pairs (p, q) with p, q in $E_{0,n}$ satisfying $t(p) = t(q)$, for $n \geq 1$. We define

$$\beta_{p,q} = \{(x, y) \in X_E \times X_E \mid x_i = p_i, y_i = q_i, 1 \leq i \leq n, x_i = y_i, i > n\},$$

for (p, q) in I_n.

Lemma 2.2. *Let $(p, q), (p', q')$ be in I_n, $n \geq 1$. We have*

1. *$\beta_{p,p}$ is the identity function of $U(p)$,*
2. *$\beta_{p,q}^{-1} = \beta_{q,p}$,*
3. *$\beta_{p,q} \circ \beta_{p',q'}$ is empty unless $q = p'$ and equals $\beta_{p,q'}$ in that case,*
4. *$\beta_{p,q} \cap \beta_{p',q'}$ is empty unless $p = p'$ and $q = q'$.*
5. $$\beta_{p,q} = \cup_{i(e)=t(p)} \beta_{pe,qe},$$
   and the sets on the right are pairwise disjoint.

The proof is trivial.

Definition 2.3. *Let (V, E) be a Bratteli diagram. We let $\mathcal{B}_E$ denote the collection of all sets $\beta_{p,q}$, where p, q are in I_n with $n \geq 1$, along with the empty set and we denote the associated étale equivalence relation by R_E.*

It is an easy matter to see that the collection $\mathcal{B}_E$ satisfies the conditions of Theorem 2.1. The third and fourth conditions follow from Lemma 2.2, provided one considers $\gamma_1 = \beta_{p,q}$ and $\gamma_2 = \beta_{p',q'}$ where p, q, p', q' are all in the same $E_{0,n}$. To deal with the case (p, q) is in I_n and (p', q') is in I_m, it suffices to observe that repeated application of the last part of 2.2 allows us to write $\beta_{p,q}$ as a union of $\beta_{p',q'}$ with p', q' in any $E_{0,m}$ with $m > n$.

For a Bratteli diagram as above, we let $b_{p,q}$ denote the characteristic function of the compact, open set $\beta_{p,q}$ for (p, q) in I_n. This is a partial isometry in $C^*(R_E)$.

It is a simple matter to see that, for fixed v in V_n,

$$B_v = span\{b_{p,q} \mid p, q \in E_{0,n}, t(p) = t(q) = v\}$$

is isomorphic to $M_{k(v)}$, where $k(v)$ denotes the number of paths p in $E_{0,n}$ with $t(p) = v$. It is also easy to check that

$$B_n = span\{b_{p,q} \mid (p, q) \in I_n\} = \oplus_{v \in V_n} B_v.$$

Finally, one checks that $B_n \subseteq B_{n+1}$, for all n and that the union of the B_n is dense in $C^*(R_E)$.

We next describe the construction of the equivalence relation R whose existence is claimed in Theorem 1.1. We begin by choosing a Bratteli diagram, (V, E), whose associated AF-equivalence relation R_E on X_E has $K_0(C^*(R_E)) \cong G_0$, as ordered abelian groups with order unit. We note that (V, E) is simple as G_0 is, in the sense that, after telescoping to a subsequence, we can assume that between every vertex at some level n and another at level $n + 1$, there is at least one edge. Also, as G_0 is not a cyclic group, we may assume that every edge set in E has at least two edges. Moreover, by telescoping and symbol splitting the diagram further, we can make the cardinality of both vertex sets and edge sets grow arbitrarily. (See [6] for more discussion on telescoping of Bratteli diagrams.)

Next, as G_1 is a countable torsion-free abelian group, it has an embedding into $\mathbb{R}$. This can be seen as follows: first, the fact it is torsion free means that the natural inclusion into $G_1 \otimes_{\mathbb{Z}} \mathbb{Q}$ is injective. This is followed by an inclusion of this countable, rational vector space into $\mathbb{R}$. With the relative order, it becomes a simple dimension group and we can find a Bratteli diagram for it, (W, F). (The actual order does not matter, only that we can write it as an inductive limit of free abelian groups with maps between them given by positive matrices.)

Having chosen (W, F), we adjust (V, E) so that $\#V_n > \#W_n$, for all $n \geq 1$ and so that the number of edges between any two vertices in V_{n-1} and V_n exceeds $2\#F_n$. In consequence, we may choose graph embeddings $\xi : W \to V$ and $\xi^0, \xi^1 : F \to E$ with disjoint images. Observe that ξ^0, ξ^1 also define continuous maps from X_F into X_E.

We let $\beta_{p,q}, \mathcal{B}_E, R_E$ be as described above for the diagram (V, E).

Recall that for any étale equivalence relation R on compact space X, a Borel measure μ on X is R-invariant if, for any Borel set $U \subseteq R$ such that $r|_U, s|_U$ are injective, we have $\mu(r(U)) = \mu(s(U))$.

Lemma 2.4. *Let μ be any R_E-invariant probability measure on X_E. Then we have*

$$\mu(\xi^0(X_F)) = \mu(\xi^1(X_F)) = 0.$$

Proof. Let $i = 0, 1$. It is well-known (see [4]) that a probability measure on an AF-equivalence relation R_E is determined uniquely by a function $\mu : V \to [0, 1]$ satisfying $\mu(v_0) = 1$ and $\mu(v) = \sum_{i(e)=v} \mu(t(e))$, for all v in V. From this and the hypothesis on the number of edges in F_n that

$$\mu(\xi^i(F_{0,n})) \leq 2^{-n},$$

for all $n \geq 1$. As $\xi^i(X_F) = \cap_n \xi^i(F_{0,n})$ and the measure is regular, this completes the proof. □

As before, we denote the set of all pairs p, q in $E_{0,n}$ with $t(p) = t(q)$ by I_n and now also denote the set of all such pairs with $t(p) = t(q) \in \xi(W)$ by I_n^W.

Now suppose that p, q are two paths in I_n^W, for some $n \geq 1$. We define subsets $\lambda_{p,q}^{1,0}, \lambda_{p,q}^{0,1}, \delta_{p,q}^{1,0}, \delta_{p,q}^{0,1}$ of $X_E \times X_E$ as follows. First, we define $\lambda_{p,q}^{1,0}$ to be the set of all pairs:

$$\begin{gathered}(p_1, \ldots, p_n, \xi^1(f_{n+1}), \xi^1(f_{n+2}), \ldots) \\ (q_1, \ldots, q_n, \xi^0(f_{n+1}), \xi^0(f_{n+2}), \ldots) \\ f_i \in F_i, i > n\end{gathered}$$

It is a simple matter to see that $\lambda_{p,q}^{1,0}$ is a bijection. In fact, its inverse is given by using the same formula, simply interchanging p and q and ξ^0 and ξ^1. We denote $(\lambda_{p,q}^{1,0})^{-1}$ by $\lambda_{q,p}^{0,1}$.

We define $\delta_{p,q}^{1,0}$ to be the union of $\lambda_{p,q}^{1,0}$ with all pairs of sequences of the form:

$$\begin{gathered}(p_1, \ldots, p_n, \xi^1(f_{n+1}), \ldots, \xi^1(f_k), \xi^0(f_{k+1}), x_{k+2}, \ldots) \\ (q_1, \ldots, q_n, \xi^0(f_{n+1}), \ldots, \xi^0(f_k), \xi^1(f_{k+1}), x_{k+2}, \ldots) \\ k \geq n+1, f_{n+1}, \ldots, f_{k+1} \in F,\end{gathered}$$

and also all pairs of the form:

$$\begin{gathered}(p_1, \ldots, p_n, \xi^1(f_{n+1}), \ldots, \xi^1(f_k), x_{k+1}, x_{k+2}, \ldots) \\ (q_1, \ldots, q_n, \xi^0(f_{n+1}), \ldots, \xi^0(f_k), x_{k+1}, x_{k+2}, \ldots) \\ k > n+1, f_{n+1}, \ldots, f_k \in F, x_{k+1} \notin \xi^0(F) \cup \xi^1(F)\end{gathered}$$

We first observe that the three types of pairs are distinct. In fact, their images under the source map (and also under the range map) are pairwise disjoint. Secondly, the image under s of $\delta^{1,0}_{p,q}$ is all sequences $(p_1, p_2, \ldots, p_n, e_{n+1}, e_{n+2}, \ldots)$ in X_E such that e_{n+1} is in $\xi^1(F_{n+1})$, which is a clopen set we denote by $U^1(p)$. Its image under r is all sequences $(q_1, q_2, \ldots, q_n, e_{n+1}, e_{n+2}, \ldots)$ in X_E such that e_{n+1} is in $\xi^0(F_{n+1})$, which is also a clopen set we denote by $U^0(q)$. It is also a simple matter to see that $\delta^{1,0}_{p,q}$ is a bijection. In fact, its inverse is given by using the same formula, simply interchanging p and q and ξ^0 and ξ^1. We denote $(\delta^{1,0}_{p,q})^{-1}$ by $\delta^{0,1}_{q,p}$. We can summarize this in the following way.

Lemma 2.5. *Let $n \geq 1$, (p, q) be in I^W_n and $i = 0, 1$. As defined above, $\delta^{i,1-i}_{p,q}$ is a local homeomorphism of X_E. Moreover, we have*

$$\delta^{i,1-i}_{p,q}\delta^{1-i,i}_{q,p} = \cup_{\{f \in F_{n+1}, \xi(i(f))=t(p)\}}\beta_{p\xi^i(f),p\xi^i(f)}$$

where the sets in the union are pairwise disjoint.

The following is obvious from the definitions, but worth stating explicitly.

Lemma 2.6. *For p, q in I^W_n and $i = 0, 1$, we have*

$$\delta^{i,1-i}_{p,q} - R_E = \lambda^{i,1-i}_{p,q}.$$

We need an analogue of property 5 of Lemma 2.2 for our elements $\delta^{i,1-i}_{p,q}$, which is the following.

Lemma 2.7. *Let (p, q) be in I^W_n with $t(p) = \xi(w)$, for some w in W_n. Define*

$$\begin{aligned} A &= \{f \in F_{n+1} \mid i(f) = w\} \\ B &= \{(f, f') \in F_{n+1} \times F_{n+2} \mid \\ &\quad i(f) = w, t(f) = i(f')\} \\ C &= \{(f, e) \in F_{n+1} \times E_{n+2} \mid \\ &\quad i(f) = w, \xi(t(f)) = i(e), e_{k+1} \notin \xi^0(F_{n+2}) \cup \xi^1(F_{n+2})\}. \end{aligned}$$

Then we have

$$\begin{aligned} \delta^{1,0}_{p,q} &= \left(\cup_{f\in A}\delta^{1,0}_{p\xi^1(f),q\xi^0(f)}\right) \\ &\cup \left(\cup_{(f,f')\in B}\beta_{p\xi^1(f)\xi^0(f'),q\xi^0(f)\xi^1(f')}\right) \\ &\cup \left(\cup_{(f,e)\in C}\beta_{p\xi^1(f)e,q\xi^0(f)e}\right) \end{aligned}$$

Moreover, the three sets on the right are pairwise disjoint.

Proof. If (x, y) is in $\delta^{1,0}_{p,q}$, then $(x_1, \ldots, x_n) = p$, and $x_{n+1} = \xi^1(f_{n+1})$, for some f_{n+1} in F_{n+1}. Now, there are three mutually distinct possibilities for x_{n+2}. The first is x_{n+2} is in $\xi^1(F_{n+2})$. In this case, (x, y) is in $\delta^{1,0}_{p\xi^1(f_{n+1}),q\xi^0(f_{n+1})}$. The second is that x_{n+2} is in $\xi^0(F_{n+2})$. In this case, (x, y) is in $\beta_{p\xi^1(f_{n+1})\xi^0(f_{n+2}),q\xi^0(f_{n+1})\xi^1(f_{n+2})}$. Finally, if x_{n+2} is in neither $\xi^0(F)$ nor $\xi^1(F)$, then (x, y) is in $\beta_{p\xi^1(f_{n+1})x_{n+2},q\xi^1(f_{n+1})x_{n+2}}$. The reverse inclusions are clear. □

The first immediate consequence of the lemma is the following.

Theorem 2.8. *The collection*

$$\begin{aligned}\mathcal{B} &= \{\beta_{p,q} \mid (p,q) \in I_n, n \geq 1\} \\ &\quad \cup \{\delta^{1,0}_{p,q}, \delta^{0,1}_{p,q} \mid (p,q) \in I^W_n, n \geq 1\} \\ &\quad \cup \{\emptyset\}\end{aligned}$$

satisfies the hypotheses of Theorem 2.1. We let R be the associated étale equivalence relation. The equivalence relation R_E is an open subequivalence relation of R.

Proof. The first three conditions are clear. The fourth and fifth must be dealt with in several cases. If γ_1 and γ_2 are both of the form $\beta_{p,q}$, both properties follow from Lemma 2.2.

The next case is to consider $\beta_{p,q}$ and $\delta^{1,0}_{p',q'}$, with (p, q) in I_n and (p', q') in I^W_m. First assume that $n = m$. In this case, it is a simple matter to check that

$$\beta_{p,q} \circ \delta^{1,0}_{p',q'} = \begin{cases} \delta^{1,0}_{p,q'}, & q = p' \\ \emptyset, & q \neq p' \end{cases}$$

and

$$\delta^{1,0}_{p',q'} \circ \beta_{p,q} = \begin{cases} \delta^{1,0}_{p',q}, & q' = p \\ \emptyset, & q' \neq p \end{cases}$$

It is also simple to observe that for any (x, y) in $\delta^{1,0}_{p'q'}$, $x_{n+1} \neq y_{n+1}$ and hence $\beta_{p,q} \cap \delta^{1,0}_{p',q'} = \emptyset$. Similar arguments apply if we replace $\delta^{1,0}_{p',q'}$ with $\delta^{0,1}_{q',p'}$.

Now let us suppose that $n < m$. Using repeated applications of part 5 of Lemma 2.2, we can replace $\beta_{p,q}$ with a union of $\beta_{p'',q''}$ with (p'', q'') in I_m and then the previous case applies, for both properties.

Finally, let us suppose that $m < n$. Here, repeated applications of Lemma 2.7 allow us to replace $\delta^{1,0}_{p',q'}$ with a union of $\delta^{1,0}_{p'',q''}$ with (p'', q'') in I^W_n and some

$\beta_{p''',q'''}$ (it is not necessary to specify the lengths of p'''). Then the previous case and part 5 of 2.1 yields both of the desired properties.

The next case to consider is $\gamma_1 = \delta^{1,0}_{p,q}$ and $\gamma_2 = \delta^{1,0}_{p',q'}$. We again begin with $n = m$. Here, $r(\delta^{1,0}_{p,q}) = U^0(q)$ and $s(\delta^{1,0}_{p,q}) = U^1(p)$. It follows easily that $\delta^{1,0}_{p,q} \circ \delta^{1,0}_{p',q'} = \emptyset$ while $\delta^{1,0}_{p,q} \cap \delta^{1,0}_{p',q'}$ is empty unless $p = p'$ and $q = q'$.

Next, we suppose that $n < m$. We use Lemma 2.2 to replace $\delta^{1,0}_{p,q}$ with a union of $\delta^{1,0}_{p'',q''}$ with (p'', q'') in I^W_m and $\beta_{p''',q'''}$ with (p''', q''') in $I_{m'}$, for varying m'. Here both results follow from the previous case and the cases already established above.

The final case to consider is $\gamma_1 = \delta^{1,0}_{p,q}$ and $\gamma_2 = \delta^{0,1}_{p',q'}$. We again begin with $n = m$. Here, $r(\delta^{1,0}_{p,q}) = U^0(q)$ and $s(\delta^{1,0}_{p,q}) = U^1(p)$. It follows easily that $\delta_{p,q} \cap \delta^{-1}_{p',q'} = \emptyset$ while $\delta^{1,0}_{p,q} \circ \delta^{0,1}_{p',q'}$ is empty unless $p = p'$ and $q = q'$ and then the composition is simply the identity function on $U^1(p)$.

Next, we suppose that $n < m$. We use Lemma 2.2 to replace $\delta^{1,0}_{p,q}$ with a union of $\delta^{1,0}_{p'',q''}$ with (p'', q'') in I^W_m and $\beta_{p''',q'''}$ with (p''', q''') in $I_{m'}$, for varying m'. Here, both results follow from the previous case and the cases already established above. □

The first desired property of our equivalence relation R is easy: as R_E is a subequivalence relation and is minimal, every R_E-equivalence class is dense in X and hence the same is true for R.

Theorem 2.9. *The equivalence relation R is minimal.*

Although we do not need it immediately, we record the following useful result.

Proposition 2.10. *A probability measure μ on X_E is R-invariant if and only if it is R_E-invariant.*

Proof. The 'only if' part is clear. Now suppose that μ is R_E-invariant. If we let Y be the set of all points which are R_E-equivalent to some point of $\xi^0(X_F) \cup \xi^1(X_F)$, this Borel set has $\mu(Y) = 0$. On the other hand, R_E and R agree on the complement of Y and it follows that μ is R-invariant as well. □

To study R in greater detail, for each $n \geq 1$, we let $\mathcal{B}_n$ be the collection of all sets $\beta_{p,q} \subseteq R$, with (p, q) in I_n, along with all $\delta^{1,0}_{p,q}$ and $\delta^{0,1}_{p,q}$, with (p, q) in I^W_n. We also define R_n to be the union of these sets.

Lemma 2.11. *1. For each $n \geq 1$, R_n, with the relative topology of R, is a compact, open, étale equivalence relation. We let $[x]_{R_n}$ be the equivalence class of x in R_n, for any x in X_E.*

2. For any x in X_E, we have

$$\#[x]_n = \begin{cases} 2\#\{p \in E_{0,n} \mid t(p) = t(x_n)\}, & x_{n+1} \in \xi^0(F_{n+1}) \cup \xi^1(F_{n+1}) \\ \#\{p \in E_{0,n} \mid t(p) = t(x_n)\}, & x_{n+1} \notin \xi^0(F_{n+1}) \cup \xi^1(F_{n+1}) \end{cases}$$

3. For $l > n$, we have $R_n - R_l$ is the union of

$$\beta_{p\xi^1(f)\xi^0(f'),q\xi^0(f)\xi^1(f')} \cup \beta_{q\xi^0(f)\xi^1(f'),p\xi^1(f)\xi^0(f')},$$

over all (p, q) in I_n^W and (f, f') in $F_{n,l} \times F_{l+1}$ with $t(f) = i(f')$.

4. For fixed $n \geq 1$, the sets $R_n - R_l, l > n$, are pairwise disjoint.

5. For all $n \geq 1$, $R_n \subseteq R_{n+1} \cup R_{n+2}$.

6. The union of $R_n, n \geq 1$, is R.

Proof. Each of the sets in $\mathcal{B}$ is compact in the topology of $X \times X$ and hence also in R. They are also open in R since they are part of the basis for its topology. Thus, R_n is the union of a finite number of compact open sets and is then compact and open.

Let x be any point in X. If x_{n+1} is not in $\xi^0(F_{n+1}) \cup \xi^1(F_{n+1})$, then its images under the elements of $\mathcal{B}_n$ is exactly the set of all y such that $y_i = x_i$, for all $i > n$. If $x_{n+1} = \xi^0(f_{n+1})$, for some f_{n+1} in F_{n+1}, then its images under the elements of $\mathcal{B}_n$ is exactly the set of all y such that $y_i = x_i$, for all $i > n$, and all z with $z_{n+1} = \xi^1(f_{n+1})$ and $z_i = x_i$, for all $i > n + 1$. Similarly, if $x_{n+1} = \xi^1(f_{n+1})$, for some f_{n+1} in F_{n+1}, then its images under the elements of $\mathcal{B}_n$ is exactly the set of all y such that $y_i = x_i$, for all $i > n$, and all z with $z_{n+1} = \xi^0(f_{n+1})$ and $z_i = x_i$, for all $i > n + 1$. From this, it is clear that R_n is an equivalence relation.

The second part follows immediately from the description of the equivalence classes we have just provided.

For the third part, we consider an element of $\mathcal{B}_n$. If it is $\beta_{p,q}$ for some (p, q) in I_n, then by part 5 of Lemma 2.2, it is contained in R_l and so $\beta_{p,q} - R_l$ is empty. For $\delta^{1,0}_{p,q}$ with (p, q) in I_n^W, we apply Lemma 2.7 $l - n$ times. In the end, we have a collection of $\delta^{1,0}_{p',q'}$ where (p', q') is in I_l^W, which are all in

R_l. We also have many $\beta_{p',q'}$. Almost all of these are contained in R_l, with the only exception being those listed in the statement where the (p', q') are in I^W_{l+1}. The reverse containment is clear.

From the description of $R_n - R_l$ which is given in the third part, we see that any pair (x, y) in $R_n - R_l$ has $x_{l+1} \neq y_{l+1}$, but $x_i = y_i$, for all $i > l + 1$. This implies the fourth part.

The fifth part follows from Lemma 2.7. The last part is immediate from the definitions. □

The last two parts of this result are rather curious: if the R_n actually formed an increasing sequence of compact, open subequivalence relations, then $C^*(R)$ would be an AF-algebra. The sets $R_n \cup R_{n+1}$ *do* form an increasing sequence of compact open subsets, but they are not subequivalence relations.

Theorem 2.12. *The étale equivalence relation R is amenable.*

Proof. For $l \geq 1$, we define h_l as follows:

$$h_l(x, y) = \begin{cases} (\#[x]_{R_l})^{-1}, & (x, y) \in R_l \\ 0, & (x, y) \notin R_l \end{cases}$$

This is a continuous function on the space R since R_l is étale. It is compactly supported as R_l is compact and it is clearly non-negative. We consider the two conditions given in the first section for amenability. The first of these conditions is trivially satisfied.

We consider h_l on R_n, for $l > n$. Let (x, y) be in R_n. If (x, y) actually lies in R_l as well, then we have

$$h_l(x, z) = h_l(y, z) = (\#[x]_{R_l})^{-1}$$

for every (x, z) in R_l. Now suppose that (x, y) is not in R_l. Let z be any point with (x, z) in R. If neither (x, z) nor (y, z) are in R_l, then

$$h_l(x, z) - h_l(y, z) = 0 - 0 = 0.$$

If z is such that one of (x, z) and (y, z) is in R_l, then as (x, y) is not in R_l,

(x,z) and (y,z) cannot both be so exactly one of them is and we have

$$\begin{aligned}\sum_{(x,z)\in R} |h_l(x,z) - h_l(y,z)| &= \sum_{(x,z)\in R_l} |h_l(x,z) - h_l(y,z)| \\ &\quad + \sum_{(y,z)\in R_l} |h_l(x,z) - h_l(y,z)| \\ &= \sum_{(x,z)\in R_l} |(\#[x]_{R_l})^{-1} - 0| \\ &\quad + \sum_{(y,z)\in R_l} |0 - (\#[y]_{R_l})^{-1}| \\ &= 2.\end{aligned}$$

Now we define $g_l = l^{-1}\sum_{i=1}^{l} h_l$. This is clearly non-negative, continuous, compactly supported and satisfies the first of the two conditions. As for the second, we consider the value of g_l on R_n:

$$\begin{aligned}\sum_{(x,z)\in R} |g_l(x,z) - g_l(y,z)| &\leq l^{-1}\sum_{i=1}^{n}\sum_{(x,z)\in R} |h_l(x,z) - h_l(y,z)| \\ &\quad + l^{-1}\sum_{i=n+1}^{l}\sum_{(x,z)\in R} |h_l(x,z) - h_l(y,z)| \\ &\leq l^{-1}2n + l^{-1}2\sum_{i=n+1}^{l}\chi_{R_n - R_i} \\ &\leq l^{-1}2(n+1)\end{aligned}$$

where we have used the fact that the sets $R_n - R_i, n < i \leq l$, are pairwise disjoint from part 4 of Lemma 2.11 in the last step. This clearly tends to zero uniformly on R_n as l tends to infinity. This completes the proof, using the characterization of amenability given in the introduction. □

3 K-theory

In this section, our single goal is to compute the K-theory of the C^*-algebra $C^*(R)$. The main tool will be Theorem 2.4 of [13]. Translating the notation from [13], the groupoid G there will be our R. It will be useful to have some notation for the elements of $C^*(R)$. As in the last section, For each pair (p, q) in I_n, we let $b_{p,q}$ be the charactersitic function of $\beta_{p,q}$. If (p, q) is in I_n^W and $i = 0, 1$, we let $d_{p,q}^{i,1-i}$ be the charactersitic function of $\delta_{p,q}^{i,1-i}$.

The C^*-subalgebra of $C^*(R)$ generated by the $b_{p,q}$ is the AF-algebra $C^*(R_E)$.

Let discuss relative K-groups for C^*-algebras. If A is a C^*-algebra and A' is a C^*-subalgebra, the mapping cone of the inclusion is

$$C(A'; A) = \{f : [0, 1] \to A \mid f \text{ continuous}, f(0) = 0, f(1) \in A'\}$$

which is a C^*-algebra with point-wise operations. The relative K-theory $K_i(A'; A)$ is usually defined as $K_{i+1}(C(A'; A))$, for $i = 0, 1$. As there is an exact sequence

$$0 \to C_0(0, 1) \otimes A \to C(A'; A) \to A' \to 0$$

and using the canoncial isomorphism $K_i(C_0(0, 1) \otimes A) \cong K_{i+1}(A)$, there is a six-term exact sequence

$$\begin{array}{ccccc} K_0(A') & \longrightarrow & K_0(A) & \longrightarrow & K_0(A'; A) \\ \uparrow & & & & \downarrow \\ K_1(A'; A) & \longleftarrow & K_1(A) & \longleftarrow & K_1(A') \end{array}$$

In [12], a different description of the relative group $K_1(A'; A)$ is given. We let $\tilde{A}$ denote the C^*-algebra obtained by adding a unit to A and $\tilde{A}' \subseteq \tilde{A}$ is the unital inclusion. We also let $M_n(A)$ denote the $n \times n$-matrices with entries from A. One considers partial isometries v in $M_n(\tilde{A})$, such that v^*v is in $M_n(\mathbb{C})$ (the subalgebra generated by the unit) and vv^* is in $M_n(\tilde{A}')$. Following [12], we let $V_n(A'; A)$ denote the set of all such elements v. The relative group $K_1(A'; A)$ is described in terms of equivalence classes of such partial isometries, which we denote by $[v]_r$ (r for 'relative').

One obvious advantage of this description is that two of the maps in the six-term exact sequence shown above become rather obvious: the one from $K_1(A)$ into $K_1(A'; A)$ simply takes $[u]_1$, where u is a unitary in $M_n(\tilde{A})$, to $[u]_r$ in $K_1(A'; A)$ and the one from $K_1(A'; A)$ to $K_0(A')$ takes $[v]_r$ to $[vv^*]_0 - [v^*v]_0$ in $K_0(A')$.

We would like to apply the results of [13] to our C^*-algebras
$C^*(R_E) \subseteq C^*(R)$. To do so, we must define the set $L \subseteq R$ from page 1492 of [13]. We let L be the union of all sets $\lambda_{p,q}^{1,0}$, with (p, q) in I_n^W and $n \geq 1$.

Observe that every point in $s(\lambda_{p,q}^{1,0})$ has all but finitely many edges in $\xi^1(F)$, while every point in $r(\lambda_{p,q}^{1,0})$ has all but finitely many edges in $\xi^0(F)$. It follows that $r(L) \cap s(L)$ is empty. It is clear then that L^{-1} is simply the union of all $\lambda_{p,q}^{0,1}$.

We know from Lemma 2.6 that, for any such p, q, we have $\delta_{p,q}^{1,0} - R_E = \lambda_{p,q}^{1,0}$. Also, as R_E is the union of all $\beta_{p,q}$, we know that $\beta_{p,q} - R_E$ is empty. It follows then that the complement of R_E in R is exactly $L \cup L^{-1}$. As the equivalence relation R_E is open in R, it follows that L is closed.

Following [13], we let

$$H_0 = L^{-1}L, H_1 = LL^{-1}, H' = H_0 \cup H_1, H = H_0 \cup H_1 \cup L \cup L^{-1}.$$

This defines these objects simply as subgroupoids. They are not locally compact in general. The main point of [13] is that each can be endowed with natural new topology which is finer than the relative topology from R and is étale.

In view of Lemma 2.6, we see that $\lambda_{p,q}^{1,0} = \delta_{p,q}^{1,0} \cap L$ and this means that the sets $\lambda_{p,q}^{1,0}$ are all clopen in the relative topology of L. It follows that, for (p, q) in I_n^W, $n \geq 1$, the sets

$$\alpha_{p,q}^{0,0} = \lambda_{p,p}^{0,1} \circ \lambda_{p,q}^{1,0} = \lambda_{p,q}^{0,1} \circ \lambda_{q,q}^{1,0}$$

are compact and open and a base for the topology of H_0. We also define

$$\alpha_{p,q}^{1,1} = \lambda_{p,p}^{1,0} \circ \lambda_{p,q}^{0,1} = \lambda_{p,q}^{1,0} \circ \lambda_{q,q}^{0,1}$$

which are compact and open and a base for the topology of H_1. The set $L \cup L^{-1}$ keeps its topology from R and H is given the disjoint union topology.

For each (p, q) in I_n^W, we define $a_{p,q}^{i,i}$ to be the characteristic function of $\alpha_{p,q}^{i,i}$ which is in $C^*(H_i) \subseteq C^*(H)$. The span of these elements is dense in $C^*(H_i)$. We also let $a_{p,q}^{1,0}$ be the characteristic function of $\lambda_{p,q}^{1,0}$ and $a_{p,q}^{0,1}$ be the characteristic function of $\lambda_{p,q}^{0,1}$, which are both in $C^*(H)$.

The following is essentially the same as Lemma 3.8 of [13] and so we do not provide a proof. On the other hand, it is a simple matter to see that the $a_{p,q}^{i,j}$ form a systems of matrix units for the various finite-dimensional algebras and this can be proved directly using arguments following Definition 2.3. It does seem worthwhile to point out that for any $n \geq 1$, (p, q) in I_n^W, $i, j = 0, 1$ and $m > n$, we have

$$a_{p,q}^{i,j} = \sum_{p' \in F_{n,m}, \xi(i(p')) = t(p)} a_{p\xi^i(p'), q\xi^j(p')}^{i,j},$$

since this is a slightly different relation than satisfied by the $b_{p,q}$.

Theorem 3.1. *1. The C^*-algebras $C^*(H_0), C^*(H_1)$ and $C^*(H)$ are all AF-algebras.*

2. We have

$$K_0(C^*(H_0)) \cong K_0(C^*(H)) \cong K_0(C^*(H_1)) \cong G_1$$

and the first two isomorphism are induced by the inclusion maps.

3. The map from $K_0(C^*(H_0))$ to $K_1(C^*(H'); C^*(H))$ which sends $[a^{0,0}_{p,p}]_0$ to $\begin{bmatrix} a^{1,0}_{p,p} & 0 \\ 1 - a^{0,0}_{p,p} & 0 \end{bmatrix}_r$ is an isomorphism.

The important relation between the $C^*(H)$ and our earlier $C^*(R)$ is the fact that elements of $C^*(R)$ act as multipliers of $C^*(H)$. When considering those elements which are continuous functions of compact support on R, this is a matter of realizing that $H \subseteq R$ and one simply restricts the functions. The topology which H has been given is not the relative topology from R, but it is finer so that a continuous function on R restricts to a continuous function on H. If it has compact support on R, its support will no longer be compact on H, but the function will be bounded.

We need to perform this product in a few simple cases.

Lemma 3.2. *Let $m \geq 1$, (p, p') be in I_m, q in $E_{0,m}$ and $i, j = 0, 1$. We have*

1.
$$d^{i,1-i}_{p,p'} a^{j,j}_{q,q} = \begin{cases} a^{i,1-i}_{p,p'} & \text{if } 1 - i = j, p' = q \\ 0 & \text{otherwise} \end{cases}$$

2.
$$a^{j,j}_{q,q} d^{i,1-i}_{p,p'} = \begin{cases} a^{i,1-i}_{p,p'} & \text{if } i = j, p = q \\ 0 & \text{otherwise} \end{cases}$$

3.
$$b_{p,p'} a^{j,j}_{q,q} = \begin{cases} a^{j,j}_{p,p'} & \text{if } p' = q \\ 0 & \text{otherwise} \end{cases}$$

4.
$$a^{j,j}_{q,q} b_{p,p'} = \begin{cases} a^{j,j}_{p,p'} & \text{if } p = q \\ 0 & \text{otherwise} \end{cases}$$

Proof. The proof of the first part involves computing the composition

$$\delta_{p,p'}^{i,1-i} \circ \alpha_{q,q}^{j,j} = \delta_{p,p'}^{i,1-i} \circ \lambda_{q,q}^{j,1-j} \circ \lambda_{q,q}^{1-j,j}.$$

It follows from the defintions that this is empty unless $j = 1 - i$ and $p' = q$. In that case, it is simply $\lambda_{p,p'}^{i,1-i}$. This completes the proof. The second part is done in a similar way and we omit the details. The others parts can be obtained from these by taking adjoints. □

Lemma 3.3. *For each $m \geq 1$, define*

$$e_m = \sum_{q \in E_{0,m}, t(q) \in \xi(W)} a_{q,q}^{0,0} + a_{q,q}^{1,1},$$

in $C^*(H_0) \oplus C^*(H_1)$.

1. *The sequence $e_m, m \geq 1$ is an approximate identity for $C^*(H)$.*
2. *If $n < m$ and (p, q) is in I_n^W, then*

   $$b_{p,q} e_m = e_m b_{p,q},$$

   is in $C^*(H_0) \oplus C^*(H_1)$.
3. *If $n < m$ and (p, p) is in I_n^W, then*

   $$d_{p,p}^{1,0} e_m = e_m d_{p,p}^{1,0} = a_{p,p}^{1,0} + c,$$

   where c is a partial isometry in $C^*(H_0) \oplus C^*(H_1)$. Moreover, $a_{p,p}^{1,0}$ and c have orthogonal ranges and orthogonal sources; that is, $c^* a_{p,p}^{1,0} = 0 = a_{p,p}^{1,0} c^*$.

Proof. The proof of the first part is straightforward and we omit it. For the second part, we use part 5 of Lemma 2.2 to write $b_{p,p'}$ as a sum of $b_{q,q'}$ over pairs (q, q') in I_m. We also write e_m as a sum of $a_{q,q}^{i,i}$ as in its definition. We then expand out the product $b_{p,q} e_m$, applying part 2 of Lemma 3.2 to each term. A similar approach to $e_m b_{p,q}$ yields the result.

For the third part, repeated application of Lemma 2.7 means that we may write

$$d_{p,p}^{1,0} = \sum_{p' \in F_{n,m}, \xi(s(p')) = t(p)} d_{p\xi^1(p'), p\xi^0(p')}^{1,0} + b,$$

where b is a sum of terms of the form $\beta_{q',q''}$ with (q',q'') in I_{m+1}. Moreover, these are two partial isometries with orthogonal initial projections and final projections. We now take the product with e_m, again writing it as a sum of $a^{i,i}_{q,q}$ and expanding. By using Lemma 3.2, we we first see that $d^{1,0}_{p,p}e_m = e_m d^{1,0}_{p,p}$. For a fixed $p' \in F_{n,m}, \xi(s(p')) = t(p)$, we also have

$$\begin{aligned} d^{1,0}_{p\xi^1(p'),p\xi^0(p')}e_m &= \sum_q d^{1,0}_{p\xi^1(p'),p\xi^0(p')}(a^{0,0}_{q,q} + a^{1,1}_{q,q}) \\ &= a^{1,0}_{p\xi^1(p'),p\xi^0(p')}, \end{aligned}$$

using Lemma 3.2. Taking the sum over all p' now yields

$$\left(\sum_{p' \in F_{n,m}, \xi(s(p'))=t(p)} d^{1,0}_{p\xi^1(p'),p\xi^0(p')} \right) e_m = a^{1,0}_{p,p}.$$

The term $be_m = c$ lies in $C^*(H_0) \oplus C^*(H_1)$ in consequence of Lemma 3.2. This completes the proof. □

Proposition 3.4. *With the isomorphism*

$$\alpha : K_1(C^*(R_E); C^*(R)) \to K_1(C^*(H'); C^*(H))$$

given in Theorem 3.1 of [12], for any $n \geq 1$ and p in $E_{0,n}$ with $t(p)$ in $\xi(W)$, we have

$$\alpha \begin{bmatrix} d^{1,0}_{p,p} & 0 \\ 1 - (d^{1,0}_{p,p})^* d^{1,0}_{p,p} & 0 \end{bmatrix}_r = \begin{bmatrix} a^{1,0}_{p,p} & 0 \\ 1 - a^{0,0}_{p,p} & 0 \end{bmatrix}_r.$$

Proof. The map α is constructed as follows. One finds an approximate unit, $e_t, t \in [0,\infty)$, for the algebra $C^*(H)$. Then for a partial isometry v in $M_n(C^*(R)^\sim)$, such that v^*v and vv^* are in $M_n(C^*(R_E)^\sim)$, we form

$$\alpha(v)_t = \begin{bmatrix} ve_t & 0 \\ (v^*v - e_t v^* v e_t)^{1/2} & 0 \end{bmatrix}.$$

Notice here that we write e_t rather than $1_n \otimes e_t \in M_n(C^*(H))$, for simplicity. For sufficently large values of t, this defines a class in $K_1(C^*(H_0) \oplus C^*(H_1); C^*(H))$ which is the image of $[v]_r$ under α.

We extend our approximate unit e_m extend to real values by setting $e_t = (m+1-t)e_m + (t-m)e_{m+1}$, for $m \leq t \leq m+1$.

It will be convenient to denote

$$b = (d^{1,0}_{p,p})^* d^{1,0}_{p,p} = \sum_{f \in F_{n+1}, \xi(i(f)) = t(p)} b_{p\xi^0(f), p\xi^0(f)}.$$

Let $v = \begin{pmatrix} d^{1,0}_{p,p} & 0 \\ 1-b & 0 \end{pmatrix}$. Then, for $m > n$, using the facts that e_m and b are projections and commute, we have

$$\begin{aligned}
\alpha(v)_m &= \begin{bmatrix} ve_m & 0 \\ (v^*v - e_m v^* v e_m)^{1/2} & 0 \end{bmatrix} \\
&= \begin{bmatrix} d^{1,0}_{p,p} e_m & 0 & 0 & 0 \\ (1-b)e_m & 0 & 0 & 0 \\ (1-e_m) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} a^{1,0}_{p,p} + a' & 0 & 0 & 0 \\ (1-b)e_m & 0 & 0 & 0 \\ (1-e_m) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\end{aligned}$$

We will use part (iii) of Lemma 2.2 of [12], which states that left multiplication by a unitary in $M_4((C^*(H_0) \oplus C^*(H_1))^{\sim})$ does not change the class of this element in the relative K-group. We then note that

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & e_m & 1-e_m & 0 \\ 0 & 1-e_m & e_m & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a^{1,0}_{p,p} + a' & 0 & 0 & 0 \\ (1-b)e_m & 0 & 0 & 0 \\ (1-e_m) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} a^{1,0}_{p,p} + a' & 0 & 0 & 0 \\ 1-be_m & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then, we compute, noting that c is in $C^*(H_0) \oplus C^*(H_1)$,

$$\begin{bmatrix} 1-cc^* & c & 0 & 0 \\ c^* & 1-c^*c & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a^{1,0}_{p,p} + c & 0 & 0 & 0 \\ 1-be_m & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} a^{1,0}_{p,p} & 0 & 0 & 0 \\ 1-a^{0,0}_{p,p} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

□

Proposition 3.5. *In the six-term exact sequence of Theorem 2.4 of [13], the map from* $K_0(C^*(H))$ *to* $K_0(C^*(R_E))$ *is zero.*

Proof. The map is question is the composition of the isomorphism $K_0(C^*(H)) \cong K_0(C^*(H_0))$, the isomorphism $K_0(C^*(H_0)) \cong K_1(C^*(H'); C^*(H))$ of Theorem 3.1 and inverse of the map α of Proposition 3.4. We begin $[a^{0,0}_{p,p}]_0$, with p in $E_{0,n}$ with $t(p) = \xi(w)$, $w \in W$. By Theorem 3.1, the second isomorphism sends this to $\left[\begin{array}{cc} a^{1,0}_{p,p} & 0 \\ 1 - a^{0,0}_{p,p} & 0 \end{array}\right]_r$ in $K_1(C^*(H'); C^*(H))$. By Proposition 3.4, the inverse of α carries this to $\left[\begin{array}{cc} d^{1,0}_{p,p} & 0 \\ 1 - (d^{1,0}_{p,p})^* d^{1,0}_{p,p} & 0 \end{array}\right]_r$. Finally, the map from $K_1(C^*(R_E); C^*(R))$ to $K_0(C^*(R_E))$ sends this to

$$\left[\begin{array}{cc} d^{1,0}_{p,p}(d^{1,0}_{p,p})^* & 0 \\ 0 & 1 - (d^{1,0}_{p,p})^* d^{1,0}_{p,p} \end{array}\right]_0 - \left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right]_0 = [d^{1,0}_{p,p}(d^{1,0}_{p,p})^*]_0 - [(d^{1,0}_{p,p})^* d^{1,0}_{p,p}]_0.$$

As $d^{1,0}_{p,p}$ is the characteristic function of $\delta^{1,0}_{p,p}$, $d^{1,0}_{p,p}(d^{1,0}_{p,p})^*$ is the characteristic function of the source of $\delta^{1,0}_{p,p}$ while $(d^{1,0}_{p,p})^* d^{1,0}_{p,p}$ is the characteristic function of the range of $\delta^{1,0}_{p,p}$. The former is the set $U^1(p)$, while the latter is $U^0(p)$. We let

$$w = \cup_{i(f)=w} b_{p\xi^1(f), p\xi^0(f)}.$$

This is clearly in R_E and it is a simple matter to see that

$$ww^* = d^{1,0}_{p,p}(d^{1,0}_{p,p})^*, w^*w = (d^{1,0}_{p,p})^* d^{1,0}_{p,p}.$$

We conclude that the two projections in $C^*(R_E)$ are equivalent in $C^*(R_E)$. This completes the proof. □

We are now ready to prove the third and fourth parts of the main theorem, 1.1 .

We consider the six-term exact sequence given in Theorem 2.4 of [13].

$$\begin{array}{ccccc} K_0(C^*(H)) & \longrightarrow & K_0(C^*(R_E)) & \longrightarrow & K_0(C^*(R)) \\ \uparrow & & & & \downarrow \\ K_1(C^*(R)) & \longleftarrow & K_1(C^*(R_E)) & \longleftarrow & K_1(C^*(H)) \end{array}$$

We know that $C^*(H)$ and $C^*(R_E)$ are both AF-algebras and their K-one groups are trivial. In addition, we know that $K_0(C^*(R_E)) \cong G_0$ and $K_0(C^*(H)) \cong G_1$. Putting in all of this information, we have

$$\begin{array}{ccccc} G_1 & \longrightarrow & G_0 & \longrightarrow & K_0(C^*(R)) \\ \uparrow & & & & \downarrow \\ K_1(C^*(R)) & \longleftarrow & 0 & \longleftarrow & 0 \end{array}$$

Finally, the last Proposition tells us that the horizontal map from $K_0(C^*(H)) \cong G_1$ is zero. It follows that the two remaining maps are isomorphisms.

We must still show that the map on K_0 is an order isomorphism. As it arises from the inclusion of $C^*(R_E)$ in $C^*(R)$, it sends positive elements to positive elements. Conversely, suppose that an element in $K_0(C^*(R_E))$ has a positive image in $K_0(C^*(R))$. Then the image under that element under every trace on $C^*(R)$ is strictly positive. On the other hand, these traces all arise from R-invariant measures on X_E, which are precisely the R_E-invariant measures by 2.10. As $C^*(R_E)$ is a simple AF-algebra, it follows that the element is positive in $K_0(C^*(R_E))$.

Remark 3.6. *For readers who might be disappointed by the fact that our computation of $K_1(C^*(R))$ did not explicitly produce a single unitary in $C^*(R)$, let us note the following. If we fix a path p with $t(p)$ in $\xi(W)$, and define*

$$v_p = \left(\sum_{f \in F_{n+1}, \xi(i(f)) = t(p)} b_{p\xi^0(f), p\xi^1(f)} \right) d^{1,0}_{p,p},$$

then v_p is a partial isometry in $C^*(R)$ and

$$v_p^* v_p = v_p v_p^* = \sum_{f \in F_{n+1}, \xi(i(f)) = t(p)} b_{p\xi^0(f), p\xi^0(f)} \leq b_{p,p},$$

and $v_p + (1 - v_p^* v_p)$ is a unitary. Under the natural map from $K_1(C^*(R))$ to $K_1(C^*(R_E), C^*(R))$, it is sent to $\left[\begin{array}{cc} d^{1,0}_{p,p} & 0 \\ 1 - (d^{1,0}_{p,p})^* d^{1,0}_{p,p} & 0 \end{array} \right]_r$.

Pursuing this slightly further, it is not difficult to show that

$$C^*(B_{t(p)}, v_p + (b_{p,p} - v_p^* v_p)) \cong C(\mathbb{T}) \otimes B_{t(p)},$$

where $B_{t(p)}$ is as described earlier and $\mathbb{T}$ denotes the circle. Repeating this construction at every vertex of $\xi(W_n)$, we can construct a C^*-subalgebra of $C^*(R)$ which we denote by D_n, containing B_n, and with

$$D_n \cong \left(\oplus_{w \in W_n} C(\mathbb{T}) \otimes B_{\xi(w)} \right) \oplus \left(\oplus_{v \in V_n - \xi(W_n)} B_v \right).$$

This looks remarkably like an approximating subalgebra with the correct K-theory. The difficulty lies in the fact that D_n is not *contained, or even approximately contained, in D_{n+1}. (This is related to a similar problem with R_n and R_{n+1} is the last section.)*

Presumably, D_n is *approximately contained in D_m, for some m sufficiently large, but this would seem to need powerful classification tools to prove.*

Let us finish this section with a pair of relatively simple applications of Theorem 2.4 from [13]. We consider a Cantor set X with a minimal homeomorphism φ of X. Let R_φ denote the orbit relation of φ. That is, each equivalence class is $\{\varphi^n(x) \mid n \in \mathbb{Z}\}$, for some x in X. We select two points y, z having distinct orbits and let R be the subequivalence relation with exactly the same equivalence classes except $\{\varphi^n(y) \mid n \leq 0\}, \{\varphi^n(y) \mid n \geq 1\}, \{\varphi^n(z) \mid n \leq 0\}$ and $\{\varphi^n(z) \mid n \geq 1\}$ are all distinct equivalence classes [11].

Also, let R_1 be the subequivalence relation with exactly the same equivalence classes as R_φ except $\{\varphi^n(z) \mid n \leq 0\}$ and $\{\varphi^n(z) \mid n \geq 1\}$ are distinct equivalence classes. It follows from the results of [11] that both R and R_1 are AF-equivalence relations.

Next, draw a Bratteli diagram for R, so we may identify X with the path space of this diagram. Put an order on the edges (as in [7]) such that y is the unique infinite maximal path, while $\varphi(y)$ is the unique infinite minimal path. Let ψ be the associated Bratteli-Vershik map and R_ψ be its orbit relation.

It is an easy matter to see that R_ψ and R_1 are actually the same equivalence relation. They carry different topologies, however. The latter is AF, while the former is not. Both contain R as an open subequivalence relation and the two topologies agree on R. In each case, we may apply the results of [13] using

$$L = \{(\varphi^n(y), \varphi^m(y)) \mid n \leq 0, m \geq 1\}.$$

The set L is discrete in both topologies and in both cases the groupoids H_0, H_1 and H are the same. It is a simple matter to check that $C^*(H) \cong \mathcal{K}$, the C^*-algebra of compact operators, and $K_0(C^*(H)) \cong \mathbb{Z}$. If we consider the six-term exact sequences in Theorem 2.4 of [13], one involving $C^*(R), C^*(R_1)$ and $C^*(H)$ and the other involving $C^*(R), C^*(R_\psi)$ and $C^*(H)$, four of the terms are identical:

$$\begin{array}{ccccc} \mathbb{Z} & \longrightarrow & K_0(C^*(R)) & \longrightarrow & K_0(C^*(R_1)) \\ \uparrow & & & & \downarrow \\ K_1(C^*(R_1)) & \longleftarrow & 0 & \longleftarrow & 0 \end{array}$$

$$\begin{array}{ccccc} \mathbb{Z} & \longrightarrow & K_0(C^*(R)) & \longrightarrow & K_0(C^*(R_\psi)) \\ \uparrow & & & & \downarrow \\ K_1(C^*(R_\psi)) & \longleftarrow & 0 & \longleftarrow & 0 \end{array}$$

The difference arises in the maps between them. In the first case, $K_1(C^*(R_1)) = 0$ since it is an AF-algebra and the $\mathbb{Z}$ in the upper left corner maps injectively under the map to its right. In the second case, $K_1(C^*(R_\psi)) \cong \mathbb{Z}$ and the vertical map into $\mathbb{Z}$ is an isomorphism.

4 Tracially AF

The aim of this section is to prove that the C^*-algebra $C^*(R)$ is tracially AF.

We know that $C^*(R)$ is the closed linear span of the elements $b_{p,q}$, with (p,q) in I_m, along with $d^{i,1-i}_{p,q}$, over (p,q) in I^W_m and $i = 0, 1$. (We are forced to switch to m since n is already in use.)

Given the presence of the ϵ in the definition, it suffices for us to consider $\mathcal{F}$ to be a finite subset of such elements and then we may as well assume that $\mathcal{F}$ consists of all such $b_{p,q}, d^{i,1-i}_{p,q}$, for some fixed m. However, we may *not* assume that x_1 is of this form. But we can assume that it is in the span of such elements.

In addition, we have a faithful conditional expectation E from $C^*(R)$ onto $C(X_E)$, considered as the diagonal functions in $C_c(R)$. As a is strictly positive we may assume that $E(a)$ is strictly positive. We may choose a clopen subset in the interior of the support of $E(a)$ and let p_a be its charactersitic function, which we regard as an element of $C^*(R_E) \subseteq C^*(R)$. It suffices for us to choose p such that $1 - p$ is equivalent to a subprojection of p_a.

For $l > m$ and $i = 0, 1$, let Q^i_l denote the set of all paths q in $E_{0,l}$ such that $(q_{m+1}, \ldots q_l)$ is in $\xi^i(F_{m,l})$. Then we define

$$\bar{e}^i_l = \sum_{q \in Q^i_l} b_{p,p}, \bar{e}_l = \bar{e}^0_l + \bar{e}^1_l,$$

which is a projection in $C^*(R_E)$. We will use $p = 1 - \bar{e}_l$, for suitably chosen $l > m$. (We use $\bar{e}_l$ so that it is not confused with the e_m of the last section.)

Let us make some easy observations. First, it follows easily from Lemma 2.2 that we have

$$\bar{e}_l b_{p,p'} = b_{p,p'} \bar{e}_l$$

for all (p, p') in I_m and that the product lies in finite-dimensional C^*-algebra B_l as described following Definition 2.3, provided $l \geq m$. Next, we consider similar facts for products with the $d^{i,1-i}_{p,p'}$.

Lemma 4.1. *For all $m < l$, (p, p') in I_m and $i = 0, 1$, we have*

$$\bar{e}_l d^{i,1-i}_{p,p'} = d^{i,1-i}_{p,p'} \bar{e}_l$$

and

$$(1 - \bar{e}_l) d^{i,1-i}_{p,p'} \in B_l.$$

Proof. The proof will be by induction on $l - m$. In the case, $l - m = 1$ or $l = m + 1$, we will actually prove that $\bar{e}_l d^{i,1-i}_{p,p'} = d^{i,1-i}_{p,p'}$. For simplicity, we will assume that $i = 1$.

It is a consequence of Lemma 2.7 that we may write

$$\begin{aligned} d^{1,0}_{p,p'} &= \sum_{f \in A} d^{1,0}_{p\xi^1(f), p'\xi^0(f)} \\ &\quad + \sum_{(f,f') \in B} b_{p\xi^1(f)\xi^0(f'), q\xi^0(f)\xi^1(f')} \\ &\quad + \sum_{(f,e) \in C} b_{p\xi^1(f)e, q\xi^0(f)\xi^1 e} \end{aligned}$$

For fixed q in Q^i_l, we have seen earlier that the proof of Theorem 2.8 that $b_{q,q} d^{1,0}_{p\xi^1(f), p'\xi^0(f)}$ is equal to $d^{1,0}_{p\xi^1(f), p'\xi^0(f)}$ if $q = p\xi^1(f)$ and is zero otherwise. As $p\xi^1(f)$ is in Q^1_l and not in Q^0_l, summing over all q and i gives

$$\bar{e}_l d^{1,0}_{p\xi^1(f), p'\xi^0(f)} = d^{1,0}_{p\xi^1(f), p'\xi^0(f)}.$$

A similar argument, using the results of Lemma 2.2, shows the same conclusion for the second and third sum. By taking adjoints, we have the same conclusion for right multiplication by $\bar{e}_l$. That is, we have shown that $(1 - \bar{e}_l) d^{i,1-i}_{p,p'} = d^{i,1-i}_{p,p'} (1 - \bar{e}_l) = 0$.

Now, we assume that $l > m+1$. Again, we use the same decomposition of $d^{i,1-i}_{p,p'}$ as above. In the first sum, we use the obvious fact that $(p\xi^1(f), p'\xi^0(f))$ is in I^W_{m+1} so that we may apply the induction hypothesis. The conclusion follows at once for the first sum.

For the second sum, the path $p\xi^1(f)\xi^0(f')$ is in neither Q^1_{m+2} nor Q^0_{m+2} and it follows that $b_{q,q}b_{p\xi^1(f)\xi^0(f'),q\xi^0(f)\xi^1(f')} = 0$, for all q in $Q^j_l, j = 0,1$. Similarly, since e is in neither $\xi^0(F_{m+2})$ nor $\xi^1(F_{m+2})$, the path $p\xi^1(f)e$ is in neither Q^1_{m+2} nor Q^0_{m+2} and it follows that $b_{q,q}b_{p\xi^1(f)e,q\xi^0(f)e} = 0$, for all q in $Q^j_l, j = 0,1$. We have shown that left multiplication by $\bar{e}_l$ on the second and third sum yields zero. Hence, multiplication by $1-\bar{e}_l$ leaves these unchanged and they are already in $B_{m+2} \subseteq B_l$. □

Now we can verify the three properties to show that $C^*(R)$ is tracially AF. Our finite-dimensional C^*-subalgebra is $(1-\bar{e}_l)B_l(1-\bar{e}_l)$. We have already shown the first of the three properties.

Next, we turn to the conclusion $\|px_1p\| > \|x_1\| - \epsilon$. Let x be any point of X_E with none of its edges in either $\xi^0(F)$ or $\xi^1(F)$. We consider the representation π of $C^*(R)$ on $\ell^2[x]_R$ [14]. As $C^*(R)$ is simple and amenable, this representation is faithful. So we may find a unit vector η such that $\|\pi(x_1)\eta\| > \|x_1\| - \epsilon/2$. It is clear that the sequence $\pi(\bar{e}_l)\eta, l \geq 1$ converges to 0. So we may find l_0 sufficiently large that $\|\pi(\bar{e}_l)\eta\|_2 < \epsilon(2\|x_1\|)^{-1}$ for all $l \geq l_0$. For such l, we have

$$\begin{aligned}
\|x_1(1-\bar{e}_l)\| &\geq \|\pi(x_1)(1-\pi(\bar{e}_l))\eta\|_2 \\
&\geq \|\pi(x_1)\eta\|_2 - \|\pi(x_1)\pi(\bar{e}_l)\eta\|_2 \\
&\geq \|x_1\| - \epsilon/2 - \|x_1\|\epsilon(2\|x_1\|)^{-1} \\
&= \|x_1\| - \epsilon.
\end{aligned}$$

Finally, we turn to the issue of making $\bar{e}_l$ small in the third condition. It is clear from our choice of the embeddings ξ^i, that we have $[\bar{e}^i_m]_0 \geq 2^{l-m}[\bar{e}^i_l]_0$, for every $i = 0,1$, $l > m$. After summing over i, we have $[\bar{e}_m]_0 \geq 2^{l-m}[\bar{e}_l]_0$, and the desired conclusion follows. This completes the proof.

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