# AN EXCISION THEOREM FOR THE K-THEORY OF $C^*$-ALGEBRAS, WITH APPLICATIONS TO GROUPOID $C^*$-ALGEBRAS

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Abstract. We discuss the relative K-theory for a $C^*$-algebra, $A$, together with a $C^*$-subalgebra, $A' \subseteq A$. The relative group is denoted $K_i(A'; A), i = 0, 1$, and is due to Karoubi. We present a situation of two pairs $A' \subseteq A$ and $B' \subseteq B$ are related so that there is a natural isomorphism between their respective relative K-theories. We also discuss applications to the case where $A$ and $B$ are $C^*$-algebras of a pair of locally compact, Hausdorff topological groupoids, with Haar systems.

## 1. Introduction

The goal of this paper is the computation of K-theory groups of the reduced $C^*$-algebras of groupoids, meaning locally compact, Hausdorff groupoids with a Haar system. To be more specific, we will be concerned with a pair of groupoids which are related in some way so that one reduced $C^*$-algebra is a $C^*$-subalgebra of the other. Our results allow computation of the relative K-theory of this pair.

Results along this line have already been obtained in [13] and [14], but under very restrictive hypotheses. In particular, the groupoids there are principal and étale. Moreover, the relation between the pair of groupoids is very limited. Our aim here is to extend the generality of these results. At the same, we give a much simpler, more conceptual description of the isomorphism between relative groups that is our main objective.

As the theory of groupoid $C^*$-algebras becomes rather technical quite quickly, we will devote this section to a discussion of the the principal ingredients in the paper along with a couple of rather simple examples which nicely illustrate some of the main ideas.

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Our first key ingredient is the notion of *relative* K-theory. Given a $C^*$-algebra, $A$, and a $C^*$-subalgebra, $A'$, Karoubi [7] defined relative groups $K_i(A';A), i = 0,1$. We will review his definition (at least for $i = 0$) in the next section. (This can be defined for any $*$-homomorphism $\alpha : A' \to A$. We will not need the definition in this generality, but we refer the reader to [5] for more details.) A key consequence is the existence of a six-term exact sequence:

$$\begin{array}{ccccc} K_0(A';A) & \xrightarrow{\nu} & K_0(A') & \xrightarrow{i_*} & K_0(A) \\ \mu\uparrow & & & & \downarrow\mu \\ K_1(A) & \xleftarrow[i_*]{} & K_1(A') & \xleftarrow[\nu]{} & K_1(A';A) \end{array}$$

where $i$ denotes the inclusion map.

Our main results may be described as *excision*, meaning that the relative K-theory of a pair $A' \subseteq A$ depends only on $A - A'$. Of course, if $A', A$ are topological spaces $A - A'$ makes perfect sense as a topological space, but this doesn't make so much sense for $C^*$-algebras. In that setting, a nice first example of excision is the following: suppose that $A'$ is a closed two-sided ideal in $A$, then $K_i(A';A) \cong K_{i+1}(A/A')$, where $A/A'$ is the usual quotient $C^*$-algebra.

Let us re-state that result in a way which will be useful for comparison later. Suppose that $A, B, C$ are $C^*$-algebras and $\alpha : A \to C, \beta : B \to C$ are $*$-homomorphisms. If $\alpha(A) = \beta(B)$, then the fact that the kernels are ideals implies that

$$K_*(\ker(\alpha); A) \cong K_*(\alpha(A)) = K_*(\beta(B)) \cong K_*(\ker(\beta); B).$$

Our main results will be concerned with replacing $*$-homomorphisms in this statement with bounded $*$-derivations. At this point, we merely note that the kernel of a bounded $*$-derivation is a $C^*$-subalgebra, although not an ideal.

Let us give a very easy example using commutative $C^*$-algebras. Let $X$ be any compact, Hausdorff space. Choose two distinct points, $y_1, y_2$, in $X$ and let $X'$ be the quotient space obtained by identifying them and $\pi : X \to X'$ be the quotient map. This means that $\pi$ induces an injection of $C(X')$ in $C(X)$. Alternately, we can write

$$C(X') = \{f \in C(X) \mid f(y_1) = f(y_2)\}.$$

These two algebras differ only at the points $Y = \{y_1, y_2\}$ in $X$, or at $Y' = \{[y_1]\}$ in $X'$. We have the diagram:

$$\begin{tikzcd}
0 \arrow[r] & C_0(X-Y) \arrow[r] & C(X) \arrow[r] & C(Y) \arrow[r] & 0 \\
0 \arrow[r] & C_0(X'-Y') \arrow[r] \arrow[u, "="] & C(X') \arrow[r] \arrow[u, "\subseteq"] & C(Y') \arrow[r] \arrow[u, "\subseteq"] & 0
\end{tikzcd}$$

which is commutative and has exact rows. There is an associated six-term exact sequence involving the three relative K-groups, and since in one of them, the inclusion is actually an equality, that relative group is zero. We conclude that there is an isomorphism

$$K_i(C(X'); C(X)) \cong K_i(C(Y'); C(Y)), i = 0, 1.$$

The latter group is rather easy to compute using the exact sequence above.

This computation can also be regarded as an excision result. While considering this example, let us see how the notion of $*$-derivation can be useful. Assuming that $X$ is separable, let us choose a countable dense subset, $Y \subset Z \subset X$. There is an obvious representation of $C(X)$ on $\ell^2(Z)$ and a slightly less obvious one of $C(Y)$, which is zero on $\ell^2(Z-Y)$. Let $F$ be the self-adjoint operator which is the identity on $\ell^2(Z-Y)$ and such that $F\xi(y_i) = \xi(y_{3-i})$, for $i = 1, 2$ and $\xi$ in $\ell^2(Z)$. Also, let $\delta(a) = i[F, a]$, for any bounded operator $a$ on $\ell^2(Z)$. Observe that $\delta$ is a bounded $*$-derivation with

$$\begin{aligned}
\ker(\delta) \cap C(X) &= C(X') \\
\ker(\delta) \cap C(Y) &= C(Y') \\
\delta(C(X)) &= \delta(C(Y)).
\end{aligned}$$

Put in this way, the excision result above now looks similar to the earlier result on the kernels of $*$-homomorphisms.

We now look at another example, which has many similarities with the last, but displays some important new features. Let $X = \{0, 1\}^{\mathbb{N}}$, $X' = [0, 1]$ and $\pi : X \to X'$ be defined by

$$\pi(x) = \sum_{n=1}^{\infty} x_n 2^{-n},$$

for $x = (x_n)_{n=1}^{\infty}$ in $X = \{0, 1\}^{\mathbb{N}}$. This can also be described as the restriction of the devil's staircase to the Cantor ternary set. It is also known less formally as base 2 expansion of real numbers.

We let $Y' = \{k2^{-n} \mid n \geq 1, 0 < k < 2^n\}$ and $Y = \pi^{-1}(Y')$. It is a rather simple matter to check that $\pi$ is one-to-one on $X - Y$ and is two-to-one on $Y$. In fact, for $y'$ in $Y'$, $\pi^{-1}\{y'\}$ consists of two points, $(x_1, x_2, \ldots, x_n, 1, 0, 0, \ldots)$ and $(x_1, x_2, \ldots, x_n, 0, 1, 1, \ldots)$. At this point, the situation is very much like our last example. The significant difference is that $Y$ and $Y'$ are no longer closed and the diagram we had above is no longer available.

The solution here is to introduce new topologies on $Y \subseteq X$ and $Y' \subseteq [0,1]$ which are finer than the relative topologies from $X$ and $[0,1]$, respectively, in which they are locally compact (and still Hausdorff). In this case, the obvious choice is the discrete topologies. Let us continue the development with derivations we had in the earlier case. We represent $C(X)$ and $C_0(Y)$ as multiplication operators on $\ell^2(Y)$ (noting that $Y$ is conveniently dense in $X$). Observe that $C(X)$ acts as multipliers of $C_0(Y)$. This is a result of the fact that for any $f$ in $C(X)$, its restriction to $Y$ will remain continuous in any finer topology. Further, define $F$ to be the operator $F\xi(y) = \xi(\bar{y})$, where $\bar{y}$ is the unique point in $Y$ with $\pi(\bar{y}) = \pi(y)$ and $\bar{y} \neq y$. Again define $\delta(a) = i[F, a]$ for any bounded operator on $\ell^2(Y)$. It is a simple matter to check that

$$\begin{aligned}
\ker(\delta) \cap C(X) &= C(X') \\
\ker(\delta) \cap C_0(Y) &= C_0(Y') \\
\delta(C(X)) &= \delta(C_0(Y)).
\end{aligned}$$

In this situation, the conclusion that

$$K_i(C(X'); C(X)) \cong K_i(C_0(Y'); C_0(Y)), i = 0, 1.$$

follows from our main result, Theorem 3.4.

The key feature in this last example, which differs from the first we gave, is the idea that the subset where the two algebras of functions differ must be endowed with a new, finer topology. At the same time, we are interested in groupoid $C^*$-algebras and the issue of endowing a subgroupoid with a new finer topology so that the original algebra acts as multipliers of the smaller one is a considerable technical one.

The two examples we have listed above are part of a general class which we refer to as 'factor groupoids'. We develop the theory in some generality in section 7. The idea, avoiding many technical issues, is to take a surjective morphism of groupoids $\pi : G \to G'$. Under some hypotheses, we show that this induces an inclusion $C_r^*(G') \subseteq C_r^*(G)$. We then consider $H \subseteq G$ and $H' \subseteq G'$ to be the subgroupoids where the map $\pi$ fails to be one-to-one. Under a number of technical hypotheses, we first show that $H$ and $H'$ may be given new, finer topologies and prove that we have

$$K_*(C_r^*(G'); C_r^*(G)) \cong K_*(C_r^*(H'); C_r^*(H)),$$

the latter being significantly simpler to compute in many examples.

The other situation which is considered in section 6 is one we refer to an 'subgroupoids'. Here, we suppose that $G$ is a groupoid and $G^0 \subseteq G' \subseteq G$ is an open subgroupoid. Again we have an inclusion $C_r^*(G') \subseteq C^*(G)$. In this situation, we introduce $H \subseteq G$ and $H' \subseteq G'$ as the subgroupoids where the groupoids $G$ and $G'$ differ. Again, under a number of technical hypotheses we first show that $H$ and $H'$ may be given new, finer topologies and prove that we have

$$K_*(C_r^*(G'); C_r^*(G)) \cong K_*(C_r^*(H'); C_r^*(H)),$$

the latter again being significantly simpler to compute in many examples.

Let us mention that, if the groupoids are all amenable, then the Baum-Connes conjecture holds [18]. It seems likely that a proof of our results could be given by using this and conventional excision results in topology. We believe there is some virtue in working with the $C^*$-algebras themselves. In particular, this is preferable for doing the computations in most applications.

Let us briefly mention some applications of the results. In some cases, these will follow from the earlier paper [14].

For the subgroupoid situation, the simplest example of this are the so-called orbit-breaking subalgebras, $A_Y \subseteq C(X) \ltimes \mathbb{Z}$ first introduced in [12]. Indeed, we give a considerable generalization of this construction at the end of the section 6 in Theorem 6.18 and Corollary 6.19.

Another application was given in [15]. The main question is, given some $K$-theory data, can one construct an étale groupoid whose associated $C^*$-algebra falls in the Elliott classification scheme and has the given $K$-theory groups. In this case, assuming the $K$-zero group is a simple , acyclic dimension group $K$-one is torsion free, one begins with $G'$ as the AF-equivalence relation with that $K$-zero group and constructs $G' \subseteq G$ so that $K_0(G^*(G))$ remains the same, while $K_1(C^*(G))$ becomes the desired $K$-one group.

The subgroupoid results are also used in [4] for similar purposes, including the case of non-zero real rank $C^*$-algebras.

In [3], examples were given of non-homogeneous extensions of minimal Cantor $\mathbb{Z}$-actions. The $K$-theory of these extensions can be computed in specific examples [6], using the factor groupoid situation. Additionally, [6] considers quotients which may be constructed rather analogously to the extensions given in [15].

Finally, we mention work in progress with Rodrigo Treviño. This is based on work of Lindsey and Treviño [8] which begins with a bi-infinite ordered Bratteli diagram and constructs from it a flat surface with vertical foliation. Typically, the surface is infinite genus. The foliation $C^*$-algebra is actually a subalgebra of the AF-algebra associated with the Bratteli diagram. In fact, their groupoids can be related by a two-step process through a third groupoid. The first step is that the intermediate groupoid is a factor of the AF-equivalence relation. The second is that the foliation groupoid is a subgroupoid of the intermediate groupoid. An interesting consequence of these $K$-theory computations is that, if the $K$-zero group of the Bratteli diagram is not finitely-generated, then the surface is necessarily infinite genus.

The paper is organized as follows. The next section outlines basic facts about relative K-theory for $C^*$-algebras. The third section is our excision result. It is stated in considerable generality for derivations between $C^*$-algebras. Its proof is rather long and technical, so it appears separately in section 4.

In section 5, we turn to the rather general question: given a groupoid, $G$, and a subgroupoid, $H \subseteq G$, endowed with a finer topology, what conditions ensure that the reduced groupoid $C^*$-algebra of $G$ acts as multipliers of that of $H$?

In section 6, we combine the excision results of section three and those of section five to consider the situation of an open subgroupoid. In section 7, we do the same for the situation of a factor groupoid.

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## 2. Relative $K$-theory

In this section, we discuss a relative K-theory for $C^*$-algebras introduced by Karoubi [7]. Most of the basic ideas are already in [7], but we will add to them slightly.

The idea is to consider a $C^*$-algebra, $A$, together with a $C^*$-subalgebra, $A' \subseteq A$, and to define a relative group for the pair, denoted $K_0(A'; A)$. Let us mention that this definition has a generalization to the case where $\varphi : A' \to A$ is a $*$-homomorphism, but we need only consider the case when $\varphi$ is the inclusion map. More information can be found in [7] or [5].

We also remark that there is a definition of a second relative group $K_1(A'; A)$ (perhaps it would be more accurate to say a sequence of relative groups satisfying Bott periodicity). Our results hold for these groups as well, but we will not need that for our applications. Again, we refer the reader to [5] for more information.

Let $A$ be a unital $C^*$-algebra. As usual, for $n \geq 1$, we let $M_n(A)$ be the $n \times n$-matrices over $A$, regarded as a $C^*$-algebra. We use the usual (non-unital) inclusions $M_n(A) \subseteq M_{n+1}(A)$, for all $n \geq 1$ and let $\mathcal{M}(A)$ denote the union, regarded as a normed $*$-algebra. It is convenient to regard the elements of $\mathcal{A}$ as matrices, indexed by the positive integers, with only finitely many non-zero entries. We also let $\mathcal{P}(A)$ denote the set of all projections (self-adjoint idempotents) in $\mathcal{M}(A)$.

We consider the category whose objects are the elements of $\mathcal{P}(A)$. If $p$ and $q$ are in $\mathcal{P}(A)$, then the morphisms from $p$ to $q$ are the elements of $q\mathcal{M}(A)p$. Composition of morphisms is given by their product and the element $p$ is the identity morphism from $p$ to itself. We denote this category by $\mathcal{P}(A)$. It is an additive category in an obvious sense. Moreover, each set of morphisms is actually a Banach space in an obvious way and $\mathcal{P}(A)$ is a Banach category (II.2.1 and II.2.6 of [7]). Then $K_0(A)$ is defined to be the $K$-theory of this category, as in II.1 of [7]. (Some caution must be used: a homotopy of morphisms in this category takes place inside a single $q\mathcal{M}(A)p$, which is slightly different from a homotopy inside $\mathcal{M}(A)$.)

We let $\widetilde{A}$ denote the unitization of $A$. In the case that the $C^*$-algebra $A$ is not unital, $K_0(A)$ is defined as the kernel of the map induced from the usual homomorphism from $\widetilde{A}$ to $\mathbb{C}$. Conveniently, this conclusion also holds for unital $C^*$-algebras.

Every element of $\widetilde{A}$ can be written as a sum of a complex multiple of the unit and an element of $A$. If $a$ is in $\widetilde{A}$, we let $\dot{a}$ denote the complex number involved. We extend this notation to elements $a$ in matrices over $\widetilde{A}$, so that $\dot{a}$ is a complex matrix of the same size.

We now suppose that $A$ is a $C^*$-algebra and $A'$ is a $C^*$-subalgebra. To define a relative group, we follow the ideas of [7], using $\varphi : \mathcal{P}(A') \to \mathcal{P}(A)$ being the inclusion map, but make some minor alterations. First, we would like to include non-unital $A$ and $A'$, so we consider the obvious unital inclusion of $\widetilde{A}'$ in $\widetilde{A}$. Second, we will suppress this map in our notation. We consider triples $(p, q, a)$, where $p$ and $q$ are objects in $\mathcal{P}(\widetilde{A}')$ and $a$ is an invertible morphism from $p$ to $q$ in $\mathcal{P}(\widetilde{A})$. Specifically, if $p$ is in $\mathcal{M}(\widetilde{A}')$ and $q$ is in $\mathcal{M}(\widetilde{A}')$, then $a$ is in $q\mathcal{M}(\widetilde{A})p$ and there is $b$ in $p\mathcal{M}(\widetilde{A})q$ such that $ab = q$ and $ba = p$. We let $\Gamma(A'; A)$ denote the set of all such triples. Although it is likely to raise a storm of controversy, we note that 0 is an invertible morphism from $\{0\}$ to $\{0\}$, so $(0, 0, 0)$ is in $\Gamma(A'; A)$.

We say two such triples $(p, q, a)$ and $(p', q', a')$ are *isomorphic* if there are isomorphisms $c$ from $p$ to $p'$ and $d$ from $q$ to $q'$ in $\mathcal{P}(\widetilde{A}')$ such that $da = a'c$. In this situation, we also say they are *isomorphic via* $c, d$. In particular, if $(p, q, a)$ is in $\Gamma(A'; A)$ and $a$ lies in $\mathcal{M}(\widetilde{A}')$, then $(p, q, a)$ is isomorphic to $(q, q, q)$ via the pair $a, q$.

Let us briefly mention that there are some small difficulties in taking direct sums of elements of $\mathcal{M}(A)$: if the elements are regarded as $a \in M_m(A)$ and $b \in M_n(A)$, then $a \oplus b \in M_{m+n}(A)$. This is not quite consistent with the identification of elements of $M_n(A)$ with those in $M_{n+1}(A)$. On the other hand, the result $a \oplus b$ is well-defined up to isomorphism as above and this ambiguity will not cause any confusion.

A triple $(p, q, a)$ is *elementary* if $p = q$ and $a$ is homotopic to $p$ within the automorphisms of $p$ in $\mathcal{P}(\widetilde{A})$. If, in addition, $a$ is actually a unitary in $p\mathcal{M}(\widetilde{A})p$, then there exists a homotopy from $p$ to $a$ within the unitaries. We also make the observation that $(p, p, a)$ is elementary if and only if $a$ is obtained as an invertible element of the $C^*$-algebra

$$\{f \in C([0, 1], p\mathcal{M}(\tilde{A})p) \mid f(0) \in \mathbb{C}p\},$$

when evaluated at 1.

Finally, we introduce an equivalence relation $\sim$ on $\Gamma(A'; A)$ as follows. Two triples $(p, q, a) \sim (p', q', a')$ if there are elementary triples $(p_1, p_1, a_1)$ and $(p_2, p_2, a_2)$ such that

$$(p, q, a) \oplus (p_1, p_1, a_1) = (p \oplus p_1, q \oplus p_1, a \oplus a_1)$$

is isomorphic to $(p', q', a') \oplus (p_2, p_2, a_2)$. Clearly, isomorphic triples are equivalent and any elementary triple is equivalent to $(0, 0, 0)$.

We define $K_0(A'; A)$ as the set of equivalence classes of the elements of $\Gamma(A'; A)$ in relation $\sim$. We denote the equivalence class of $(p, q, a)$ by $[p, q, a]$. It is a simple matter to check that

$$[p, q, a] + [p', q', a'] = [p + p', q + q', a + a'],$$

for $(p, q, a), (p', q', a')$ in $\Gamma(A'; A)$ with $pp' = qq' = 0$, is a well-defined binary operation. Alternately, we could define

$$[p, q, a] + [p', q', a'] = [p \oplus p', q \oplus q', a \oplus a'].$$

The element $[0, 0, 0]$ is the identity and any element $[p, q, a]$ has inverse $[q, p, b]$, where $b$ satisfies $ab = q, ba = p$. Hence, $K_0(A'; A)$ is a group.

Suppose that $\pi : A \to B$ is any $*$-homomorphism between two $C^*$-algebras and $A' \subseteq A, B' \subseteq B$ are two subalgebras satisfying $\pi(A') \subseteq B'$, then we may extend $\pi$ to a unital map from $\tilde{A}$ to $\tilde{B}$ and to matrices over $\tilde{A}$ and it follows that $\pi$ induces a group homomorphism $\pi_* : K_0(A'; A) \to K_0(B'; B)$.

We will not give a proof of the following result, but refer the reader to Theorem 2.1 of [5].

**Theorem 2.1.** *Let $A$ be a $C^*$-algebra and let $A'$ be a subalgebra. There is an exact sequence*

$$K_1(A') \xrightarrow{i_*} K_1(A) \xrightarrow{\mu} K_0(A'; A) \xrightarrow{\nu} K_0(A') \xrightarrow{i_*} K_0(A) ,$$

*where $i : A' \to A$ denotes the inclusion map, $\mu([u]_1) = [1_n, 1_n, u]$, for any unitary $u$ in $M_n(\widetilde{A})$ and $\nu[p, q, a] = [p]_0 - [q]_0$, for any $(p, q, a)$ in $\Gamma(A'; A)$.*

As an immediate consequence, we note that in the special cases, $\mu : K_1(A) \to K_0(0, A)$ is an isomorphism, while $K_0(A; A) = 0$.

**Example 2.2.** *Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{N}$ be a closed subspace with $\mathcal{N}^\perp$ its orthogonal complement. Assume that $\mathcal{N} \neq 0, \mathcal{H}$. We consider $A = \mathcal{K}(\mathcal{H})$, the $C^*$-algebra of compact operators on $\mathcal{H}$, and $A' = \mathcal{K}(\mathcal{N}) \oplus \mathcal{K}(\mathcal{N}^\perp)$. From the short exact sequence in Theorem 2.1 and the well-known result that $K_0(\mathcal{K}(\mathcal{H})) \cong \mathbb{Z}$, via the usual trace, and $K_1(\mathcal{K}(\mathcal{H})) \cong 0$, we see that $K_0(A'; A)$ is isomorphic to the kernel of the map $i_*$, which consists of pairs $([p]_0 - [p']_0, [q]_0 - [q']_0)$, where $p, p'$ are finite rank projections on subspaces of $\mathcal{N}$, $q, q'$ are finite rank projections on subspaces of $\mathcal{N}^\perp$ and $Rank(p) - Rank(p') = Rank(q') - Rank(q)$. Associating to such an element $Rank(p) - Rank(p')$ is an isomorphism from this subgroup to $\mathbb{Z}$.*

*We can give a useful classification of many elements in the relative group as follows. Suppose $S, S'$ are finite rank operators on $\mathcal{H}$ with $S(\mathcal{N}) \subseteq \mathcal{N}^\perp$, $S(\mathcal{N}^\perp) = 0$ and and $S'(\mathcal{N}) = 0, S'(\mathcal{N}^\perp) \subseteq \mathcal{N}$. Define $P, Q, P', Q'$ as the projections onto the initial space of $S$, the range of $S$, the initial space of $S'$ and the range of $S'$, respectively. Then $(P + P', Q + Q', S + S')$ determines an element of $\Gamma(A'; A)$. The composition of the map $\nu$ with the isomorphism described above sends the class of this element to*

$$Rank(P) - Rank(P') = Rank(S) - Rank(S').$$

We want to establish a few simple properties of the triples under consideration. We refer the reader to Proposition 3.3 of [5] for a proof of the following.

**Proposition 2.3.** *Let $A$ be a $C^*$-algebra and $A'$ be a $C^*$-subalgebra of $A$.*

(1) *If $(p, q, a)$ is in $\Gamma(A'; A)$ and $a$ lies in $\mathcal{M}(\widetilde{A'})$, then $[p, q, a] = 0$.*
(2) *If $a(t), 0 \leq t \leq 1$ is a continuous path of invertible elements in $q\mathcal{M}(\widetilde{A})p$ with $p, q$ in $\mathcal{M}(\widetilde{A'})$, for all $0 \leq t \leq 1$, then $[p, q, a(0)] = [p, q, a(1)]$.*
(3) *Every element of $\Gamma(A'; A)$ is equivalent to one of the form $(1_m, q, a)$, where $a$ is a partial isometry with $a^*a = 1_m$, $aa^* = q$ and $\dot{a} = \dot{q} = 1_m$. We refer to such an element as being in standard form.*
(4) *For $(1_m, q, a)$ in $\Gamma(A'; A)$ is in standard form, $[1_m, q, a] = 0$ if and only if there exists $n \geq 1$, and elementary triples $(1_n, 1_n, a_1)$ and $(1_{m+n}, 1_{m+n}, a_2)$ in standard form such that $a_2(a \oplus a_1)^*$ is in $\mathcal{M}(\widetilde{A'})$.*
(5) *If $(p, q, a), (p', q', b)$ are in $\Gamma(A'; A)$ and satisfy $q = p'$, then*
$$[p, q, a] + [p', q', b] = [p, q', ba].$$
(6) *If $(p, q, a)$ is in $\Gamma(A', A)$, then $-[p, q, a] = [q, p, a^*]$.*

As a final item for this section, we need a result that links relative groups for a pair of short exact sequences. This will be used in a key way in the next section in defining our excision map.

**Theorem 2.4.** *Let*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\pi} & C & \xrightarrow{\rho} & D & \longrightarrow & 0 \\ & & \cup\!| & & \cup\!| & & \| & & \\ 0 & \longrightarrow & A' & \xrightarrow{\pi} & C' & \xrightarrow{\rho} & D & \longrightarrow & 0 \end{array}$$

*be a commutative diagram with exact rows. Then the natural map $\pi_* : K_0(A'; A) \to K_0(C'; C)$ is an isomorphism.*

This is a consequence of a more general result which begins with a diagram like the one above, but without the equality on the right, and shows there is a six-term exact sequence of relative groups. We refer the reader to Theorem 2.2 of [5]. Our conclusion follows from this and the fact that $K_*(D; D) = 0$ (which follows from Theorem 2.1 of [5]).

**Remark 2.5.** *Let us make a few concluding remarks on the subject of relative K-theory. First of all, there is also a relative group $K_1(A'; A)$. Without going into many details, one considers triples $(p, a, g)$, where $p$ is a projection in $\widetilde{\mathcal{A}'}$, $a$ is an invertible in $p\mathcal{M}(\widetilde{A'})p$ and $g(t), 0 \leq t \leq 1$,* *is a continuous path of invertibles in* $p\mathcal{M}(\widetilde{A})p$ *with* $b(0) = p, b(1) = a$, *We refer the reader to* [5] *for more details. The only important result for us here is that*

$$K_1(A'; A) \cong K_0(C_0(\mathbb{R}) \otimes A'; C_0(\mathbb{R}) \otimes A),$$

*in a natural way so that all of our results here for the relative* $K_0$*-group pass to the relative* $K_1$*-group as well. Of course, the exact sequence of Theorem 2.1 actually becomes a six-term exact sequence.*

*Furthermore, the setting of relative K-theory in* [5] *considers a* $*$*-homomorphism* $\varphi : A' \to A$ *and defines a relative group for the map,* $K_0(\varphi)$*. Our situation is simply the special case of the inclusion map.*

*There is an alternate approach to relative K-theory, which is to define* $K_0(A'; A)$ *to be the K-zero group of the mapping cylinder*

$$\{f : [0, 1] \to A \mid f \text{ continuous }, f(0) = 0, f(1) \in A'\}.$$

*This actually gives the same answer (see* [5]*). However, it is not really useful for us; while our excision map can be defined in quite general terms, we really need Karoubi's description of the cycles in order to prove it is an isomorphism.*

## 3. Excision: the main result

We begin with the following result, which is a minor variant of a well-known fact. The important part of the set-up is that we do not suppose that either $C^*$-algebra is acting non-degenerately.

**Proposition 3.1.** *Let* $\mathcal{H}$ *be a Hilbert space. If* $A, B \subseteq \mathcal{B}(\mathcal{H})$ *are* $C^*$*-algebras and* $AB \subseteq A$*, then* $A + B$ *is a* $C^*$*-algebra,* $A$ *is a closed two-sided ideal in* $A + B$ *and the quotient is isomorphic to* $B/A \cap B$.

*Proof.* It is clear that $A + B$ is a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ and that $A$ is a closed, two-sided ideal. We will show that $A + B$ is closed and the proof will be complete.

Let $\mathcal{N}$ be the closure of $A\mathcal{H}$. The condition that $AB \subseteq A$ means that $\mathcal{N}$ is invariant under $B$. Hence, writing $\mathcal{H} = \mathcal{N} \oplus \mathcal{N}^\perp$, each element $b$ of $B$ may be written as $b|_{\mathcal{N}} \oplus b|_{\mathcal{N}^\perp}$. As $A$ acts non-degenerately on $\mathcal{N}$, $b|_{\mathcal{N}}$ lies in $M(A)$, the multiplier algebra of $A$, for every $b$ in $B$. We regard $M(A)$ as a subalgebra of $\mathcal{B}(\mathcal{N})$.

Let $q : M(A) \to M(A)/A$ be the quotient map and $\theta : (M(A) \oplus 0) + B \to M(A)/A \oplus \mathcal{B}(\mathcal{N}^\perp)$ be the map sending $c$ to $q(c|_{\mathcal{N}}) \oplus c|_{\mathcal{N}^\perp}$. Since this is a $*$-homomorphism, it is continuous and $\theta(B)$ is closed. We note that $A + B = \theta^{-1}(\theta(B))$ is closed also. $\square$

We will assume throughout this section that $A$ and $B$ are related as above and we let $i : A \to A + B$ and $j : B \to A + B$ denote the two inclusion maps.

**Theorem 3.2.** *Let $A, B \subseteq \mathcal{B}(\mathcal{H})$ be $C^*$-algebras with $AB \subseteq A$.*

*Let $E$ be a Banach $A + B$-bimodule which is also a $C^*$-algebra and let $\delta : A + B \to E$ be a bounded $*$-derivation satisfying $\delta(B) \subseteq \delta(A)$.*

*Then*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{i} & A+B & \longrightarrow & (A+B)/A & \longrightarrow & 0 \\ & & \cup\vert & & \cup\vert & & \Vert & & \\ 0 & \longrightarrow & \ker(\delta) \cap A & \xrightarrow{i} & \ker(\delta) & \longrightarrow & (A+B)/A & \longrightarrow & 0 \end{array}$$

*is a commutative diagram with exact rows. In consequence,*

$$i_* : K(\ker(\delta) \cap A; A) \to K(\ker(\delta); A + B)$$

*is an isomorphism and*

$$\alpha = (i_*)^{-1} \circ j_* : K(\ker(\delta) \cap B; B) \to K(\ker(\delta) \cap A; A).$$

*is a homomorphism.*

*Proof.* Exactness of the top row follows from Proposition 3.1 and commutativity is obvious. Exactness of the bottom row is easy except in showing that the map from $\ker(\delta)$ is surjective. It is clear that any element of $(A + B)/A$ can be represented as $b + A$, with $b$ in $B$. By hypothesis, there exists $a$ in $A$ with $\delta(a) = \delta(b)$. It follows that $b - a$ is in $\ker(\delta)$ and its image in $(A + B)/A$ is simply $b + A$.

The rest follows from Theorem 2.2. □

Let us consider a special case of interest, just to see that the hypotheses are quite general and that the map $\alpha$ is non-trivial. Let $B$ be any $C^*$-algebra and suppose that $(\pi, \mathcal{H}, F)$ is a Fredholm module for $B$: that is, $\mathcal{H}$ is a separable Hilbert space, $\pi : B \to \mathcal{B}(\mathcal{H})$ is a representation of $B$ on $\mathcal{H}$ and $F$ is a bounded linear operator on $\mathcal{H}$ such that

$$\pi(b)(F - F^*), \pi(b)(F^2 - 1), [\pi(b), F] = \pi(b)F - F\pi(b),$$

are all compact, for any $b$ in $B$. We assume the slightly stronger conditions that $F = F^*, F^2 = 1$ and, for simplicity, that $B$ and $\pi$ are unital. Let $P = \frac{1}{2}(F + 1)$, which is a self-adjoint projection.

Such a Fredholm module induces a natural homomorphism from $K_1(B)$ to the integers, which sends $[u]_1$, where $u$ is a unitary in $M_n(B)$, to

$$Ind((1_n \otimes P)(id_n \otimes \pi)(u)|_{\mathbb{C}^n \otimes P\mathcal{H}}),$$

where $Ind$ denotes Fredholm index. We denote this map by $Ind$ for simplicity.

We can apply our Theorem 3.2 in this situation as follows. Let $A = \mathcal{K}(\mathcal{H})$, use $\pi(B)$ as the $C^*$-algebra $B$ in 3.2, $E = \mathcal{B}(\mathcal{H})$, which is obviously an $A + \pi(B)$ bimodule and define

$$\delta(x) = i[x, F], x \in A + \pi(B).$$

It is a simple matter to see that

$$\ker(\delta) \cap A = \mathcal{K}(P\mathcal{H}) \oplus \mathcal{K}((I - P)\mathcal{H})$$

and the relative group $K_0(\ker(\delta) \cap A; A) \cong \mathbb{Z}$ has already been computed in Example 2.2.

**Theorem 3.3.** *Let $B$ be a unital $C^*$-algebra and $(\pi, \mathcal{H}, F)$ be a Fredholm module for $B$ with $\mathcal{H}$ separable, $F = F^*, F^2 = I, F \neq \pm I$ and $\pi$ unital. With $E, \delta, A, A', P$ as above, the following diagram is commutative*

$$\begin{tikzcd}
K_1(B) \arrow[rr, "Ind"'] \arrow[d, "\mu\circ\pi"] & & \mathbb{Z} \\
K_0(\ker(\delta) \cap \pi(B); \pi(B)) \arrow[r, "\alpha"] & K_0(\ker(\delta) \cap A; A) \arrow[ur, "\cong"'] &
\end{tikzcd}$$

*where $\mu$ is as in Theorem 2.1, $\alpha$ is as in Theorem 3.2 and the vertical arrow on the right is the isomorphism described in Example 2.2.*

*Proof.* For simplicity, we consider a unitary $u$ in $B$, instead of $M_n(B)$, for some $n$. Let $P_1, P_2, P_3, P_4$ be the orthogonal projections of $\mathcal{H}$ onto each of the following four subspaces:

$$\begin{array}{rl} P\mathcal{H} \cap \pi(u)^* P\mathcal{H}, & P\mathcal{H} \cap \pi(u)^*(I - P)\mathcal{H}, \\ (I - P)\mathcal{H} \cap \pi(u)^*(I - P)\mathcal{H}, & (I - P)\mathcal{H} \cap \pi(u)^* P\mathcal{H}. \end{array}$$

As $\pi(u)$ is unitary, $P_1 + P_2 + P_3 + P_4 = I$. The projections $P_2$ and $P_4$ are both finite rank from the Fredholm module condition and the index of $P\pi(u)|_{P\mathcal{H}}$ is simply $\dim(P_2\mathcal{H}) - \dim(P_4\mathcal{H})$. Also let $Q_i = \pi(u)P_i\pi(u)^*, 1 \leq i \leq 4$.

Letting $j_*$ be as in Theorem 3.2, $j_* \circ \mu \circ \pi[u]_1$ is represented by the class of $(I, I, \pi(u))$ in $\Gamma(\ker(\delta), \pi(B) + A)$. It follows that

$$\begin{aligned} [I, I, \pi(u)] &= [P_1 + P_3, Q_1 + Q_3, \pi(u)(P_1 + P_3)] \\ &\quad + [P_2 + P_4, Q_2 + Q_4, \pi(u)(P_2 + P_4)]. \end{aligned}$$

On the other hand, $\pi(u)(P_1+P_3)$ commutes with $P$ and, hence, is in $\ker(\delta)$, so $[P_1+P_3, Q_1+Q_3, \pi(b)(P_1+P_3)] = 0$. We have

$$\alpha \circ \mu \circ \pi([u]_1) = i_*^{-1} \circ j \circ \mu \circ \pi([u]_1) = [P_2+P_4, Q_2+Q_4, \pi(u)(P_2+P_4)]$$

and the isomorphism of Example 2.2 (using $\mathcal{H}^+ = P\mathcal{H}$) maps this to

$$rank(\pi(u)P_2) - rank(\pi(u)P_4) = \dim(P_2\mathcal{H}) - \dim(P_4\mathcal{H}).$$

□

We will continue to assume throughout that $A, B, \delta, E$ are as in Theorem 3.2. Our main goal is to provide extra conditions under which the map $\alpha$ is actually an isomorphism.

**Theorem 3.4.** *Let $A, B, E, \delta$ be as in 3.2. Suppose that $A$ has a dense $*$-subalgebra, $\mathcal{A}$ satisfying the following.*

C1 *There is a constant $K \geq 0$ such that, for every $a$ in $\mathcal{A}$, there is $a'$ in $\ker(\delta) \cap \mathcal{A}$ such that*

$$\|a - a'\| \leq K\|\delta(a)\|.$$

C2 *For every $a_1, \ldots, a_I$ in $\mathcal{A}$, there is $0 \leq e \leq 1$ in $M(A)$ and $b_1, \ldots, b_I$ in $B$ such that such that*
(a) *$a_i = ea_i = a_ie = eb_i = b_ie$, and*
(b) *$\delta(b_i) = \delta(a_i)$,*
*for all $1 \leq i \leq I$.*

*Then $\delta(A) = \delta(B)$ is closed and the map $\alpha$ of Theorem 3.2 is an isomorphism.*

All of our later applications will be to groupoid $C^*$-algebras. In such cases, the dense $*$-subalgebras of continuous compactly supported functions on the groupoid will form the $*$-subalgebra, $\mathcal{A}$.

Let us remark that it seems reasonable to conjecture that one could replace conditions C1 and C2 of Theorem 3.4 with the hypothesis that $\delta(A) = \delta(B)$ is closed. It certainly makes for a cleaner result. On the other hand, if one wants to verify this condition in the case of groupoid $C^*$-algebras, then employing C1 and C2 to do this (and using the dense subalgebras of continuous compactly-supported functions) is not an unreasonable route.

We comment that the relation between the statement of the result and its applications is rather similar to the first isomorphism theorem for groups. That statement is, given a surjective group homomorphism, $\alpha : G \to H$, the map induces an isomorphism between the quotient group $G/ker(\alpha)$ and $H$. In most applications, however, one is not given $\alpha, G$ and $H$, but rather $G$ and a normal subgroup $N$ and then tries to cook up an $\alpha$ and $H$ such that $N = \ker(\alpha)$, so that one may identify

$G/N$ with the (hopefully) more familiar group, $H$. Our applications usually involve starting with $B' \subseteq B$ and trying to cook up $\delta, A$ and $E$ so as to apply Theorem 3.4 and with $B' = B \cap \ker(\delta)$. The hope, in this case, is that $A \cap \ker(\delta) \subseteq A$ is simpler than $B' \subseteq B$.

## 4. The proof

This section is devoted to a proof of Theorem 3.4. We start by expanding on condition C1. We remark that, informally, this means that *almost* being in $\ker(T)$ ($Tx$ is small) implies *nearly* being in $\ker(T)$ (near an element in $\ker(T)$). The following result is quite standard and we omit the proof.

**Proposition 4.1.** *Let $T : X \to Y$ be a bounded linear map between two Banach spaces. The following are equivalent.*

(1) *$T(X)$ is closed in $Y$.*
(2) *There is a constant $K \geq 1$ such that, for every $x$ in $X$, there is $x'$ in $\ker(T)$ such that*

$$\|x - x'\| \leq K\|Tx\|.$$

(3) *There is a dense linear subspace $\mathcal{X} \subseteq X$ and a constant $K \geq 1$ such that, for every $x$ in $\mathcal{X}$, there is $x'$ in $\ker(T)$ such that*

$$\|x - x'\| \leq K\|Tx\|.$$

Observe that our condition C1 is stronger than the third condition above because C1 requires $a'$ to be in $\ker(\delta) \cap \mathcal{A}$, and not simply in $\ker(\delta)$.

An immediate nice consequence is the following.

**Lemma 4.2.** *If $\mathcal{A}$ is a dense subalgebra of $A$ satisfying condition C1 of Theorem 3.4 and there is a dense $*$-subalgebra, $\mathcal{B}$ in $B$ such that $\delta(\mathcal{B}) \subseteq \delta(\mathcal{A})$, then we have $\delta(B) \subseteq \delta(A)$.*

*Proof.* It follows from C1 and Proposition 4.1 that the extension of $\delta$ to $A$ has closed range. As $\delta$ is bounded and $\mathcal{B}$ is dense in $B$, we are done. □

As we will need to deal with elements of the relative $K$-groups, it will be useful to have the following, which considers the unitization of algebras, matrices over algebras and the cones over algebras.

**Proposition 4.3.** *Suppose that $A, B, E, \delta$ be as in Theorem 3.2 and $\mathcal{A}$ satisfy conditions C1 and C2 of Theorem 3.4.*

(1) *If we define* $\delta : \widetilde{A+B} \to E$ *by*

$$\delta(\lambda + a + b) = \delta(a + b), a \in A, b \in B, \lambda \in \mathbb{C},$$

*then* $\delta$ *is a bounded* $*$*-derivation and* $\delta(\tilde{B}) \subseteq \delta(\tilde{A})$*.*

(2) *For any integer* $n \geq 1$*,* $M_n(A), M_n(B), M_n(E), id_{M_n} \otimes \delta$ *satisfy the conditions of Theorem 3.2 and* $M_n(\mathcal{A})$ *satisfy conditions C1 (although* $K$ *may depend on* $n$*) and C2 of Theorem 3.4. Moreover, the* $e$ *of condition C2 may be chosen to be of the form* $1_n \otimes e$*, where* $e$ *is in* $M(A)$*.*

(3) $C_0(0,1] \otimes A, C_0(0,1] \otimes B, C_0(0,1] \otimes E, id_{C_0(0,1]} \otimes \delta$ *satisfy the conditions of Theorem 3.2. In addition, let* $C_c(0,1] \odot \mathcal{A}$ *denote the algebraic tensor product of the continuous, compactly supported functions on* $(0,1]$ *with* $\mathcal{A}$*. That is, it consists of all functions of the form*

$$f(t) = \sum_{i=1}^{I} f_i(t) a_i,$$

*where* $f_1 \ldots, f_I$ *are in* $C_c(0,1]$ *and* $a_1, \ldots, a_I$ *are in* $\mathcal{A}$*. Then* $C_c(0,1] \odot \mathcal{A}$ *is a dense* $*$*-subalgebra of* $C_0(0,1] \otimes A$ *which satisfies conditions C1 and C2 of Theorem 3.4.*

*Proof.* The first part is trivial and we omit the proof. For the second part, the proof of C1 is trivial.

Let us sketch the proof of C2 for the second part. If we are given a collection of $m$ elements of $M_n(\mathcal{A})$, their individual entries provide $n^2 m$ elements of $\mathcal{A}$. If we select an appropriate $e$ and $n^2 m$ elements of $B$, it is fairly easy to check that $1_n \otimes e$ and the corresponding $m$ elements of $M_n(B)$ satisfy the desired conclusion.

We now consider the third part, beginning with C1. Let

$$f(t) = \sum_{i=1}^{I} f_i(t) a_i,$$

be in $C_c(0,1] \odot \mathcal{A}$. If $(1 \otimes \delta)(f) = 0$, we are done. Otherwise, let $0 < \epsilon = \|(1 \otimes \delta)(f)\|$. We may choose $N$ sufficiently large so that $\|f(s) - f(t)\| < \epsilon$, whenever $|s - t| < N^{-1}$. Also choose $N$ sufficiently large so that $f(t) = 0$, for all $0 \leq t \leq N^{-1}$. For each $1 \leq n \leq N$, let $g_n$ be the function which is 0 on $[0, (n-1)/N] \cup [(n+1)N, 1]$, 1 at $n/N$, and linear on $((n-1)/N, n/N)$ and $(n/N, (n+1)/N)$. Let

$$g(t) = \sum_{n=1}^{N} g_n(t) f(n/N), t \in [0,1].$$

It is an easy computation to see that $\|g - f\| < \epsilon$. For $2 \leq n \leq N$, we may find $a'_n$ in $\mathcal{A} \cap \ker(\delta)$ with

$$\|a'_n - f(n/N)\| \leq K\|\delta(f(n/N))\| \leq K\|1 \otimes \delta(f)\|, 1 \leq n \leq N.$$

We note that $f(1/N) = 0$ and so set $a'_1 = 0$. We then define $f'(t) = \sum_{n=1}^{N} g_n(t)a'_n$. It is clear that $f'$ is in $\ker(1 \otimes \delta)$ and also

$$\begin{aligned}
\|f - f'\| &\leq \|f - g\| + \|g - f'\| \\
&\leq \|(1 \otimes \delta)(f)\| + \sum_{n=1}^{N} g_n(t)\|f(n/N) - a'_n\| \\
&\leq \|(1 \otimes \delta)(f)\| + \sum_{n=1}^{N} g_n(t)K\|1 \otimes \delta(f)\| \\
&\leq (K + 1)\|(1 \otimes \delta)(f)\|.
\end{aligned}$$

This completes the proof of C1.

For C2, if we have a finite collection of functions, $f^{(1)}, \ldots, f^{(I)}$ in $C_c(0, 1] \odot \mathcal{A}$, we may write all of them in the form $f^{(i)}(t) = \sum_{j=1}^{J} f_j^{(i)}(t)a_j$, for all $1 \leq i \leq I$, simply by using all the possible elements of $\mathcal{A}$ involved in each and using $f_j^{(i)} = 0$ as needed.

We then choose $b_j, 1 \leq j \leq J, e$ as in C2 for $\mathcal{A}$. As each $f_j^{(i)}$ is compactly supported, we may find $0 < \epsilon$ such that each is zero on $[0, \epsilon]$. Let $g(t)$ be the continuous function that is zero on $[0, \epsilon/2]$, 1 on $[\epsilon, 1]$ linear on $[\epsilon/2, \epsilon]$. It is now easy to that the set of elements $\sum_{j=1}^{J} f_j^{(i)} b_j$ and $e(t) = g(t)e$ satisfies the desired conclusion. □

**Lemma 4.4.** *Suppose $a, b, e$ are elements of a $C^*$-algebra satisfying $e^* = e$,*

$$a = ae = ea = be = eb$$

*and $\delta$ is a $*$-derivation on that $C^*$-algebra with $\delta(a) = \delta(b)$. It follows that for any continuous function $f$ on the positive real numbers with $f(0) = 0$, we have $\delta(bf(b^*b)) = \delta(af(a^*a))$. In addition, for any continuous function $f$ on the positive real numbers and complex numbers $\lambda, \mu$, we have*

$$\begin{aligned}
\delta((\lambda + b)(f((\mu + b)^*(\mu + b))) &= \delta((\lambda + a)(f((\mu + b)^*(\mu + b))) \\
&= \delta((\lambda + a)f((\mu + a)^*(\mu + a))), \\
\delta((\lambda + b)(f((\mu + b)(\mu + b)^*)) &= \delta((\lambda + a)(f((\mu + b)(\mu + b)^*)) \\
&= \delta((\lambda + a)f((\mu + a)(\mu + a)^*))
\end{aligned}$$

*Proof.* We observe from the hypotheses on $b, e$ that $ab = aeb = aea = a^2$. Similar equations holds using $a$ and $b^*$. It follows that in any polynomial in $a, b, a^*, b^*$ is unchanged if we replace all $b, b^*$ by $a, a^*$, provided each term in the sum has at least one $a$ or $a^*$. We also note that $\delta(a)b = \delta(ab) - a\delta(b) = \delta(a^2) - a\delta(a) = a\delta(a)$, so the same type of statement holds for polynomials involving $\delta(a), \delta(a)^*, a, a^*, b, b^*, \delta(b), \delta(b)^*$, provided each term in the sum has at least one of $a, a^*, \delta(a), \delta(a)^*$.

From these observations and the Leibniz rule, it follows that if $f(t)$ is any polynomial with 0 constant term, we have $bf(b^*b)) = af(a^*a)$. For a continuous functions $f$ with $f(0) = 0$, by finding a sequence of polynomials that converge uniformly to $f(t)$, the same conclusion follows.

The conclusions are done in a similar fashion. □

**Proposition 4.5.** *If $A, B, \delta, E$ satisfy conditions C1 and C2 of Theorem 3.4, then $\delta(B) = \delta(A)$ is closed.*

*Proof.* First, we claim that if $a$ is any element of $\mathcal{A}$, then there is $b$ in $B$ with $\delta(b) = \delta(a)$ and $\|b\| \leq \|a\|$. To see this, first select $b$ in $B$ and $e$ in $M(A)$ as in condition C2. Let $f : [0, \infty) \to \mathbb{R}$ be the continuous function

$$f(t) = \begin{cases} 1 & 0 \leq t \leq \|a\|^2 \\ t^{-1/2}\|a\| & \|a\|^2 < t. \end{cases}$$

Then, using Lemma 4.4, we have

$$\|bf(b^*b)\| = \|f(b^*b)b^*bf(b^*b)\|^{1/2} = \|tf(t)^2\|_\infty^{1/2} = \|a\|.$$

It also follows from Lemma 4.4 that

$$\delta(bf(b^*b)) = \delta(af(a^*a)) = \delta(a).$$

We know that $\delta(B) \subseteq \delta(A)$. To prove the reverse inclusion, let $a$ be in $A$. With $a_0 = 0$, inductively choose $a_n, n \geq 1$, in $\mathcal{A}$ such that

$$\|(a - a_0 - \cdots - a_{n-1}) - a_n\| \leq 2^{-n}\|a\|$$

and

$$\|a_n\| \leq \|a - a_0 - \cdots - a_{n-1}\|.$$

For each $n$, we may find $b_n$ in $B$ with $\delta(b_n) = \delta(a_n)$ and $\|b_n\| \leq \|a_n\|$. The series $b = \sum_n b_n$ is then convergent in $B$ and $\delta(b) = \delta(a)$ by continuity of $\delta$. □

As a final preliminary step, we observe the following important consequence of the fact that $\delta(A)$ is closed.

**Lemma 4.6.** *Suppose that $\delta : A \to E$ is a bounded $*$-derivation with closed range. For each $n \geq 1$, there is $\epsilon_n > 0$ such that if $a$ is a partial isometry in $M_n(\tilde{A})$, $a^*a = p, aa^* = q$ with $\delta(p) = \delta(q) = 0$ and $\|\delta(a)\| < \epsilon_n$, then there are partial isometries $a_1, a_2$ in $M_n(\widetilde{A})$ with $a_1^*a_1 = a_2a_2^* = a_2^*a_2 = p$, $a = a_1a_2$, $\delta(a_2) = 0$ and $(q, q, a_1)$ is elementary.*

*Proof.* A standard argument using functional calculus (see for example, Proposition 2.2.4 of [17]) proves that there is $\epsilon_0 > 0$ such that if $c$ is a partial isometry any $C^*$ algebra $C$ and $d$ is any other element of $C$ with $\|c - d\| < \epsilon_0$, then $dd^*$ is invertible in $cc^*Ccc^*$ and

$$\|d(d^*d)^{-1/2} - c\| < 1.$$

Let $K$ be the constant associated with $\delta : M_n(\tilde{A}) \to M_n(E)$ in condition C1 and set $\epsilon_n = K^{-1}\epsilon_0$, as above.

Given $a, p, q$ as in the statement, we may find $a_3$ in $qM_n(\tilde{A})p$ with $\delta(a_3) = 0$ and $\|a_3 - a\| < K\epsilon_n$. Let $a_2 = a_3(a_3^*a_3)^{-1/2}$ and $a_1 = aa_2^*$. All of the conclusions are immediate except that $(q, q, a_1)$ is elementary. To see this, we first observe that

$$\|a_1 - q\| = \|aa_2^* - aa^*\| \leq \|a_2 - a\| < 1.$$

It follows that $a_1$ is homotopic to $q$ in the invertibles of $qM_n(\widetilde{A})q$. $\square$

**Lemma 4.7.** *Assume that $A, B, \delta, E$ satisfy conditions C1 and C2 of 3.4. Let $(1_m, q, a)$ be in standard form in $\Gamma(A \cap \ker(\delta); A)$ and assume that $a$ is in $M_n(\widetilde{A})$. Given $\epsilon > 0$, there exists a partial isometry $b$ in $M_{2n}(\widetilde{B})$ with $\delta(b^*b) = 0, bb^* = 1_m$ and $\|\delta(ab)\| < \epsilon$.*

*Proof.* We let $\|\delta\|$ be the norm of $\delta : M_n(A+B) \to M_n(E)$. We also let $K$ be a constant such that C1 holds, for both $\delta : M_{2n}(\tilde{A}) \to M_{2n}(E)$ and $\delta : M_{2n}(\tilde{B}) \to M_{2n}(E)$, making use of Proposition 4.3, Theorem 4.5 and Proposition 4.1.

Let $\epsilon' > 0$ be such that

$$\|\delta\| \left[(8K\epsilon')^{1/2} + 4K\epsilon' + 7\epsilon'\right] < \epsilon$$

and so that $K\epsilon' < 1/2$.

We find $a_1$ in $M_n(\mathcal{A})$ such that $\|a - 1_m - a_1\| < \epsilon'$ and $\|1_m + a_1\| \leq \|a\| = 1$. It follows that

$$\|(1_m + a_1)^*(1_m + a_1) - 1_m\|, \|(1_m + a_1)(1_m + a_1)^* - q\| \leq 2\epsilon'.$$

If we replace $a_1$ by $a_1 \cdot 1_m$, the same estimate still holds and so we may assume that $a_1 1_m = a_1$.

We use condition C2 to find $b_1$ in $M_n(B)$ and $e$ in $M_n(\mathcal{M}(A))$, commuting with $1_m$, with

$$a_1 = a_1 e = e a_1 = b_1 e = e b_1, \delta(a_1) = \delta(b_1).$$

By replacing $b_1$ with $b_1 1_m$, we may assume that $b_1 1_m = b_1$.

Let

$$f(t) = \begin{cases} 1 & t \leq 1, \\ t^{-1/2} & t \geq 1 \end{cases}$$

and $b_2 = (1_m + b_1) f((1_m + b_1)^*(1_m + b_1))$. It is clear that $b_2$ is in $M_n(\tilde{B})$, $b_2 1_m = b_2$ and $\|c\| \leq 1$. It follows that

$$b_3 = \left[ \begin{array}{cc} b_2^* & (1_m - b_2^* b_2)^{1/2} \\ 0 & 0 \end{array} \right]$$

is in $M_{2n}(\tilde{B})$ and satisfies $b_3 b_3^* = 1_m$.

We define the partial isometry

$$a b_3 = \left[ \begin{array}{cc} a b_2^* & a\,(1_m - b_2^* b_2)^{1/2} \\ 0 & 0 \end{array} \right].$$

First, we note

$$\begin{aligned} \|\delta(ab_3)\| &\leq \|\delta((1_m + a_1)b_3)\| + \|\delta\| \|a - (1_m + a_1)\| \\ &\leq \|\delta((1_m + a_1)b_3)\| + \|\delta\| \epsilon'. \end{aligned}$$

To deal with the first of the two terms, we make extensive use of Lemma 4.4. We compute

$$\begin{aligned} \|\delta((1_m + a_1)b_2^*)\| &= \|\delta(1_m + a_1)b_2^* + (1_m + a_1)\delta(b_2)^*\| \\ &= \|\delta(1_m + a_1) f((1_m + b_1)^*(1_m + b_1))(1_m + b_1)^* \\ &\quad + (1_m + a_1)\delta(a_1)^*\| \\ &= \|\delta(1_m + a_1) f((1_m + a_1)^*(1_m + a_1))(1_m + a_1)^* \\ &\quad + (1_m + a_1)\delta(a_1)^*\| \\ &= \|\delta((1_m + a_1)^*(1_m + a_1))\| \\ &\leq \|\delta\| 2\epsilon'. \end{aligned}$$

Finally, letting $h(t) = (1 - t f(t)^2)^{1/2}$, as $0 \leq (1_m + a_1^*(1_m + a_1) \leq 1$,

$$\|h((1_m + a_1)^*(1_m + a_1))\| = \|1_m - (1_m + a_1)^*(1_m + a_1)\| \leq 2\epsilon'.$$

We also have

$$
\begin{aligned}
\delta\left((1_m + a_1)\,(1_m - b_2^* b_2)^{1/2}\right) &= \delta(1_m + a_1)\,(1_m - b_2^* b_2)^{1/2} \\
&\quad + (1_m + a_1)\delta\left((1_m - b_2^* b_2)^{1/2}\right) \\
&= \delta\left((1_m + a_1)\right) h((1_m + b_1)^*(1_k + b_1)) \\
&\quad + (1_m + a_1)\delta\left(h(1_m + b_1)^*(1_m + b_1))\right) \\
&= \delta\left(a_1\right) h((1_m + a_1)^*(1_m + a_1)) \\
&\quad + (1_m + a_1)\delta\left(h((1_m + a_1)^*(1_m + a_1))\right)
\end{aligned}
$$

Together, we conclude that

$$\|\delta\left((1_m + a_1)\,(1_m - b_2^* b_2)^{1/2}\right)\| \leq \|\delta\| 4\epsilon'.$$

Putting this together implies that

$$\|\delta(ab_3)\| \leq \|\delta\| 2\epsilon' + \|\delta\| 4\epsilon' + \|\delta\|\epsilon' = 7\|\delta\|\epsilon'.$$

At this point, the only property lacking for $b_3$ is that $\delta(b_3^* b_3)$ may not be zero. Let $p' = b_3^* b_3$. We have

$$\|\delta(p')\| = \|\delta((ab_3)^*(ab_3))\| \leq 2\|\delta(ab_3)\| \leq \epsilon'.$$

So we may find $b_4$ in $M_{2n}(\tilde{B})$ with $\delta(b_4) = 0$ and $\|b_4 - p'\| \leq K\epsilon'$. We may also assume that $b_4$ is self-adjoint. By Lemma 2.2.3 of [17], the spectrum of $b_4$ is contained in $[-K\epsilon', K\epsilon'] \cup [1 - K\epsilon', 1 + K\epsilon']$. As $K\epsilon' < 1/2$, $p = \chi_{(1/2,\infty)}(b_4)$ is a projection in $M_{2n}(\tilde{B}) \cap \ker(\delta)$ and $\|p - p'\| \leq 2K\epsilon'$. It follows from routine estimates that we may find a partial isometry $b_5$ in $M_{2n}(\tilde{B})$ such that $b_5 b_5^* = p'$ and $b_5^* b_5 = p$ and

$$\|b_5 - p\| \leq (8K\epsilon')^{1/2} + 4K\epsilon'.$$

From this, it follows that

$$\|\delta(b_5)\| \leq \|\delta\| \left[(8K\epsilon')^{1/2} + 4K\epsilon'\right].$$

Letting $b = b_3 b_5$ now satisfies all the desired properties. $\square$

**Lemma 4.8.** *Suppose that $(1_m, q, a)$ in $\Gamma(A \cap \ker(\delta); A)$ is in standard form and that $\epsilon > 0$. Then there exists $(1_m, p, b)$ in $\Gamma(B \cap \ker(\delta); B)$ also in standard form and $(q, q, \tilde{a})$ in $\Gamma(A \cap \ker(\delta); A)$, elementary with $a^* a = aa^* = q$ such that $(1_m, q, \tilde{a}a)$ is isomorphic to $(1_m, q', b)$ in $\Gamma(\ker(\delta), A + B)$ via $1_m, b(\tilde{a}a)^*$.*

*Proof.* By Lemma 4.7, we may find a partial isometry $b$ in $M_{2n}(\tilde{B})$ with $bb^* = 1_m$, $p = b^* b$ in $\ker(\delta)$ and $\|\delta(ab)\| < \epsilon_{4n}$.

Let $\rho : \widetilde{A + B} \to \widetilde{A + B}/A$ denote the quotient map. As we observed in the proof of Theorem 3.2, we have $\rho(\tilde{B}) = \rho(\ker(\delta))$. Hence, we may find $c$ in $M_{2n}(\ker(\delta))$ such that $\rho(c) = \rho(ab)$ and $\|c\| \leq \|ab\| \leq 1$.

It follows that

$$c' = \begin{bmatrix} qc & (q - qcc^*q)^{1/2} \\ 0 & 0 \end{bmatrix}$$

is in $M_{4n}(\ker(\delta))$, $c'c'^* = q$ and $\rho(c') = \rho(ab)$. Hence, $c'^*ab$ is a partial isometry. Let $p'$ denote its initial projection, which is also in $M_{4n}(\ker(\delta))$. Moreover, we have

$$\rho(c'^*ab) = \rho(c')^*\rho(ab) = \rho(ab)^*\rho(ab) = \rho(p) = \dot{p},$$

as $p$ is in $M_n(\tilde{A} \cap \ker(\delta))$. This means that $c'^*ab$ is in $M_{4n}(\tilde{A})$; let us denote it by $a_1$. We have $a_1^*a_1 = b^*a^*c'c'^*ab = p$ and $a_1a_1^* = q$. Therefore, $(p, q, ab)$ is isomorphic to $(p', q, a_1)$ via $c', q$. Using the fact $c'$ is in $\ker(\delta)$, we have

$$\|\delta(a_1)\| = \|\delta(c'^*ab)\| = \|c'^*\delta(ab)\| < \epsilon_{4n}.$$

It now follows from Lemma 4.6 that $a_1 = a_2a_3$, with $a_2^*a_2 = a_3a_3^* = a_3^*a_3 = p, q_1 = a_2a_2^*$, $\delta(a_2) = 0$ and $(p, p, a_3)$ elementary. It follows that

$$a = aba_1^*a_1b^* = (aba_1^*)a_2a_3b^*$$

and hence

$$(aba_1^*a_2)a_3^*(aba_1^*a_2)^*a = (aba_1^*a_2)b^*.$$

Observe that $\tilde{a} = (aba_1^*a_2)a_3^*(aba_1^*a_2)^*$ is also elementary and $(1_m, q', \tilde{a}a)$ is isomorphic to $(1_m, p, b^*)$ via $1_m, b^*a^*\tilde{a}^*$. □

We are now ready to prove Theorem 3.4. We begin with surjectivity. Let $(1_m, q, a)$ be in $\Gamma(\ker(\delta) \cap A; A)$ and assume it is in standard form and that $a$ is in $M_n(\widetilde{A})$. We apply Lemma 4.6 to the map $\delta : A+B \to E$ to obtain $\epsilon_{2n}$. By Lemma 4.7, we may find a partial isometry $b$ in $M_{2n}(\tilde{B})$ such that $\delta(b^*b) = 0$, $bb^* = 1_m$ an d $\|\delta(ab)\| < \epsilon_{2n}$. Let $p = b^*b$. From Proposition 2.3, we know that

$$[1_m, q, a] + [p, 1_m, b] = [p, q, ab].$$

By Lemma 4.6, we may write $ab = c_1c_2$, where $c_1, c_2$ are partial isometries in $M_{2n}(\widetilde{A + B})$ with $\delta(c_2) = 0$ and $(q, q, c_1)$ elementary. It follows that $(p, q, ab)$ is isomorphic to $(q, q, c_1)$ via $c_2, q$ and hence it represents the zero element of $K_0(\ker(\delta); A + B)$. It follows from Theorem 3.2 that $\alpha[1_m, p, b^*] = [1_m, q, a]$.

We now turn to the issue in injectivity. In view of Theorem 3.2, it suffices to prove that the map from $K_0(B \cap \ker(\delta); B)$ to $K_0(\ker(\delta), A+B)$ is injective. Let $(1_m, q, b)$ be an element of $\Gamma(\ker(\delta) \cap B; B)$ which is in standard form and $b$ is in $M_n(\tilde{B})$ and such that its class in $K_0(\ker(\delta), A + B)$ is zero. From Proposition 2.3, there exist $(1_k, 1_k, c_1)$ and $(1_{k+m}, 1_{k+m}, c_2)$ which are elementary and in standard form such that $c_2(b \oplus c_1)^*$ is in $\ker(\delta)$.

As $c_1, c_2$ are elementary, we may find a partial isometry $c_1(t)$ in $M_{2k}((C_0(0,1] \otimes (A+B))^\sim)$ such that $(1_k, 1_k, c_1(t))$ is in standard form in $\Gamma(\ker(1_{C_0(0,1]} \otimes \delta) \cap C_0(0,1] \otimes (A+B); C_0(0,1] \otimes (A+B))$. We may also find $c_2(t)$ in an analogous way in $M_{2(k+m)}(C_0(0,1] \otimes (A+B))^\sim)$.

By part 3 of Proposition 4.3, we know that conditions C1 and C2 also hold for $C_0(0,1] \otimes A$ and $C_0(0,1] \otimes B$. We apply Lemma 2.3 to find partial isometries $a_1(t)$ in $M_{2k}((C_0(0,1] \otimes A)^\sim)$ and $a_2(t)$ in $M_{2(k+m)}((C_0(0,1] \otimes A)^\sim)$ such that

$$\begin{aligned} \delta(a_1 c_1^*) &= \delta(a_2 c_2^*) = 0, \\ a_1^* a_1 &= 1_k, \\ a_2^* a_2 &= 1_{k+l}. \end{aligned}$$

Note that $a_i a_i^* = (a_i c_i^*)(a_i c_i^*)^*$ is in $\ker(\delta)$.

With $\epsilon = \epsilon_{4(k+m)}/2$, we now appeal to Lemma 4.7 to find $b_1(t)$ in $M_{4k}((C_0(0,1] \otimes B)^\sim)$ and $b_2(t)$ in $M_{4(k+m)}((C_0(0,1] \otimes B)^\sim)$ such that

$$\begin{aligned} (1 \otimes \delta)(b_i(t)^* b_i(t)) &= 0, \\ b_1(t) b_1(t)^* &= 1_k, \\ b_2(t) b_2(t)^* &= 1_{k+m}, \\ \|(1 \otimes \delta)(a_i(t) b_i(t))\| &< \epsilon. \end{aligned}$$

Now there is a small problem in that $b_1(t)^* b_1(t)$ may not be constant in $t$. But it is a continuous path of projections in $M_{4k}(\ker(\delta) \cap B)$ which begins at $1_k$ at $t = 0$. Hence we may find a continuous path of partial isometries $b_3(t)$ in $M_{4k}(\ker(\delta) \cap B)$ with $b_3(t) = 1_k$, $b_3(t) b_3(t)^* = b_1(t)^* b_1(t), b_3(t)^* b_3(t) = 1_k$. By replacing $b_1(t)$ by $b_1(t) b_3(t)$, we may assume $b_1(t)^* b_1(t) = 1_k$ for all $t$, without changing the other properties of $b_1(t)$. We do the same for $b_2(t)$.

It follows that if we define $b_1 = b_1(1), b_2 = b_2(1)$, then $(1_k, 1_k, b_1)$ and $(1_{k+m}, 1_{k+m}, b_2)$ are elementary in $\Gamma(\ker(\delta) \cap B; B)$. Moreover, we can write

$$\begin{aligned} b_2^*(b \oplus b_1^*)^* &= b_2^*(b^* \oplus b_1) \\ &= b_2^* a_2^* a_2 c_2^* c_2 (b^* \oplus c_1^* c_1 a_1^* a_1 b_1) \\ &= (a_2 b_2)^* (a_2 c_2^*) c_2 (b^* \oplus c_1^*)(1_m \oplus (c_1 a_1^*))(a_1 b_1) \end{aligned}$$

Observe that the terms are grouped so that each group is in the kernel of $\delta$, except the first and last, $a_2 b_2, a_1 b_1$. It follows that

$$\|\delta(b_2^*(b \oplus b_1^*)^*)\| < 2\epsilon.$$

By Lemma 4.6, we may find partial isometries $b_4, b_5$ in $\mathcal{M}(\tilde{B})$ such that

$$b_2^*(b \oplus b_1^*)^* = b_4 b_5$$

$\delta(b_5) = 0$ and $(1_{k+m}, 1_{k+m}, b_4)$ is elementary. It follows that $(1_m \oplus 1_k, q \oplus 1_k, b \oplus b_1)$ is isomorphic to $(1_{k+m}, 1_{k+m}, b_4^* b_2^*)$ via $1_{k+m}, b_5$ As the latter is elementary, it follows that $[1_m, q, b] = 0$ in $K_0(\ker(\delta) \cap B; B)$.

## 5. Groupoid $C^*$-algebras

We are interested in applying our excision result to various groupoid $C^*$-algebras. The first question which arises, even before we get to some of the more subtle hypotheses, is when the $C^*$-algebra of one groupoid lies in the multiplier of another. This issue is the focus of this section. We refer the reader to [16] and [19] as standard references on groupoids and their $C^*$-algebras.

The question for us will reduce to the following: suppose that $G$ is a locally compact, Hausdorff groupoid with a Haar system and $H$ is a subgroupoid of $G$, endowed with its own locally compact, Hausdorff topology which is finer than the relative topology of $G$. Under what circumstances can we conclude that $C_r^*(G)$ lies in the multiplier algebra of $C_r^*(H)$.

If this situation does not seem intuitive, let us provide some justification. The first is that it arises quite naturally as we will show in the next two sections. Secondly, let us offer the following fairly simple example, which is a special case.

It is well-known that the category of unital, commutative $C^*$-algebras is isomorphic to that of compact, Hausdorff spaces with continuous maps. If we enlarge this to include non-unital commutative $C^*$-algebras, we replace 'compact' by 'locally compact' and also require the maps to be proper. On the other hand, there is an alternative (less categorical) possibility. Suppose that $X, Y$ are locally compact, Hausdorff spaces and $\alpha : Y \to X$ is simply continuous. Then $\alpha$ induces a $*$-homomorphism, $\rho$, from $C_0(X)$ to the algebra of continuous bounded functions on $Y$, which is multiplier algebra of $C_0(Y)$. The point is that, for $f$ in $C_0(X)$, while $f \circ \alpha$ may not be compactly supported, it is continuous, and its product with any compactly supported function on $Y$ will be compactly supported and continuous.

If we additionally assume that $\alpha$ is injective, then we can simply identify $\alpha(Y)$ with its image in $X$, which we will simply denote $Y$. But the map $\alpha$ induces a topology on $Y$ which is possibly finer than the relative topology from $X$. The map $\rho$ we described above is now simply the restriction of functions from $X$ to $Y$. If we consider $X$ and $Y$ with the co-trivial groupoid structures (that is, the equivalence relation which is equality), then this is exactly the situation we outlined above and we have a positive answer to our question. In general, of course, more hypotheses are needed.

We first discuss the structure and relations in the setting of $H \subseteq G$, but at the end of the section we will provide some constructions for obtaining $H$ from $G$ which are inspired by the seminal work of Muhly, Renault and Williams [9].

Let us begin by setting out some standard notation for groupoids and groupoid $C^*$-algebras.

Let $G$ be a groupoid. That is, there there is a set of composable pairs, $G^2 \subseteq G \times G$ with a product from $G^2$ to $G$. We assume this is associative, as in [16]. It will sometimes be convenient to denote the product map as $\mu_G$ from $G^2$ to $G$. The units of $G$ are denoted by $G^0$ and the range and source maps $r_G, s_G : G \to G^0$, $r_G(g) = gg^{-1}, s_G(g) = g^{-1}g$, for all $g$ in $G$. When no confusion can arise, we will drop the subscripts. We let

$$G^u = r_G^{-1}\{u\}, G_u = s_G^{-1}\{u\},$$

for each $u$ in $G^0$. A subset $X$ of $G^0$ is called $G$-invariant if, whenever $g$ is in $G$ with $r_G(g)$ in $X$, then $s_G(g)$ is also in $X$.

We also assume that $G$ is a topological groupoid; that is, it has a topology, $\mathcal{T}_G$, in which the inverse and product are continuous. Moreover, we assume throughout that it is second countable, locally compact and Hausdorff.

Recall the definition of a Haar system for $G$: for each $u$ in $G^0$, there is a measure $\nu^u$ on $G^u$ with full support and such that the function

$$u \to \int_{G^u} f(x) d\nu^u(x)$$

is continuous, for every $f$ in $C_c(G)$, the continuous compactly-supported functions on $G$. We also assume that $\nu$ is left-invariant in the sense that, for any $g$ in $G$ and $f$ in $C_c(G)$, we have

$$\int_{G^{s(x)}} f(xy) d\nu^{s(x)}(y) = \int_{G^{r(x)}} f(y) d\nu^{r(x)}(y).$$

The following result is well-known, but worth recording here.

**Lemma 5.1.** *Let $G$ be a locally compact, Hausdorff groupoid. The following are equivalent:*

(1) *The map $r_G : G \to G^0$ is open.*
(2) *The map $s_G : G \to G^0$ is open.*
(3) *The product map $\mu_G : G^2 \to G$ is open.*

*If the groupoid has a Haar system, then all three conditions are satisfied.*

*Proof.* The equivalence of the first two follows from the facts that $x \to x^{-1}$ is a homeomorphism and $r(x^{-1}) = s(x)$.

Let us assume the first condition holds and prove the third. Let $(x, y)$ be in $G^2$, $x \in U, y \in V$, open and suppose that $\mu(U \times V \cap G^2)$ is not open. Then there exists a sequence $z_k$ converging to $xy$ which is not in $\mu(U \times V \cap G^2)$. Choose a decreasing sequence of open sets, $U_l, l \geq 1$, contained in $U$ and with intersection $\{x\}$. Since $r$ is open, $r(U_l)$ is open and contains $r(x)$, which is the limit of the sequence $r(z_k)$, as $r$ is continuous. Hence, for each $l$, we may find $k_l$ and $x_l$ in $U_l$ such that $r(x_l) = r(z_{k_l})$. Hence, the sequence $x_l$ is converging to $x$ and $x_l^{-1} z_{k_l}$ is converging to $x^{-1}(xy) = y$. So for $l$ sufficiently large, $x_l^{-1} z_{k_l}$ is in $V$ and hence $z_{k_l} = x_l(x_l^{-1} z_{k_l})$ is in $\mu(U \times V \cap G^2)$, a contradiction.

The third condition implies the first on general topological grounds.

The last statement is Proposition 2.4 of Chapter 2 of [16]. □

We now begin to consider the situation we outlined of $H \subseteq G$. The following is an extension result that will be needed shortly.

**Lemma 5.2.** *Let $(G, \mathcal{T}_G)$ and $(H, \mathcal{T}_H)$ be locally compact, Hausdorff topological groupoids and that $\nu^u, u \in G^0$ is a Haar system for $G$. Assume that $H$ is a subgroupoid of $G$ such that the topology $\mathcal{T}_H$ is finer than the relative topology on $H$ of $\mathcal{T}_G$ and that, for every $u$ in $H^0$, $H^u = G^u$ and the relative topologies from $\mathcal{T}_H$ and $\mathcal{T}_G$ agree on this set.*

*Let $Y \subseteq H$ be a subset which is closed in both topologies and such that the two topologies agree on this set. Let $K \subseteq H^0$ be compact, $M \geq 1$ and suppose that $a : H \to \mathbb{C}^M$ is continuous, compactly supported and $r(supp(a)) \cup s(supp(a)) \subseteq K$. Then there exists $b : G \to \mathbb{C}^M$ continuous and compactly supported such that $b(x) = a(x)$, for all $x$ in $Y \cup r^{-1}(K) \cup s^{-1}(K)$.*

*Proof.* Let $U = \{x \in H \mid a(x) \neq 0\}$. It follows that $\overline{U}$ is compact in $H$, hence also in $G$ and the two topologies agree on this set. As $Y$ and $\overline{U}$ are closed in both $H$ and $G$, the two topologies agree on $Y \cup \overline{U}$.

Let $X = r^{-1}(K) \cup s^{-1}(K)$, which is closed in $G$.

We define $b(x) = a(x)$, for all $x$ in $\overline{U} \cup Y$ and we define $b(x) = 0$, for all $x$ in $X - U$. First, observe that this is well-defined, for if $x$ is in $X - U$, then $a(x) = 0$. Secondly, both of these sets are closed in $G$. Thirdly, $b(x)$ is continuous on both: on the first because the two relative topologies agree there and on the second because it is constant there. Finally, the support of $b(x)$ is contained in $U$, which is pre-compact. Hence, by the Tietze extension theorem (Proposition 1.5.8 of [11]), $b$ may be extended to a continuous function of compact support on $G$.

It is clear that $b(x) = a(x)$, for any $x$ in $Y$. Now suppose that $r(x)$ is in $K$. If $x$ is in $U$, then $b(x) = a(x)$, by definition. If $x$ is not in $U$, then $a(x) = 0$ by the hypothesis on $U$, while $b(x) = 0$ as $x$ is in $X - U$. The case for $s(x) \in K$ is similar. □

**Theorem 5.3.** *Let $(G, \mathcal{T}_G)$ and $(H, \mathcal{T}_H)$ be locally compact, Hausdorff topological groupoids and that $\nu^u, u \in G^0$ is a Haar system for $G$. Assume that $H$ is a subgroupoid of $G$ such that the topology $\mathcal{T}_H$ is finer than the relative topology on $H$ of $\mathcal{T}_G$ and that, for every $u$ in $H^0$, $H^u = G^u$ and the relative topologies from $\mathcal{T}_H$ and $\mathcal{T}_G$ agree on this set.*

*Then the following are equivalent:*

1. *the map $r : H \to H^0$ is open (in $\mathcal{T}_H$).*
2. *the map $s : H \to H^0$ is open (in $\mathcal{T}_H$).*
3. *$\nu^u, u \in H^0$, is a Haar system for $H$.*

*Proof.* The first two conditions are clearly equivalent since $r(x^{-1}) = s(x)$ and $x \to x^{-1}$ is a homeomorphism. The third condition implies the first from Lemma 5.1 or Proposition 2.4, Chapter 2 of [16].

It remains for us to prove that the first condition implies the third. First, it is clear from the fact that the two topologies agree on all sets $H^u = G^u$ that the measures are well-defined, have the desired support and are left-invariant. It remains for us to verify the continuity property.

Let $h$ be any continuous function of compact support on $H$. Let $U = \{x \in H \mid h(x) \neq 0\}$. Its closure $\bar{U}$ is compact in $H$ and so is $K = r(\bar{U}) \cup s(\bar{U})$. We apply Lemma 5.2 to find $g$ in $C_c(G)$ such that $g(x) = h(x)$ for all $x$ with $r(x)$ or $s(x)$ in $K$.

We consider the function $I(u) = \int_{H^u} h(x) d\nu^u(x)$ defined on $H^0$ and show it is continuous. We appeal to the same general topology result as in the last proof. First, on the set $r(\bar{U})$, it agrees with the function $J(u) = \int_{G^u} g(x) d\nu^u(x)$. As $\nu^u$ is a Haar system, this function is continuous on $G^0$ in the topology $\mathcal{T}_G$ and so its restriction to $r(\bar{U})$ is continuous in the topology $\mathcal{T}_H$. Secondly, on the set $H^0 \setminus r(U)$, it is clearly 0, which is continuous in any topology. These two sets cover $H^0$. The first is closed as we observed above that it is compact. The second is closed since $U$ is open and our hypothesis is that $r$ is an open map. □

We consider the left regular representation of $G$, $\lambda_G$. For each unit, $u$ in $G^0$, we define the measure $\nu_u$ on $G_u$ by $\nu_u(E) = \nu^u(E^{-1})$, for any Borel set $E$ in $G_u$. We let $L^2(G_u, \nu_u)$ be the corresponding Hilbert space and define (with a slight abuse of notation) $L^2(G, \nu) = \oplus_{u \in G^0} L^2(G_u, \nu_u)$. It is worth noting that the elements can be seen as functions on $G$. For each $f$ in $C_c(G)$ and $u$ in $G^0$, we define the operator $\lambda_u(f)$ on $L^2(G_u, \nu_u)$

$$(\lambda_u(f)\xi)(x) = \int_{y \in G_u} f(xy^{-1})\xi(y)\sigma(xy^{-1}, y)d\nu_u(y),$$

for $f$ in $C_c(G)$, $\xi$ in $L^2(G, \nu)$ and $x$ in $G_u$. We also let $\lambda_G(f)$ be $\oplus_{u \in G^0}\lambda_u(f)$.

We let $\lambda_H$ denote the left regular representation of $H$ on $L^2(H, \nu)$. From the fact that, for every unit $u$, $G_u = H_u$ and assuming they have the same topology $G$ and in $H$, the Hilbert space $L^2(H, \nu)$ is a closed subspace of $L^2(G, \nu)$. We use this inclusion implicitly. Furthermore, we can regard the operators $\lambda_H(f), f \in C_c(H)$, and also those in the operator-norm closure of these as being defined on $L^2(G, \nu)$ by setting them to be zero on the orthogonal complement of $L^2(H, \nu)$.

**Theorem 5.4.** *Let $(G, \mathcal{T}_G)$ be a locally compact, Hausdorff topological groupoid with Haar system, $\nu^u, u \in G^0$, and 2-cocycle $\sigma$. Let $H \subseteq G$ be a subgroupoid with a topology, $\mathcal{T}_H$, in which it is also locally compact and Hausdorff. Suppose that*

(1) *$H^0 \subseteq G^0$ is $G$-invariant,*
(2) *the topology $\mathcal{T}_H$ is finer than the relative topology of $\mathcal{T}_G$ on $H$,*
(3) *For every $u$ in $H^0$, the topologies $\mathcal{T}_G$ and $\mathcal{T}_H$ agree on $H^u = G^u$ and $G_u = H_u$.*
(4) *$\nu^u, u \in H^0$ is a Haar system for $H$.*

*Then the following hold.*

(1) *For each $f$ in $C_c(G)$ and $g$ in $C_c(H)$, the functions*
$$\begin{aligned}(fg)(x) &= \int_{H^u} f(y)g(y^{-1}x)\sigma(y, y^{-1}x)d\nu^u(y) \\ (gf)(x) &= \int_{H^u} g(y)f(y^{-1}x)\sigma(y, y^{-1}x)d\nu^u(y)\end{aligned}$$
*are well-defined and in $C_c(H)$.*
(2) *For each $f, g$ as above we have*
$$\begin{aligned}\lambda_G(f)\lambda_H(g) &= \lambda_H(fg), \\ \lambda_H(g)\lambda_G(f) &= \lambda_H(gf),\end{aligned}$$
(3) *This defines a map sending $f$ in $C_c(G)$ to $\rho(f)$ in $M(C_c(H))$ which extends to a $*$-homomorphism from $C_r^*(G, \sigma)$ to $M(C_r^*(H, \sigma))$ which is injective if and only if $H^0$ is dense in $G^0$.*

We will not give the proof which is really quite straightforward.

It is observed in Remark (iii) on page 59 and Proposition 1.14 of [16] that there is a natural way that $C_c(H^0)$ acts as multipliers of $C_c(H)$:

$$(ef)(x) = e(r(x))f(x), (fe)(x) = f(x)e(s(x)),$$

for each $f$ in $C_c(H)$, $e$ in $C_c(H^0)$ and $x$ in $H$. In our case here, this representation of $C_c(H)$ interacts with $C_c(G)$ in a particularly nice way. Observe that the two formulae above make sense equally well if $e$ is in $C_c(H^0)$ and $f$ is in $C_c(G)$, the results being functions on $H$ and it is easy to see that they are both continuous and have compact support.

There remains one technical issue in this construction. Suppose that $f$ is a continuous function of compact support on $H$. Let us suppose for the moment that $H$ is closed in $G$ (which is rarely the case). Then the Tietze Extension Theorem guarantees the existence of a continuous function $\tilde{f}$ on $G$ such that $\tilde{f}|_H = f$. It is a more subtle question to ask if $\tilde{f}$ may be chosen so that the norm, $\|\tilde{f}\|_r$, can be controlled in some way by $\|f\|_r$, independently of $f$.

This is true if we replace the reduced $C^*$-norm by the uniform norm. The proof is quite standard but it will be helpful to examine it, as we will do in a moment.

As we noted $H$ itself is usually not closed in $G$ and it is necessary for us to restrict our attention to subsets of $H$ which *are* closed in $G$, which usually rules out $H$ itself.

**Definition 5.5.** *Let $(G, \mathcal{T}_G)$ be a locally compact, Hausdorff topological groupoid with Haar system, $\nu^u, u \in G^0$, and 2-cocycle $\sigma$. Let $H \subseteq G$ be a subgroupoid with a topology, $\mathcal{T}_H$, in which it is also locally compact and Hausdorff.*

*For any $C \geq 1$, we say that a set $X \subseteq H$ which is closed in $G$ (and hence also in $H$) has the $C$-*extension property* if, for any $f$ in $C_c(H)$ with support in $X$, there exists $\tilde{f}$ in $C_c(G)$ such that $\tilde{f}|_X = f|_X$ and*

$$\|\tilde{f}\|_r \leq C\|f\|_r.$$

Let us just check that the property holds (with $C = 1$) if we use the uniform norm instead of the reduced $C^*$-norm.

Let $X$ be a subset of $H$ which is closed in $G$. Let $f$ be in $C_c(H)$ with support in $X$. Consider $X \cup \{\infty\}$ as a closed subset of $G \cup \{\infty\}$, the one-point compactification of $G$. Extending $f|_X$ to be zero at $\infty$, this function is continuous on $X$, with the topology from $G$ and we may apply the Tietze Extension Theorem to find $\tilde{f}$, a continuous function of $G \cup \{\infty\}$ which agrees with $f$ on $X$ and is zero at $\infty$. It is a simple matter to check that this can be modified so that $\tilde{f}$ is actually compactly supported.

It remains to worry about the norm of the extension. Define the function $h(t)$ to be 1 for $t$ in the interval $[0, \|f\|^2]$, and $\|f\|^{-1}t^{-1/2}$ for $t > \|f\|^2$. It is an easy exercise to check that $\tilde{f}h(\tilde{f}^*\tilde{f})$ satisfies all the desired properties. Notice that $h(\tilde{f}^*\tilde{f})$ exists in the unitization $C(G \cup \{\infty\})$.

It is worth asking why this same argument does not suffice for general groupoids. The answer is that $h(\tilde{f}^*\tilde{f})$ exists in the unitization of $C_r^*(G)$, but not necessarily in the unitization of $C_c(G)$ and the element $\tilde{f}h(\tilde{f}^*\tilde{f})$ exists in $C_r^*(G)$, but not necessarily in $C_c(G)$.

Now we could replace the function $h$ above by some polynomial which approximates our given $h$ and our final element would indeed lie in $C_c(G)$, but then we cannot be sure of the condition $\tilde{f}|_X = f|_X$.

It follows then that in the case that $H$ and $G$ are co-trivial, $G = G^0, H = H^0$, where the two norms agree, this holds. In generality, it seem to be a subtle issue, although it does hold in many cases of interest.

Let us just observe the following positive result in a very special case.

**Proposition 5.6.** *Let $G, H$ be as in Definition 5.5 and assume that $H$ is closed in $G$. Then $G - H$ is also a locally compact groupoid with Haar system. Assume we have a short exact sequence*

$$0 \to C_r^*(G - H) \to C_r^*(G) \to C_r^*(H) \to 0.$$

*(See* [16], [1] *for further discussion.) Then $H$ has the $C$-extension property, for any $C > 1$.*

*Proof.* Let the quotient map from $C_r^*(G)$ to $C_r^*(H)$ be denoted by $\rho$. Let $f$ be in $C_c(H)$. It is a consequence of the Tietze Extension Theorem that we may find $g$ in $C_c(G)$ with $\rho(g) = g|_H = f$. It follows that

$$\|f\|_r = \inf\{\|f + b\|_r \mid b \in C_r^*(G - H)\}.$$

The desired conclusion follows for any $C > 1$ from the fact that $C_c(G - H)$ is dense in $C_r^*(G - H)$. □

I do not know if the converse holds (the $C$-extension property implies exactness), but this does suggest that the property is linked with amenability/exactness in some way.

Our next task is to provide a fairly general way of constructing groupoids $H \subseteq G$ from $G$. This follows ideas of Muhly, Renault and Williams [9].

Let $(G, \mathcal{T}_G)$ be a locally compact, Hausdorff topological groupoid with Haar system, $\nu^u, u \in G^0$, and 2-cocycle $\sigma$. Suppose that $Y$ is a closed subset of $G^0$. We can form

$$G_Y^Y = \{g \in G \mid r(g), s(g) \in Y\}$$

which is obviously a closed subgroupoid of $G$ with unit space $Y$. It also acts on the left of

$$G^Y = \{g \in G \mid r(g) \in Y\}.$$

Observe that this action is *free* in the sense that $g \cdot x = x$ for $g$ in $G_Y^Y$ and $x$ in $G^Y$ with $s(g) = r(x)$ only if $g$ is a unit. It is also proper in the sense that the map sending $(g, x)$ in $G_Y^Y \times G^Y$ with $s(g) = r(x)$ to $(gx, x)$ in $G^Y \times G^Y$ is a proper map.

We define

$$\tilde{H} = \{(x, y) \in G^Y \times G^Y \mid r(x) = r(y)\}$$

which is equipped with an action of $G_Y^Y$ by $g(x, y) = (gx, gy)$, for $(x, y)$ in $H$ and $g$ in $G_Y^Y$ with $s(g) = r(x) = r(y)$. This action is also free and proper.

**Theorem 5.7.** *Let $(G, \mathcal{T}_G)$ be a locally compact, Hausdorff topological groupoid with Haar system, $\nu^u, u \in G^0$, and a 2-cocycle, $\sigma$. Suppose that $Y$ is a closed subset of $G^0$ such that*

$$r, s : G_Y^Y \to Y$$

*are open. If we define*

$$H = \{x^{-1}y \mid (x, y) \in \tilde{H}\},$$

*and endow it with the quotient topology from the map sending $(x, y)$ in $\tilde{H}$ to $x^{-1}y$, then it satisfies the hypotheses of Theorem 5.4. In addition, $G^Y$ is a $G_Y^Y - H$-equivalence bimodule.*

We will not give a proof of this result. Most of it follows the techniques of [9] or is quite straightforward. We would like to examine a special case where $G$ is a transformation groupoid.

Suppose that $X$ is a locally compact Hausdorff space and $\Gamma$ is a locally compact Hausdorff topological group which acts on $X$ by homeomorphisms on the right: that is there is a map sending $(x, \gamma)$ in $X \times \Gamma$ to $x\gamma$ in $X$ which is continuous and satisfies $(x\gamma)\gamma' = x(\gamma\gamma')$, for all $x$ in $X$, $\gamma, \gamma'$ in $\Gamma$.

Let $G$ be the associated transformation groupoid:

$$G = X \rtimes \Gamma,$$

which is simply $X \times \Gamma$ as a set with product given by $(x_1, \gamma_1)(x_2, \gamma_2) = (x_1, \gamma_1\gamma_2)$ if $x_1 \cdot \gamma_1 = x_2$ and inverse $(x, \gamma)^{-1} = (x \cdot \gamma, \gamma^{-1})$. The unit space is $G^0 = X \times \{e_\Gamma\}$. For notational convenience, we will usually write this as simply $X$. It is given the product topology. It has a Haar system by transferring the Haar measure from $\Gamma$ to $G^{(x,e)} = \{x\} \times \Gamma$ in the obvious way, for any $x$ in $X$.

We remark that in this example, the reduced groupoid $C^*$-algebra, $C_r^*(G)$, coincides with the reduced crossed product algebra, $C_0(X) \rtimes_r \Gamma$.

**Definition 5.8.** *Suppose that $X$ is a locally compact Hausdorff space and $\Gamma$ is a locally compact, Hausdorff topological group which acts on $X$ by homeomorphisms If $Y$ is a closed subset of $X$, we say that $Y$ is* $\Gamma$-semi-invariant *if, for every $\gamma$ in $\Gamma$ either $Y\gamma = Y$ or $Y\gamma \cap Y$ is empty. In this case, we denote by $\Gamma_Y$ the set of $\gamma$ for which the first condition holds.*

Notice that this includes the case, $Y\gamma \cap Y = \emptyset$, for all $\gamma \neq e$. Observe that $\Gamma_Y$ is clearly a subgroup of $\Gamma$. Also notice that for any $\gamma$ in $\Gamma$, the set $Y\gamma$ depends only on the right coset $\Gamma_Y\gamma$.

The following is an easy consequence of the definitions and we will not give a proof.

**Theorem 5.9.** *Suppose that $X$ is a locally compact Hausdorff space and $\Gamma$ is a locally compact Hausdorf topological group which acts on $X$ by homeomorphisms. Suppose that $Y$ is a closed $\Gamma$-semi-invariant subset such that $\Gamma_Y\backslash\Gamma$ is discrete. Then $\Gamma_Y$ is open in $\Gamma$. Moreover, $G_Y^Y \cong Y \rtimes \Gamma_Y$, where the latter is regarded as a transformation groupoid and the maps $r, s : G_Y^Y \to Y$ are open. If $H$ is groupoid given in 5.7, then the unit space of $H$ is*

$$H^0 = \cup_{\Gamma_Y\gamma \in \Gamma_Y\backslash\Gamma} Y\gamma$$

*and has the inductive limit topology. With this identification, $H$ is isomorphic to the transformation group*

$$\left(\cup_{\Gamma_Y\gamma \in \Gamma_Y\backslash\Gamma} Y\gamma\right) \rtimes \Gamma.$$

These examples seem to be more accessible for the extension property mentioned earlier. While the following deals with a couple of specific closed subsets of $H$, it would seem the techniques of proof could be applied more generally and we will do so in future sections.

**Theorem 5.10.** *Let $X$ be a locally compact Hausdorff space and $\Gamma$ be a locally compact Hausdorff topological group acting on $X$ by homeomorphisms. Suppose that $Y$ is a closed $\Gamma$-semi-invariant subset of $X$. Suppose that $\Gamma_Y\backslash\Gamma$ is discrete. Let $H$ be as in 5.7.*

*If there is a short exact sequence*

$$0 \to C_0(X - Y) \rtimes_r \Gamma_Y \to C_0(X) \rtimes_r \Gamma_Y \to C_0(Y) \rtimes_r \Gamma_Y \to 0$$

*then the closed sets $G_Y, G^Y \subseteq H$ have the extension property of Definition 5.5 .*

We will not give a proof. First, we will not need the result. Secondly, the proof is very similar to the one given for the last item in Theorem 6.18.

## 6. Application to subgroupoids

In this section, we consider a groupoid $G$ and open subgroupoid $G'$ and give conditions under which our excision theorem can be applied to the reduced groupoid $C^*$-algebra, $B = C_r^*(G)$, and a $C^*$-subalgebra, $B' = C_r^*(G')$.

We assume that $G$ is a locally compact, Hausdorff, second countable groupoid with Haar system $\nu^u, u \in G^0$, and 2-cocycle $\sigma$. We suppose that $G^0 \subseteq G' \subseteq G$ is a subgroupoid (using the same algebraic operations) and is open in $G$. This means that the unit space of $G'$ coincides with that of $G$. The following is an easy result and we omit the proof.

**Theorem 6.1.** *Let $G$ be a locally compact, Hausdorff topological groupoid with Haar system, $\nu^u, u \in G^0$ and suppose $G^0 \subseteq G' \subseteq G$ is an open subgroupoid.*

(1) *The system of measures, $\nu^u|_{G^u \cap G'}, u \in G^0$, is a Haar system for $G'$.*
(2) *The inclusion $C_c(G') \subseteq C_c(G)$ obtained by extending functions to be zero on $G - G'$ extends to an inclusion of $C^*$-algebras, $C_r^*(G', \sigma) \subseteq C_r^*(G, \sigma)$.*

Let us define a notational convention: if $A \subseteq G$ is any subset of a groupoid $G$, $A^2 = A \times A \cap G^2$ and $A^n, n \geq 3$ is defined in an analogous way.

We state the following result for convenience. The proof is trivial and we omit it.

**Lemma 6.2.** *For an open subgroupoid $G^0 \subseteq G' \subseteq G$, we have*

(1) $\Delta = G - G'$ *is closed in $G$,*
(2) $\Delta = \Delta^{-1}$*,*
(3) $\Delta G', G'\Delta \subseteq \Delta$*,*
(4) $\Delta^2 = (\Delta \times \Delta) \cap G^2$ *is closed in $G^2$,*
(5) $\Delta^2 \cap \mu^{-1}(G')$ *is open in $\Delta^2$.*

We now want to construct a new pair of groupoids, $H' \subseteq H$. If $u$ is any unit of $G$, the set $s(G^u)$ is $G$-invariant and the restriction of $G$ to this set is a transitive groupoid (see 1.1 of [16]). Of course, this restriction may be disastrous, topologically. Our new groupoids will be the restrictions of $G$ and $G'$ to all sets $s(G^u)$ where they differ. In other words, all $s(G^u)$ where $u$ is in $r(\Delta)$. At least intuitively, any relative theory of $G' \subseteq G$ should be the same as that for $H' \subseteq H$. The first difficulty lies in the issue of putting suitable topologies on $H'$ and $H$. In fact, there is a canonical way to do this, but we will need certain technical conditions to proceed further.

To facilitate this, let us name some of the topologies involved. We let $\mathcal{S}$ be the topology on $G$ and $\mathcal{S}_A$ be the relative topology on any subset $A$ of $G$.

**Definition 6.3.** *We say the inclusion $G^0 \subseteq G' \subseteq G$ is* regular *if the map $r : \Delta \to r(\Delta)$ is open, when the image is given the quotient topology. We let $\mathcal{R}$ be the quotient topology on $r(\Delta)$.*

**Lemma 6.4.** *Let $G^0 \subseteq G' \subseteq G$ be a open subgroupoid. If the inclusion is regular, then $\Delta^2 \cap \mu^{-1}(G')$ is closed in $\Delta^2$.*

*Proof.* Suppose that $(x_i, y_i), i \geq 1$ is a sequence in $\Delta^2 \cap \mu^{-1}(G')$ converging to $(x, y)$, which is in $\Delta^2$, as $\Delta$ is closed. Suppose that $xy$ is in $\Delta$. We may choose a decreasing sequence of open subsets in $G$, $U_n, n \geq 1$, which intersect to $xy$. Then $U_n \cap \Delta$ is an open set in $\Delta$ and, as our inclusion is regular, $r(U_n \cap \Delta)$ is open in $\mathcal{R}$. As $x_i \in \Delta, i \geq 1$ is converging to $x$, $r(x_i), i \geq 1$ in the topology $\mathcal{R}$. So for each $n \geq 1$, we may find $x_{i_n}$ such that $r(x_{i_n})$ is in $r(U_n \cap \Delta)$. So we may find $z_n$ in $U_n \cap \Delta$ with $r(z_n) = r(x_{i_n})$. The sequence $z_n^{-1} x_{i_n} y_{i_n}$ is in $\Delta G' = \Delta$ and converges to $(xy)^{-1}xy$ which is in $G^0 \subseteq G'$. This contradicts $G'$ being open. $\square$

**Definition 6.5.** *Assume that $G^0 \subseteq G' \subseteq G$ is a open subgroupoid and that the inclusion is regular. Define*

$$H' = \mu(\Delta^2 \cap \mu^{-1}(G')) \subseteq G'.$$

*We endow $H'$ with the quotient topology from the map*

$$\mu : \Delta^2 \cap \mu^{-1}(G') \to H'$$

*which we denote by $\mathcal{T}'$.*

We observe the following for future purposes.

**Lemma 6.6.** *With $H'$ as defined in Definition 6.5, we have*

(1) *$x$ in $G'$ is in $H'$ if and only if $r(x)$ is in $r(\Delta)$,*
(2) *$H'$ is a subgroupoid of $G'$,*
(3) *the unit space of $H'$ is $r(\Delta)$.*

*Moreover, the topology on the unit space given in Definition 6.5 agrees with the quotient topology of $r : \Delta \to r(\Delta)$. That is, we have $\mathcal{T}'_{(H')^0} = \mathcal{R}$.*

*Proof.* For the first part, it is clear that

$$r(H') = r \circ \mu(\Delta^2 \cap \mu^{-1}(G')) \subseteq r \circ \mu(\Delta^2) \subseteq r(\Delta).$$

On the other hand, if $x$ is in $\Delta$,, then $(x, x^{-1})$ is in $\Delta^2$ and also $\mu^{-1}(G')$ and $r(x) = \mu(x, x^{-1})$. The second and third statements follow easily from the first.

The final part relies on the following basic topological fact: if $X$ is a topological space, $f : X \to Y$ is a surjection and $Y$ is given the quotient topology, then for any $Z \subseteq Y$, the relative topology from $Y$ agrees with the quotient topology from $f : f^{-1}(Z) \to Z$. In our case, we use $X = \Delta^2 \cap \mu^{-1}(G')$, $f = \mu$, $H' = Y$ and $Z = (H')^0$ along with the observation that the map sending $x$ in $\Delta$ to $(x, x^{-1})$ is a homeomorphism from $\Delta$ to $\mu^{-1}((H')^0$. □

**Theorem 6.7.** *Assume that $G^0 \subseteq G' \subseteq G$ is a open subgroupoid and that the inclusion is regular. Let $H', \mathcal{T}'$ be as in Definition 6.5.*

(1) *The topology $\mathcal{T}'$ on $H'$ is finer than $\mathcal{S}_{H'}$, the relative topology from $G$.*
(2) *$H'$ is topological groupoid.*
(3) *$H'$ is locally compact.*
(4) *$H'$ is Hausdorff.*
(5) *For each $u$ in $(H')^0 = H' \cap G^0$, we have*

$$G^u \cap H' = G^u \cap G', \quad G_u \cap H' = G_u \cap G'$$

*and the relative topologies from $\mathcal{T}'$and $\mathcal{S}$ are the same on each of these sets.*

(6) *The system of measures, $\nu^u|_{G^u \cap G'}, u \in (H')^0$, is a Haar system for $H'$.*

*Proof.* The first statement follows immediately from the fact that the map $\mu$ is continuous and the definition of the quotient topology.

Let us consider the set $\Delta^4 \cap \mu^{-1}(G') \times \mu^{-1}(G')$ and the map $(\mu^2 \times id)(w, x, y, z) = (wxy, z)$ defined on this set. First, we observe that $wx$ is in $G'$ and so $wxy$ is again in $\Delta$. We also note that $wxyz = (wx)(yz)$ is in $G'$ and so we have

$$\mu^2 \times id : \Delta^4 \cap \mu^{-1}(G') \times \mu^{-1}(G') \to \Delta^2 \cap \mu^{-1}(G').$$

Moreover, we have a commutative diagram:

$$\begin{tikzcd}
\Delta^4 \cap \mu^{-1}(G') \times \mu^{-1}(G') \arrow[r, "\mu^2 \times id"'] \arrow[d, "\mu \times \mu"] & \Delta^2 \cap \mu^{-1}(G') \arrow[d, "\mu"] \\
(H')^2 \arrow[r, "\mu_{H'}"] & H'
\end{tikzcd}$$

To verify $\mu_{H'}$ is continuous, we must take an open set $U$ in $H'$ and see that $\mu_{H'}^{-1}(U)$ is open in $(H')^2$. To see that, we must see that $(\mu \times \mu)^{-1}(\mu_{H'})^{-1}(U))$ is open. From the commutativity of the diagram, we have

$$(\mu \times \mu)^{-1}(\mu_{H'})^{-1}(U)) = (\mu^2 \times id)^{-1}(\mu^{-1}(U)).$$

The set $\mu^{-1}(U)$ is open due to the definition of the topology on $H'$ and $(\mu^2 \times id)^{-1}(\mu^{-1}(U))$ is open because $\mu^2 \times id$ is continuous. The continuity of the inverse in the new topology is obvious from the fact that $\mu(gh)^{-1} = \mu(h^{-1}, g^{-1})$.

Let us prove $H'$ is locally compact. Let $h$ be in $H'$, so it is in $G'$ we may write $h = gg'$ with $g, g'$ in $\Delta$. As $\mu$ is open and $G'$ is open, we may find open sets $g \subseteq U$, $g' \subseteq U'$ , each with compact closure, such that $\mu(U \times U' \cap G^2)$ is an open set in $G'$. It follows fairly easily that $\mu(U \times U' \cap \Delta^2)$ is an open set in $H'$. Let us verify its closure is compact. Let $h_n, n \geq 1$ be any sequence. It follows that, for every $n$, we can find $g_n$ in $U \cap \Delta$ and $g'_n$ in $U' \cap \Delta$ with $g_n g'_n = h_n$. As $U$ and $u'$ have compact closures, we may pass to a subsequence such that $(g_{n_i}, g'_{n_i})$ is converging. From Lemma 6.4, the limit of this subsequence must also lie in $\mu^{-1}(G') \cap \Delta^2$. It follows then that $h_n$ has a subsequence converging in $H'$.

The topology on $H'$ is Hausdorff since it is finer than the usual topology, which is Hausdorff.

For the fifth part, the containment $G^u \cap H' \subseteq G^u \cap G'$ is obvious since $H' \subseteq G'$. For the reverse containment, let $g$ be in $G^u \cap G'$. The fact that $u$ is in $H'$ means that $u = hh^{-1}$, for some $h$ in $\Delta$. We have $g = ug = (hkh^{-1})g = h(h^{-1}g)$, which is in $H'$ since $h^{-1}g$ is in $\Delta G' \subseteq \Delta$.

As we know that the topology for $H'$ is finer than that for $G'$, to see they are equal it suffices for us to take a sequence $g_n$ in $(G')^u$ converging to $g$ in the topology of $G'$ and show it also converges in the topology for $H'$. We simply write $g_n = ug_n = h(h^{-1}g_n)$, with $h$ as above. It suffices now to observe that $h, h^{-1}g_n$ and $h^{-1}g$ are in $\Delta$, and $(h, h^{-1}g_n)$ converges to $(h, h^{-1}g)$ in $G^2$. This implies their images under $\mu$ converge in the quotient topology as desired.

For the last part, by using Theorem 5.3, it suffices to prove that $r_{H'} : H' \to (H')^0$ is open. Let $x$ be in $H'$ and $U$ be set in $\mathcal{T}'$ containing $x$. This means $x$ is in $G'$ while $x = yz$, with $y, z$ is $\Delta$. It also means that $\mu^{-1}(U) \cap \Delta^2 \cap \mu^{-1}(G') = V \cap \Delta^2 \mu^{-1}(G')$, for some open set $V$ in $G^2$. We have $(x, y)$ is in $V \cap \mu^{-1}(G')$ and so we may find open sets $W, Z$ in $G$ with $s(W) = r(Z)$ such that $W \times Z \cap G^2 \subseteq V \cap \mu^{-1}(G')$. It follows from regularity and Lemma 6.6 that $r(W \cap \Delta)$ is an open subset of $(H')^0$. We claim that it is contained in $r(U)$. If $u = r(w)$ for some

$w$ in $W \cap \Delta$, then $r(w) = r(wz)$, for some $(w, z)$ in $G^2 \cap W \times Z \subseteq V$ The fact that $w$ is in $\Delta$ while $wz$ is in $G'$ means that $z$ is in $\Delta$ also. Hence $(w, z)$ is in $\Delta^2 \cap V = \mu^{-1}(U) \cap \Delta^2 \cap \mu^{-1}(G')$, $wz$ is in $U$ and $u = r(w) = r(wz) \in r(U)$, as desired. □

**Theorem 6.8.** *Assume that $G^0 \subseteq G' \subseteq G$ is a open subgroupoid and that the inclusion is regular. With $H'$ as in Definition 6.5, we define*

$$H = H' \cup \Delta.$$

*We endow $H$ with the disjoint union topology; that is, both $H'$ and $\Delta$ are clopen. We denote this topology by $\mathcal{T}$, so that $\mathcal{T}_{H'} = \mathcal{T}'$ and $\mathcal{T}_\Delta = \mathcal{S}_\Delta$.*

(1) *The topology $\mathcal{T}$ on $H$ is finer than the relative topology from $G$.*
(2) *$H$ is topological groupoid.*
(3) *$H$ is locally compact.*
(4) *$H$ is Hausdorff*
(5) *For each $u$ in $H^0 = H \cap G^0$, we have*

$$G^u \cap H = G^u, \quad G_u \cap H = G_u$$

*and the relative topologies from $H$ and $G$ are the same on each of these sets.*
(6) *The the systems of measures, $\nu^u, u \in H^0$, is a Haar system for $H$.*

*Proof.* The first, third and fourth parts follow easily from their counterparts in Theorem 6.7. For the second part, the continuity of inverses is clear. As $\Delta$ is clopen, it suffices to check the continuity of products on $H' \times H', H' \times \Delta, \Delta \times H'$ and $\Delta \times \Delta$ separately. The first is done. The second and third both have range in $\Delta$ and the continuity follows from that of the product in $G$, together with the fact the the topology on $H'$ is finer than that of $G$. For the last case, we use the fact that Lemma 6.4 implies that

$$\Delta^2 = \Delta^2 \cap \mu^{-1}(G') \cup (\Delta^2 - \Delta^2 \cap \mu^{-1}(G'))$$

are clopen. On the first set, continuity follows from Theorem 6.7, the second just uses the continuity in $G$.

For the proof of the fifth part, we can write $G^u = (G')^u \cup (\Delta \cap G^u)$ and the result follows from Theorem 6.7.

The last part follows from the last part of Theorem 6.7 and Lemmas 6.4 and 6.6. □

We remark that the groupoids $(G, \mathcal{S})$ and $(H, \mathcal{T})$ satisfy all hypotheses of Theorem 5.4 and the conclusions hold.

We turn now to the regular representations of $C_r^*(G,\sigma)$ and $C_r^*(H,\sigma)$ on $L^2(G,\nu)$. We let $p$ be the projection operator on $L^2(G,\nu)$ whose range is $L^2(G',\nu)$. That is, for $\xi$ in $L^2(G,\nu)$ and $g$ in $G$, $p\xi(g) = \xi(g)$ for $g$ in $G'$ and $p\xi(g) = 0$ for $g$ in $\Delta$.

We let $E$ denote the $C^*$-algebra $\mathcal{B}(L^2(G,\nu))$ and define $\delta : C_r^*(H,\sigma) + C_r^*(G,\sigma) \to E$ by

$$\delta(x) = i[x,p] = i(xp - px),$$

for $x$ in $C_r^*(H,\sigma) + C_r^*(G,\sigma)$. It is clear that $\delta$ is a completely contractive, $*$-derivation.

We let $\chi_\Delta$ be the function on $G$ which is 1 on $\Delta$ and 0 on $G'$. It is neither continuous, nor compactly supported, but will be useful, as follows.

**Lemma 6.9.** (1) *For any function $a$ in $C_c(H)$, the function $\chi_\Delta a$ (meaning the pointwise product) is in $C_c(H)$ and $(1-\chi_\Delta)a$ is in $C_c(H')$.*

(2) *For any function $a$ in $C_c(H)$ , we have*

$$\begin{aligned} pa(1-p) &= p\,(\chi_\Delta a) \\ (1-p)ap &= (\chi_\Delta a)\,p \\ \delta(a) &= \delta(\chi_\Delta a) \\ pap &= p\,((1-\chi_\Delta)a) \\ pap &= ((1-\chi_\Delta)a)\,p. \end{aligned}$$

(3) *For any function $b$ in $C_c(G)$, the function $\chi_\Delta b$ (meaning the pointwise product) is in $C_c(H)$.*

(4) *For any function $b$ in $C_c(G)$, we have*

$$\begin{aligned} pb(1-p) &= p\,(\chi_\Delta b) \\ (1-p)bp &= (\chi_\Delta b)\,p \\ \delta(b) &= \delta(\chi_\Delta b). \end{aligned}$$

*Proof.* The first part follows from the fact that $\Delta$ is both closed and open in $H$ and $H'$ is the complement of $\Delta$ in $H$.

For the first equation of the second part, let $\xi$ be a function in $L^2(G,\nu)$. From the definition of $p$, $(pa(1-p)\xi)(g) = (p(\chi_\Delta)a)\xi)\,(g) = 0$, if $g$ is in $\Delta$. Now, let $g$ be in $G'$ and we compute

$$\begin{aligned} (pa(1-p)\xi)(g) &= (ap\xi)(g) \\ &= \int_{h\in G^{r(g)}} a(gh^{-1})((1-p)\xi)(h)\sigma(gh^{-1},h)d\nu_{s(g)}(h). \end{aligned}$$

Now $((1-p)\xi)(h)$ is non-zero only if $h$ is in $\Delta$. As $g$ is in $G'$, this means that $gh^{-1}$ must be in $\Delta$ as well. For such values of $gh^{-1}$, $a(gh^{-1}) =$

$(\chi_\Delta a)\,(gh^{-1})$ and we have

$$\begin{aligned}
(pa(1-p)\xi)(g) &= \int_{h \in G_{s(g)}} a(gh^{-1})((1-p)\xi)(h)\sigma(gh^{-1},h)d\nu_{s(g)}(h) \\
&= \int_{h \in G_{s(g)} \cap \Delta} a(gh^{-1})((1-p)\xi)(h)\sigma(gh^{-1},h)d\nu_{s(g)}(h) \\
&= \int_{h \in G_{s(g)} \cap \Delta} (\chi_\Delta a)(gh^{-1})\xi(h)\sigma(gh^{-1},h)d\nu_{s(g)}(h) \\
&= \int_{h \in G_{s(g)}} (\chi_\Delta a)(gh^{-1})\xi(h)\sigma(gh^{-1},h)d\nu_{s(g)}(h) \\
&= ((\chi_\Delta a)\xi)\,(g).
\end{aligned}$$

The second equation is obtained by taking adjoints of both sides of the first (with $a^*$ replacing $a$). For the third equation, we compute, using the first two equations,

$$\begin{aligned}
\delta(\chi_\Delta a) &= i\,(\chi_\Delta ap - p\chi_\Delta a) \\
&= i\,((1-p)ap - pa(1-p)) \\
&= i\,(ap - pa) \\
&= \delta(a).
\end{aligned}$$

For the fifth equation, it follows from the first that

$$pap = pa - p\,(\chi_\Delta a) = p(((1-\chi_\Delta)a)\,.$$

as desired. The sixth is done similarly.

The third part follows from the fact that $\Delta$ has the same topology in $H$ as it does in $G$, and that $\Delta$ is a clopen subset of $H$. So the function $\chi_\Delta b$ is a continuous function of compact support on $H$.

The proofs of the last part are exactly the same as the first three parts of the second part and we omit the details. □

**Lemma 6.10.** *If $b$ is in $C_c(G,\sigma)$, then $\delta(b) \in \delta(C_c(H,\sigma))$.*

*Proof.* This follows at once from part 3 and the last equation in part 4 of Lemma 6.9. □

Finally, we need a reasonable extension of the notion of finite index from groups to groupoids to obtain condition C1 of Theorem 3.4.

**Definition 6.11.** *Let $G^0 \subseteq G' \subseteq G$ be an open subgroupoid. We say it has* finite index *if there is a constant $K \geq 1$ such that, for any $u$ in $G^0$, there is a finite subset $F \subseteq G_u$ with $\#F \leq K$ and $G_u = G'F$.*

We remark that if $G' \subseteq G$ is of finite index, then $H' \subseteq H$, as defined in Definition 6.5 and Theorem 6.8 is also of finite index, as it is a purely algebraic condition.

**Lemma 6.12.** *Let $H^0 \subseteq H' \subseteq H$ be an open subgroupoid of finite index. If $K$ is as in the definition and $p$ is the projection of $L^2(H,\nu)$ onto $L^2(H',\nu)$, 6.11, then for every $a$ in $C_r^*(H,\sigma)$ then $\|a\| \leq K\|pa\|$.*

*Proof.* Let us fix a unit $u$ in $H^0$ and restrict our attention to $L^2(G_u,\nu_u)$. Let $F \subseteq H^u$ be a subset of at most $K$ elements such that $FH' = H^u$. We may assume that this set is minimal, so that $f_1 f_2^{-1}$ is not in $H'$, for $f_1 \neq f_2$ in $F$. For each $f$ in $F$, the map sending $h$ in $H_u$ to $hf^{-1}$ in $H_{r(f)}$ induces a unitary operator $U_f : L^2(H_{r(f)},\nu_{r(f)}) \to L^2(H_u,\nu_u)$. This unitary operator conjugates the part of the left regular representation of $C_r^*(H,\sigma)$ on $L^2(H_u,\nu_u)$ onto that on $L^2(H_{r(f)},\nu_{r(f)})$. For convenience, we denote these two Hilbert spaces by $\mathcal{H}_u$ and $\mathcal{H}_{r(f)}$, respectively. Define $p_f = U_f p U_f^*$. It is an easy computation to check that the condition that $H'F = H_u$ implies that $\sum_{f\in F} p_f$ is the identity operator on $L^2(H_u,\nu_u)$. Hence we have

$$\begin{aligned}
\|a|_{\mathcal{H}_u}\| &= \|\sum_{f\in F} p_f a|_{\mathcal{H}_u}\| \\
&\leq \sum_{f\in F} \|p_f a|_{\mathcal{H}_u}\| \\
&= \sum_{f\in F} \|U_f p U_f^* a|_{\mathcal{H}_u}\| \\
&= \sum_{f\in F} \|p U_f^* a U_f|_{\mathcal{H}_{s(f)}}\| \\
&= \sum_{f\in F} \|pa|_{\mathcal{H}_{s(f)}}\| \\
&\leq K\|pa\|.
\end{aligned}$$

Taking the supremum over all $u$ completes the proof. □

**Corollary 6.13.** *If $G^0 \subseteq G' \subseteq G$ is an open subgroupoid and the inclusion is regular and finite index, then the $*$-algebra $C_c(H)$ satisfies condition C1 of Theorem 3.4.*

*Proof.* Let $a$ be in $C_c(H)$ and define $a' = (1-\chi_\Delta)a$ which is in $C_c(H')$ by Lemma 6.9. We have

$$\delta(a) = i(ap - pa) = i\left((1-p)ap - pa(1-p)\right).$$

The first of these terms maps $pL^2(G,\nu)$ to $(1-p)L^2(G,\nu)$ and the second the other way. It follows then that

$$\begin{aligned} K\|\delta(a)\| &= K\max\{\|(1-p)ap\|, \|pa(1-p)\|\} \\ &= \max\{K\|(\chi_\Delta a)p\|, K\|p(\chi_\Delta a)\|\} \\ &\geq \|\chi_\Delta a\| \\ &= \|a-a'\|. \end{aligned}$$

□

**Lemma 6.14.** *If $G^0 \subseteq G' \subseteq G$ is an open subgroupoid and the inclusion is regular, then the $*$-algebra $C_c(H)$ satisfies condition C2 of Theorem 3.4.*

*Proof.* Let $a_1, \ldots, a_M$ be in $C_c(H)$. Define $a = (a_1, \ldots, a_M)$ Let $U = \{x \in H \mid a(x) \neq 0\}$ so that $\overline{U}$ is compact in $H$. It follows that $r(\overline{U}) \cup s(\overline{U})$ is compact in $H^0$. We can find $e : H^0 \to [0,1]$ which is continuous, identically one on $r(\overline{U}) \cup s(\overline{U})$ and has support contained in the compact set $K \subseteq H^0$.

We apply Lemma 5.2 with $K, a$ as above and $Y = \Delta$ to each to obtain $b = (b_1, \ldots, b_M)$.

Then we have

$$\delta(b_m) = \chi_\delta b_m = \chi_\delta a_m = \delta(a_m),$$

from parts 2 and 4 of Lemma 6.9, for $1 \leq m \leq M$.

We claim that $a_m = ea_m$, for all $1 \leq m \leq M$. For any $x$ in $H$, we have $(ea_m)(x) = e(r(x))a_m(x)$. If $x$ is not in $U$, then $a_m(x) = 0 = e(r(x))a_m(x)$. On the other hand, if $x$ is in $U$, then $r(x)$ is in $r(supp(a))$ and so $e(r(x)) = 1$.

Finally, we claim that $ea_m(x) = eb_m(x)$, for all $x$ in $H$. If $r(x)$ is in $K$, this follows from the fact that $a_m(x) = b_m(x)$ for such $x$. On the other hand, if $r(x)$ is not in $K$, then $e_m(x) = 0$.

The proof that $a_m = a_m e = b_m e$ is done in a similar way. □

What remains at this point is to verify that $C_r^*(H,\sigma) \cap \ker(\delta) = C_r^*(H',\sigma)$ and $C_r^*(G,\sigma) \cap \ker(\delta) = C_r^*(G',\sigma)$. The first is relatively simple.

**Theorem 6.15.** *If $G^0 \subseteq G' \subseteq G$ is an open subgroupoid and the inclusion is regular and finite index, then*

$$C_r^*(H,\sigma) \cap \ker(\delta) = C_r^*(H',\sigma).$$

*Proof.* The containment $C_r^*(H,\sigma) \cap \ker(\delta) \supseteq C_r^*(H',\sigma)$ follows at once from part 2 of Lemma 6.9 and the continuity of $\delta$. For the reverse inclusion, let $a$ be any element of $C_r^*(H,\sigma) \cap \ker(\delta)$. We may select a sequence $a_n, n \geq 1$ in $C_c(H)$. As $\delta(a) = 0$, we know $\delta(a_n), n \geq 1$ converges to 0. By Corollary 6.13, for each $n$, we may find $a'_n$ in $C_c(H')$ such that $\|a_n - a'_n\| \leq K\|\delta(a_n)\|$, so $a'_n$ also converges to $a$. $\square$

The second equality is more subtle and we prove it only under some additional hypotheses. It would follow if we knew that $\mathcal{A} = C_c(G)$ satisfied condition C1 of 3.4, using exactly the same argument as above.

**Lemma 6.16.** *Assume that $G$ is a locally compact, Hausdorff groupoid with Haar system $\nu^u, u \in G^0$, $\sigma$ is a 2-cocycle on $G$ and that $G^0 \subseteq G' \subset G$ is a regular, open subgroupoid of finite index with $K$ as in the definition. If $a$ is in $C_c(\Delta) \subseteq C_c(H)$, then*

$$\|a\| \leq K\|\delta(a)\|.$$

*Proof.* It follows from part 2 of Lemma 6.9 that $pa = p(\chi_\Delta a) = pa(1 - p)$ and that $ap = (\chi_\Delta a)p = (1-p)ap$. We have

$$\delta(a) = iap - ipa = (1-p)iap - pia(1-p).$$

Thus $\delta(a)$ is the sum of two operators, one which is zero on $(1-p)\mathcal{H}$ and range in $(1-p)\mathcal{H}$ and the other which is zero on $p\mathcal{H}$ and range in $p\mathcal{H}$. It follows that

$$\begin{aligned} \|\delta(a)\| &= \max\{\|(1-p)iap\|, \|pia(1-p)\|\} \\ &= \max\{\|ap\|, \|pa\|\} \\ &= \max\{\|pa^*\|, \|pa\|\} \\ &\leq \max\{K\|a^*\|, K\|a\|\} \\ &= K\|a\|. \end{aligned}$$

$\square$

**Theorem 6.17.** *Assume that $G$ is a locally compact, Hausdorff groupoid with Haar system $\nu^u, u \in G^0$, $\sigma$ is a 2-cocycle on $G$ and that $G^0 \subseteq G' \subset G$ is a regular, open subgroupoid of finite index. If the closed set $\Delta = G - G'$ has the $C$-extension property of Definition 5.5 for some $C \geq 1$, then*

$$C^*_r(G, \sigma) \cap \ker(\delta) = C^*_r(G', \sigma).$$

*Proof.* The containment $C^*_r(G, \sigma) \cap \ker(\delta) \supseteq C^*_r(G', \sigma)$ follows at once from part 4 of Lemma 6.9 and the continuity of $\delta$. For the reverse inclusion, let $b$ be any element of $C^*_r(G, \sigma) \cap \ker(\delta)$. We may select a sequence $b_n, n \geq 1$ in $C_c(G)$. As $\delta(b) = 0$, we know $\delta(b_n), n \geq 1$ converges to 0. From part 2 of Lemma 6.10 above, $\chi_\Delta b_n$ lies in $C_c(\Delta) \subseteq C_c(H)$ and, from part 4 of the same Lemma, $\delta(\chi_\Delta b_n) = \delta(b_n)$. As $\Delta$

satisfies the $C$-extension property, for every $n$, we may find $c_n$ in $C_c(G)$ with

$$c_n|_\Delta = \chi_\Delta b_n|_\Delta = b_n|_\Delta$$

and

$$\|c_n\| \leq C\|\chi_\Delta b_n\| \leq CK\|\delta(\chi_\Delta b_n)\| \leq CK\|\delta(b_n)\|.$$

It follows that $\|c_n\|$ tends to zero as $n$ tends to infinity so $b_n - c_n$ also converges to $b$. These functions are in $C_c(G)$ and vanish on $\Delta$, so they can be approximated by functions in $C_c(G')$. □

We finish this section with a specific construction which generalizes the *orbit splitting* groupoids, first introduced in [12]. The group of integers still plays an important role. This has seen a number of applications.

Suppose that the locally compact Hausdorf space $X$ has an action by homeomorphisms of the locally compact Hausdorff group $\Gamma$. Also suppose that $Y \subseteq X$ is semi-invariant in the sense of Definition 5.8. We further suppose that $\Gamma_Y$ is a normal subgroup of $\Gamma$ and that the quotient group $\Gamma/\Gamma_Y$ is isomorphic to $\mathbb{Z}$. Let us denote the quotient map from $\Gamma$ to $\mathbb{Z}$ by $\zeta$.

We notice that Theorem 5.9 applies immediately to this situation. But we can go further to find an open subgroupoid.

**Theorem 6.18.** *Let $\Gamma, X, Y, \zeta$ be as above and let $G = X \rtimes \Gamma$ be the associated transformation groupoid. Define*

$$\Delta = \{(y\gamma_1, \gamma_1^{-1}\gamma_2), (y\gamma_2, \gamma_2^{-1}\gamma_1) \mid y \in Y, \gamma_1, \gamma_2 \in \Gamma, \zeta(\gamma_1) \leq 0, \zeta(\gamma_2) > 0\}$$

*Then we have the following.*

1. *$G' = G - \Delta$ is an open subgroupoid.*
2. *The inclusion $G^0 \subseteq G' \subseteq G$ is regular and finite index.*
3. *If $H$ and $H'$ are the groupoids of Definition 6.5 and Theorem 6.8, then*
$$H \cong (Y \rtimes \Gamma_Y) \times (\mathbb{Z} \rtimes \mathbb{Z})$$
*(the latter is the co-trivial groupoid) while*
$$H' \cong (Y \rtimes \Gamma_Y) \times \left(\mathbb{Z}^- \rtimes \mathbb{Z}^- \cup \mathbb{Z}^+ \rtimes \mathbb{Z}^+\right)$$
*where $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ and $\mathbb{Z}^- = \{0, -1, -2, \ldots\}$.*
4. *If there is a short exact sequence*
$$0 \to C_0(X - Y) \rtimes_r \Gamma_Y \to C_0(X) \rtimes_r \Gamma_Y \to C_0(Y) \rtimes_r \Gamma_Y \to 0$$
*then the closed set $\Delta$ has the $C$-extension property for any $C > 1$.*

*Proof.* Let us choose $a$ in $\Gamma$ with $\zeta(a) = 1$.

First observe that $Y\gamma = Ya^{\zeta(\gamma)}$, for every $\gamma$ in $\Gamma$. It follows that we may write

$$\Delta = \left(\cup_{i\leq 0, j>0} Ya^i \times \zeta^{-1}\{i+j\}\right) \cup \left(\cup_{i>0, j\leq 0} Ya^i \times \zeta^{-1}\{j-i\}\right).$$

The sets on the right are pairwise disjoint and each is clopen in $G$. It follows that $\Delta$ is closed so $G'$ is open. It is a simple computation to check that $G'$ is a subgroupoid.

For the second point, it is easy to check from the equation above that

$$r(\Delta) = \cup_{i\in\mathbb{Z}} Ya^i,$$

and that each of the sets $Ya^i$ is clopen in the quotient topology. In addition, the restriction of $r$ to a set of the form $Ya^i \times \gamma$ is actually a homeomorphism to $Ya^i$ and, in particular, $r$ is open so the inclusion is regular. It also follows that the map sending $ya^i$ to $(y, i)$ is a homeomorphism between $r(\Delta)$ and $Y \times \mathbb{Z}$.

We check that the inclusion is finite index. If $(x, e)$ is any unit in $G$ and $x \notin r(\Delta)$, then we may use $F = \{(x, e)\}$ in Definition 6.11. On the other hand, if $(x, e) = (ya^i, e)$, for some $y$ in $Y$ and integer $i$, letting $F = \{(ya^i, e), (ya^{-i+1}, a^{2i-1})\}$ satisfies $G_u = G'_u F$. (The essential point being that $i \leq 0$ if and only if $-i + 1 > 0$.)

As for the descriptions of $H'$ and $H$, recall that $(ya^i, \gamma)$ and $(y'a^{i'}, \gamma')$ are composable if and only if $ya^i\gamma = y'a^{i'}$. If this occurs then $i+\zeta(\gamma) = i'$ and $y' = a^i\gamma a^{-i'}$. We know also that if $i \leq 0$, then $i' = i + \zeta(\gamma) > 0$ while if $i > 0$, then $i' = i + \zeta(\gamma) \leq 0$. It follows then that

$$\Delta^2 \cap \mu^{-1}(G') = \cup_{(i,\gamma,\gamma')} \cup_{y\in Y} \left((ya^i, \gamma), (ya^i\gamma, \gamma')\right)$$

where the union is over $y$ in $Y$ and triples $(i, \gamma, \gamma')$ with either $i \leq 0, i+\zeta(\gamma) > 0, i+\zeta(\gamma)+\zeta(\gamma') \leq 0$ or $i > 0, i+\zeta(\gamma) \leq 0, i+\zeta(\gamma)+\zeta(\gamma') > 0$. If we just take the union over $Y$, these sets are pairwise disjoint, clopen and each is homeomorphic to $Y$.

As a set, we can write

$$H' = \{(ya^i, \gamma) \mid y \in Y, i, i + \zeta(\gamma) \leq 0, \text{or } i, i + \zeta(\gamma) > 0\}.$$

and the isomorphism of part 3 sends $(ya^i, \gamma)$ to $\left((y, a^{-i}\gamma a^{i+\zeta(\gamma)}, (i, i + \zeta(\gamma))\right)$. We omit the topological details. This is extended to $H$ by mapping $(ya^i, \gamma)$ in $\Delta$ to $\left((y, a^{-i}\gamma a^{i+\zeta(\gamma)}, (i, i + \zeta(\gamma))\right)$.

We turn our attention to the last part. Observe that, for any $n \geq 2$, the sets

$$\Delta_n = \cup_{i=1-n}^{0} \cup_{j=1}^{n} \left(Ya^i \times \zeta^{-1}\{j - i\} \cup Ya^j\zeta^{-1} \times \{i - j\}\right)$$

are all in $\Delta$. Moreover, they are increasing with $n$ and each is clopen (in $\Delta$ and hence also in $H$). Therefore, it suffices for us to consider a continuous function $f$ with compact support in $\Delta_n$, for some fixed $n$. Let us denote $Y_n = \cup_{i=1-n}^{n} Ya^i$, which we identify with $r(\Delta_n) = s(\Delta_n)$.

In fact, $\Delta_n$ is a clopen subset of the groupoid $G_{Y_n}^{Y_n} = H_{Y_n}^{Y_n}$ and by simply restricting the function we can regard $f$ is being in $C_c(G_{Y_n}^{Y_n})$ and its norm there coincides with its norm in $C_r^*(H)$.

As the sets $Ya^i, i \in \mathbb{Z}$ are pairwise disjoint, we may choose an open set $Y \subseteq U \subseteq X$ such that the sets $Ua^l, 1 - 3n \leq l \leq 3n$ are pairwise disjoint also.

The set $U$ will not necessarily be $\Gamma_Y$-invariant. In addition, $U \cap U^{a^i}$ may be non-empty for some $i \neq 0$. However, we may consider the reductions $(X \times \Gamma)_U^U$ and $(X \times \Gamma_Y)_U^U$. Their associated $C^*$-algebras are hereditary subalgebras of $C_0(X) \rtimes_r \Gamma$ and $C_0(X) \rtimes_r \Gamma_Y$, respectively.

We define $V = \cup_{i=1-n}^{n} Ua^i$ and, identifying $V$ with $V \times \{e\}$ in $G^0$,

$$G(V) = G_V^V \cap X \times \zeta^{-1}\{1 - 2n, \ldots, 2n - 1\}.$$

This is an open subset of the groupoid $G_V^V$. We claim that it is also a groupoid. Suppose that $(x, \gamma)$ is in $G(V)$. Then for some $1-n \leq i \leq n$, $x$ is in $Ua^i$ and $1-2n \leq \zeta(\gamma) \leq 2n-1$. In addition $s(x, \gamma) = (x\gamma, e)$ is in $V$. On the other hand, $x\gamma$ is in $Ua^{i+\zeta(\gamma)}$ and $1-3n \leq i+\zeta(\gamma) \leq 3n-1$. As these sets are pairwise disjoint and only in $V$ if $1-2n \leq i+\zeta(\gamma) \leq 2n-1$, we know that this inequality must hold.

The left regular representation of $C^*(G)$ also extends to a representation of the bounded Borel functions on $G^0$, as well as to a unitary representation of the group $\Gamma$ [16]. It is a simple matter to check that

$$\chi_{Ua^i} a^i = a^i \chi_U$$

and, for any $g$ in $C_c(G(V))$, $1 - n \leq i, j \leq n$,

$$a^{-j} \chi_{Ua^j} g \chi_{Ua^i} a^i$$

is in $C_c((X \times \Gamma_Y)_U^U)$. This defines an isomorphism (we will not write it explicitly) between $C_r^*(G(V))$ and $M_{2n}(C_r^*((X \times \Gamma_Y)_U^U))$ (the matrix entries are indexed by the set $\{1 - n, \ldots, n\}$). For simplicity, let us denote $X \times \Gamma_Y$ by $L$.

The set $Y_n$ is a closed invariant subset of the unit space of $G(V)$ and we have a quotient map from $C_r^*(G(V))$ to $C_r^*(G_{Y_n}^{Y_n})$. This yields the following commutative diagram

$$\begin{tikzcd}
C_r^*(G(V)) \arrow[r] \arrow[d] & C_r^*(G_{Y_n}^{Y_n}) \arrow[d] \\
M_{2n}(C_r^*(G_U^U)) \arrow[r] & M_{2n}(C_r^*(G_Y^Y))
\end{tikzcd}$$

and the vertical map on the right is also an isomorphism.

We now invoke our hypothesis to extend this diagram to

$$\begin{tikzcd}
C_r^*(G(V)) \arrow[r] \arrow[d] & C_r^*(G_{Y_n}^{Y_n}) \arrow[r] \arrow[d] & 0 \\
M_{2n}(C_r^*(L_U^U)) \arrow[r] \arrow[d] & M_{2n}(C_r^*(L_Y^Y)) \arrow[r] \arrow[d] & 0 \\
M_{2n}(C_0(X) \rtimes_r \Gamma_Y) \arrow[r] & M_{2n}(C_0(Y) \rtimes_r \Gamma_Y) \arrow[r] & 0.
\end{tikzcd}$$

We now take our function $f$ lying in $C_c(H)$, whose support is in $\Delta_n$ and assume that $f$ is non-zero. We can regard this as an element of $M_{2n}(C_c(Y \rtimes \Gamma_Y)$ and we may lift it an element $\tilde{f}$ in $M_{2n}(C_c(X \rtimes \Gamma_Y)$. From our hypothesis, we know that

$$\|f\|_r = \inf\{\|\tilde{f} + g\|_r \mid g \in C_c((X - Y) \times \Gamma_Y)\},$$

and so if $C > 1$, we may choose $g$ so that $\|\tilde{f}+g\|_r \leq C\|f\|_r$. It remains to get this function back into $M_{2n}(C_r^*(L_U^U))$. As we noted above, this is a hereditary subalgebra of $M_{2n}(C_c(X \rtimes \Gamma_Y))$ and we can multiply on both sides by a function bounded between 0 and 1, identically 1 in $Y_n$ and supported in $V$. $\square$

**Corollary 6.19.** *With $X, \Gamma, Y, \zeta$ as above, if there is a short exact sequence*

$$0 \to C_0(X - Y) \rtimes_r \Gamma_Y \to C_0(X) \rtimes_r \Gamma_Y \to C_0(Y) \rtimes_r \Gamma_Y \to 0$$

*then*

$$K_*(C_r^*(G'); C_0(X) \rtimes_r \Gamma) \cong K_*(C_0(Y) \rtimes_r \Gamma_Y).$$

## 7. Application to factor groupoids

In this section, we again have two groupoids, $G, G'$. We assume that each is locally compact and Hausdorff. In addition, we assume that $G$ has a Haar system, $\lambda_G^u, u \in G^0$ and $G'$ has a Haar system, $\lambda_{G'}^u, u \in (G')^0$. Finally, we assume that $\sigma$ is a 2-cocycle on $G$ and that $\sigma'$ is a 2-cocycle on $G'$

We assume that there is a map $\pi : G \to G'$ satisfying the following conditions, which we refer to as our standing hypotheses on $\pi : G \to G'$:

(1) $\pi$ is continuous,

(2) $\pi$ is surjective,
(3) $\pi$ is proper,
(4) $\pi$ is a morphism of groupoids,
(5) for every $u$ in $G$ and $\pi|_{G^u} : G^u \to (G')^{\pi(u)}$ is a homeomorphism and
(6) for every $u$ in $G$ and Borel set $E$ in $G^u$, we have

$$\lambda^u_G(E) = \lambda^{\pi(u)}_{G'}(\pi(E)).$$

(7) for all $(s,t)$ in $G^2$, we have $\sigma(s,t) = \sigma'(\pi(s), \pi(t))$.

We will assume this holds throughout this section.

**Example 7.1.** *As an example, consider the case where $\Gamma$ is a locally compact, Hausdorff topological group acting continuously on compact Hausdorff spaces, $X$ and $X'$. If $\pi : X \to X'$ is a continuous, $\Gamma$-invariant surjection then, $\pi \times id_\Gamma : X \rtimes \Gamma \to X' \rtimes \Gamma$ satisfies the desired conditions.*

The proof of the following result is straightforward and we omit it.

**Theorem 7.2.** *The map sending $b$ in $C_c(G')$ to $b \circ \pi$ in $C_c(G)$ extends to an injective $*$-homomorphism, also denoted $\pi$, from $C^*_r(G', \sigma)$ to $C^*_r(G, \sigma)$.*

It will be convenient (though probably not necessary) for us to assume that $d_G$ is a metric on $G$ giving rise to its topology. By simply replacing $d_G$ by the function

$$\max\{d_G(x,y), d_G(x^{-1}, y^{-1}), d_G(r(x), r(y)), d_G(s(x), s(y))\}$$

we may assume that the map $x \to x^{-1}$ is an isometry and that $r, s$ are contractions.

We will let $G(x,r)$ denote the ball centred at $x \in G$ of radius $r > 0$. In addition, if $A \subseteq G$ is any subset, we let

$$G(A,r) = \{x \in X \mid \text{ for some } y \in A, d(x,y) < r\}$$

for $r > 0$.

We will then use the same notation, $d_G$, for the Hausdorff metric on the compact subsets of $G$:

$$d_G(E,F) = \inf\{\epsilon > 0 \mid F \subseteq G(E,\epsilon), E \subseteq G(F,\epsilon)\},$$

for $E, F \subseteq G$ compact. We will use the notation

$$diam_G(E) = \sup\{d_G(x,y) \mid x,y \in E\},$$

for $E \subseteq G$ compact. We will also assume for convenience that $G'$ is second countable.

The following is an obvious consequence of the third and fourth conditions, but we find it convenient to state explicitly.

**Lemma 7.3.** *Let $\pi : G \to G'$ satisfy the standing assumptions listed at the beginning of this section. For every $x$ in $G$, we have*

$$\#\pi^{-1}\{x\} = \#\pi^{-1}\{r(x)\} = \#\pi^{-1}\{s(x)\}.$$

The following technical result will also be useful later.

**Lemma 7.4.** *Suppose the sequence $x'_k$ converges to $x'$ in $G'$. Then*

$$\lim_k diam_G(\pi^{-1}\{x'_k\}) = 0,$$

*if and only if*

$$\lim_k diam_G(\pi^{-1}\{r(x'_k)\}) = 0,$$

*Proof.* Let $X'$ consist of the sequence $x'_k$, along with its limit point, $x'$. This is evidently a compact subset of $G'$. Hence, $r(X')$ is also compact in $G'$, while $X = \pi^{-1}(X')$ and $r(X)$ are compact in $G$ as $\pi$ is proper.

As $X$ is compact, $r$ is uniformly continuous on $X$ and the 'only if' direction follows at once. Conversely, suppose the first condition fails: then there exist subsequences $x_k, y_k$ in $G$ with $\pi(x_k) = x'_k = \pi(y_k)$, $\lim_k x_k = x \neq y = \lim_k y_k$ and $\lim_k r(x_k) = \lim_k r(y_k)$. Then we have

$$r(x) = r(\lim_k x_k) = \lim_k r(x_k) = \lim_k r(y_k) = r(\lim_k y_k) = r(y).$$

On the other hand, we also have

$$\pi(x) = \pi(\lim_k x_k) = \lim_k \pi(x_k) = x' = \lim_k \pi(y_k) = \pi(\lim_k y_k) = \pi(y).$$

Thus $x, y$ are in $G^{r(x)}$ with $\pi(x) = \pi(y)$. By the fourth condition of our standing hypotheses, this means $x = y$, a contradiction. □

As a final preliminary topological result, we have the following.

**Lemma 7.5.** *Given $x'$ in $G'$ and $\epsilon > 0$, there is an open set $x' \in U' \subseteq G'$ such that $\pi^{-1}(U') \subseteq G(\pi^{-1}\{x'\}, \epsilon)$.*

*Proof.* If the conclusion is false, then we may find a sequence, $x'_k, k \geq 1$, converging to $x'$ and a sequence $x_k, k \geq 1$ with $\pi(x_k) = x'_k$, for all $k \geq 1$ such that $x_k$ is not in $G(\pi^{-1}\{x'\}, \epsilon)$. The sequence $x'_k, k \geq 1$ along with $x'$ forms a compact set in $G'$. As $\pi$ is proper, its preimage is aslo compact in $G$. So we may find a subsequence $x_{k_l}, l \geq$ converging to some $x$ in $G$. As $G(\pi^{-1}\{x'\}, \epsilon)$ is open, $x$ is also not in this set. But by continuity of $\pi$, $\pi(x) = \lim_l \pi(x_{k_l}) = x'$. This is a contradiction. □

We want to focus our attention on the parts of $G$ and $G'$ where they are actually different; that is, where $\pi$ is not injective.

**Definition 7.6.** *Let $\pi : G \to G'$ satisfy the standing hypotheses. We define*

$$H' = \{x' \in G' \mid \#\pi^{-1}\{x'\} > 1\}$$

*and $H = \pi^{-1}(H')$.*

We endow $H'$ with the metric

$$d_{H'}(x', y') = d_G(\pi^{-1}\{x'\}, \pi^{-1}\{y'\}),$$

for $x', y'$ in $H'$, and $H$ with the metric

$$d_H(x, y) = d_G(x, y) + d_{H'}(\pi(x), \pi(y)),$$

for $x, y$ in $H$.

To obtain our excision result, we will need a hypothesis on our map $\pi$. It implies the continuity of the fibres of $\pi$, but in a weak sense.

**Definition 7.7.** *Let $\pi : G \to G'$ satisfy the standing hypotheses. We say that $\pi$ is* regular *if, for every $x'$ in $H'$ and $\epsilon > 0$, there is an open set $x' \in U' \subseteq G'$ such that if $y'$ is in $U'$, then either*

$$d_G(\pi^{-1}\{x'\}, \pi^{-1}\{y'\}) < \epsilon$$

*or*

$$diam_G(\pi^{-1}\{y'\}) < \epsilon.$$

*In view of Lemma 7.5, we may also assume the conclusion there also holds for $U'$. We remark that if $\epsilon < 3^{-1}diam_G(\pi^{-1}\{x'\})$, the two conditions are mutually exclusive.*

**Proposition 7.8.** *The topologies on $H'$ and $H$ from the metrics $d_{H'}$ and $d_H$ are finer, respectively, than the relative topologies from $G'$ and $G$.*

*Proof.* Let $s'$ be in $H'$ and let $U$ be an open set in $G'$ containing it. Choose any $s$ in $\pi^{-1}\{s'\}$. As $\pi$ is continuous, there is $\epsilon > 0$ such that $\pi(G(s, \epsilon)) \subseteq U$. We claim that $H'(s', \epsilon) \subseteq U \cap H'$. If $t'$ is in $H'(s', \epsilon)$ then it is clearly in $H'$. Moreover, we know $d_G(\pi^{-1}\{s'\}, \pi^{-1}\{t'\}) < \epsilon$. It follows that there is $t$ with $\pi(t) = t'$ and $d_G(s, t) < \epsilon$. It follows that $t' = \pi(t)$ is in $U$. Hence the topology from $d_{H'}$ finer than the relative topology from $G'$.

The fact that, for all $x, y$ in $H$,

$$d_G(x, y) \leq d_{H'}(\pi(x), \pi(y)) + d_G(x, y) = d_H(x, y),$$

immediately implies the desired conclusion for $H$. □

**Theorem 7.9.** *Suppose that $\pi : G \to G'$ is regular. Then $H'$ and $H$, with the metrics $d_{H'}$ and $d_H$, are locally compact, Hausdorff topological groupoids and $\pi : H \to H'$ is an open, continuous, proper morphism of groupoids.*

*Proof.* We begin with $H'$. It follows from Lemma 7.3 that either $x', r(x')$ and $s(x')$ are all in $H'$, or none are. It follows that $H'$ is a subgroupoid of $G'$.

Next, we show that $H'$ is locally compact. Let $x'$ be in $H'$. Let $\epsilon = 2^{-1} diam_G \pi^{-1}\{x'\}$. Observe that if $y'$ is any element of $H'$ within $\epsilon$ of $x'$ then $diam_G(\pi^{-1}\{y\}) > \epsilon$. Select $x' \in U \subseteq G'$ open, as in Definition 7.7. As $G'$ is locally compact, we may assume that $U$ has compact closure. Finally, select $\epsilon > \epsilon' > 0$ such that $\pi(G(\pi^{-1}\{x'\}, \epsilon') \subseteq U$. We claim that the $d_{H'}$-ball around $x'$ of radius $\epsilon'$ has compact closure. Let $x'_k$ be any sequence in this ball. The fact that $d_{H'}(x', x'_k) < \epsilon'$ means that $\pi^{-1}\{x'_k\}$ is contained in $G(\pi^{-1}\{x'\}, \epsilon')$ and so $x'_k$ is in $U$. As $U$ has compact closure, we may pass to a subsequence which converges to $y'$ in $G'$. We claim this sequence also converges to $y'$ in $d_{H'}$. Let $\delta > 0$ be given. Without loss of generality, assume $\delta < \epsilon$. We now apply our regularity hypothesis to $y'$ to find an open set $y' \in W \subseteq G'$ such that, for all $z'$ in $W$, either

$$d_G(\pi^{-1}\{y'\}, \pi^{-1}\{z'\}) < \delta$$

or

$$diam_G(\pi^{-1}\{z'\}) < \delta.$$

As our subsequence of $x'_k$ converges to $y'$ in the usual topology, we may find $K$ such that $x'_k$ is in $W$, for all $k \geq K$. Since our subsequence is taken from the $d_{H'}$-ball, we know that $diam_G(\pi^{-1}\{x'_k\}) \geq \epsilon > \delta$. This eliminates the second possibility above and hence, we have $d_{H'}(x'_k, y') < \delta$, for $k \geq K$.

Every metric space is Hausdorff.

As $d_G$ is preserved under inverses, so is $d_{H'}$. We must now check that the product on $H'$ is continuous in $d_H$. Suppose that $(x'_k, y'_k)$ is a sequence converging to $(x', y')$ in $(H')^2$. We again let $X'$ be this sequence, together with its limit point. As $G^2$ is closed in $G \times G$, $X = (\pi \times \pi)^{-1}(X') \cap G^2$ is compact in $G^2$. The product map on $X$ is continuous and it follows that it is continuous on the compact subsets of $X$, equipped with the Hausdorff metric. The continuity of the product on $H'$ follows from this.

We now turn our attention to $H$. As $\pi \circ r = r \circ \pi, \pi \circ r = r \circ \pi$, an element $x$ in $G$ is in $H$ if and only if $r(x)$ is in $H$, if and only if $s(x)$ is in $H$. It follows that $H$ is a groupoid.

We observe first that. for any $x, y$ in $G$, we have

$$d_{H'}(\pi(x), \pi(y)) \leq d_{H'}(\pi(x), \pi(y)) + d_G(x, y) = d_H(x, y)$$

so the map $\pi$ is contractive and hence continuous.

We show that $H$ is locally compact in $d_H$. Let $x$ be in $H$. We may find $\epsilon > 0$ such that $H'(\pi(x), \epsilon)$ has compact closure. We claim the same is true of $H(x, \epsilon)$. Let $x_k$ be any sequence in $H(x, \epsilon)$. It follows from the definition of $d_H$ that that $\pi(x_k)$ is in $H'(\pi(x), \epsilon)$. Hence we may pass to a subsequence such that $\pi(x_k)$ is converging to $x'$. We use the same trick again: let $X$ denote the pre-image of of the subsequence and its limit point under $\pi$, which is compact and contains $x_k$. Now we can further extract a subsequence where the $x_k$'s converge in the usual topology of $G$, also. It then follows from the definition of $d_H$ that this subsequence is also converging in $d_H$. Again, a metric space is Hausdorff.

To check that the product in $H$ is continuous in the metric $d_H$, it suffices to observe the definition of the metric $d_H$ and the facts that the product is continuous in $d_G$, $\pi$ is continuous and the product in $H'$ is continuous in $d_{H'}$. The inverse is isometric in $d_H$: this follows from the fact that taking inverses is isometric in both $d_G$ and $d_{H'}$.

Let us show that $\pi$ is continuous. Let $x$ be in $H$ and $\epsilon > 0$. Without loss of generality, assume that $\epsilon < 3^{-1} diam_G(\pi(x))$. There is an open set $U'$ in $G'$ containing $\pi(x)$ such that, for all $x'$ in $U'$, $diam_G(x') < \epsilon$ or $d_G(\pi^{-1}\{x'\}, \pi^{-1}\{\pi(x)\}) < \epsilon$. As the topology on $H'$ from the metric $d_{H'}$ is finer than the relative topology from $G'$, we may find $\epsilon/2 > \delta > 0$ such that $H'(\pi(x), \delta) \subseteq U'$. Now suppose that $x'$ is in $H'(\pi(x), \delta)$. It follows that $d_G(\pi^{-1}\{x'\}, \pi^{-1}\{\pi(x)\}) < \delta < \epsilon/2$, implying that we may find $y$ in $H$ with $\pi(y) = x'$ and $d_G(y, x) < \delta$. It follows that $d_H(x, y) < \epsilon$ and this completes the proof.

The last thing for us to check is that $\pi$ is proper. Let $K' \subseteq H'$ be any subset which is compact in $d_{H'}$. Then it is compact in the topology of $H$ as well and hence $K = \pi^{-1}(K')$ is compact in $G$. Now let $x_k, k \geq 1$ be any sequence in $K$. It follows that there is a subsequence which is converging in $d_G$. There is a further subsequence such that $\pi(x_k)$ is converging in $d_{H'}$. This subsequence converges from the definition of $d_H$. □

**Theorem 7.10.** *Suppose that $\pi : G \to G'$ is regular.*

1. *The groupoids $G'$ and $H'$ satisfy the hypothesis of Theorem 5.4.*
2. *The groupoids $G$ and $H$ satisfy the hypothesis of Theorem 5.4.*
3. *For every $u$ in $H^0$, the map $\pi : H^u \to (H')^{\pi(u)}$ is a homeomorphism.*

*Proof.* We begin with $G'$ and $H'$. We have already seen $G'$-invariance as a consequence of Lemma 7.1. The second condition is the conclusion of Proposition 7.8.

Let us now fix $u'$ in $H'$ and verify that the relative topology from $G'$ on $(G')^{u'} = (H')^{u'}$ is finer than the topology of $d_{H'}$. Fix $s'$ in $H'$ with $r(s') = u'$ and let $\epsilon > 0$. We want to find an open set $U$ in $(G')^{u'}$ such that $U \subseteq H'(s', \epsilon)$. It is a fairly easy result in topology that there exists an open set $V$ containing $s'$ such that, for every $t'$ in $V$, $\pi^{-1}\{t'\} \subseteq G(\pi^{-1}\{s'\}, \epsilon)$. The collection of sets $G(s, \epsilon/2), s \in \pi^{-1}\{s'\}$ forms an open cover of $\pi^{-1}\{s'\}$. As $\pi$ is proper, we extract a finite subcover corresponding to points $s_1, \ldots, s_K$ in $\pi^{-1}\{s'\}$. As $\pi$ is assumed to be a homeomorphism when restricted to each $G^{r(s_i)}$, each of the sets $\pi(G(s_k, \epsilon/2) \cap G^{r(s_k)})$ is an open subset of $(G')^{u'}$. We define

$$U = V \cap \left( \cap_{k=1}^{K} \pi(G(s_k, \epsilon/2) \cap G^{r(s_k)}) \right),$$

which is an open set in $(G')^{u'}$ containing $s'$. We claim that $U$ is contained in the $d_{H'}$-ball of radius $\epsilon$ around $s'$. Let $t'$ be in $U$. As $U \subseteq V$, we have $\pi^{-1}\{t'\} \subseteq G(\pi^{-1}\{s'\}, \epsilon)$. For the other inclusion, for any $s$ in $\pi^{-1}\{s'\}$, we know that $s$ is in $G(s_k, \epsilon/2)$, for some $k$. We also know that $t'$ is in $\pi(G(s_k, \epsilon/2) \cap G^{r(s_k)})$, so we may find $t$ in $G(s_k, \epsilon/2)$ with $\pi(t) = t'$. From the triangle inequality, we have $d_G(t, s) < \epsilon$.

Finally, to check that fourth property, we must verify that $r : H' \to (H')^0$ is open in $d_{H'}$. Let $x'$ be in $H'$ and $\epsilon_0 > 0$. As $G'$ is locally compact, we may find an open set $x' \in U_0 \subseteq G'$ with compact closure. It follows that $\pi^{-1}(\overline{U_0})$ is also compact and contains $\pi^{-1}\{x'\}$. As $r_G$ is uniformly continuous on the compact set $\pi^{-1}(\overline{U_0})$, we may find $\epsilon_0 > \epsilon > 0$ such that for any $y, z \in \pi^{-1}(\overline{U_0})$ with $d_G(y, z) < \epsilon$, it follows that $d_G(r_G(y), r_G(z)) < 3^{-1} diam_G(\pi^{-1}\{r_{G'}(x')\})$. Now, we may use our hypothesis of regularity to find an open set $x' \in U \subseteq U_0$ such that, for all $y'$ in $U$, we have either $d_G(\pi^{-1}\{y'\}, \pi^{-1}\{x'\}) < \epsilon$ or $diam_G(\pi^{-1}\{y'\}) < \epsilon$. We use the fact that $r_{G'}$ is an open map to find an open set $r_{G'}(x') \in V \cap (G')^u \subseteq r_{G'}(U)$. Finally, we choose $0 < \delta < 3^{-1} diam_G(\pi^{-1}\{r_{G'}(x')\})$ such that $G(\pi^{-1}\{r_{G'}(x')\}, \delta) \subseteq \pi^{-1}(V)$. We claim that

$$H'(r_{G'}(x'), \delta) \cap (H')^u \subseteq r_{G'}(H'(x', \epsilon)) \subseteq r_{G'}(H'(x', \epsilon_0)).$$

Let $u'$ be in the leftmost set. From our choice of $\delta$, $\pi^{-1}\{u'\}$ is contained in $\pi^{-1}(V)$, so $u'$ is in $V \cap (G')^u$, which, in turn, is contained in $r_{G'}(U)$. Hence, we know that $u' = r_{G'}(y')$, with $y'$ in $U$. It follows from the choice of $U$ that either $d_G(\pi^{-1}\{y'\}, \pi^{-1}\{x'\}) < \epsilon$ or $diam_G(\pi^{-1}\{y'\}) < \epsilon$. In the former case, it follows from the definition of $d_{H'}$ that $y'$ is in $H'(x', \epsilon)$ and we are done. In the latter case, as $y'$ is in $U \subseteq U_0$, we have $diam_G(r_G(\pi^{-1}\{y'\})) < 3^{-1} diam_G(\pi^{-1}\{r_{G'}(x')\})$. As $r_G(\pi^{-1}\{y'\}) =$

$\pi^{-1}\{r_{G'}(y')\} = \pi^{-1}\{u'\}$, this contradicts the hypothesis that

$$d_{H'}(u', r_{G'}(x')) < \delta < 3^{-1} diam_G(\pi^{-1}\{r_{G'}(x')\}).$$

We turn to $G$ and $H$. The first two conditions follow from Lemma 7.1 and Proposition 7.8. For the third part, let $\mathcal{S}$ denote the usual topology on $G'$. Fix a unit $u$ in $H^0$. We know that the relative topology $\mathcal{S}|_{H'}$ and the metric topology from $d_{H'}$ agree on $(H')^{\pi(u)}$. On the other hand, $\pi$ induces a homeomorphism between $H^u$, with the metric topology from $d_G$ and $(H')^{\pi(u)}$ with the relative topology $\mathcal{S}|_{H'}$. Together, these imply that $\pi$ is a homeomorphism between the metric spaces $(H^u, d_G)$ and $((H')^{\pi(u)}, d_{H'})$. It follows immediately that the metrics $d_G$ and $d_H$ induce the same topology on $H^u$. At the same time, we see that $\pi : (H^u, d_H) \to ((H')^{\pi(u)}, d_{H'})$ is a homeomorphism.

Finally, to check the last condition of Theorem 5.4, we must see that $r : H \to H^0$ is open. If this fails, then there is an $x$ in $G$, an $\epsilon > 0$ and a sequence $u_k, k \geq 1$ in $H^0$ converging to $r(x)$ in $d_H$, while $u_k \notin H(x, \epsilon), k \geq 1$. By continuity, $\pi(u_k), k \geq 1$ converges to $\pi(r(x))$ in $d_{H'}$. We know that $r_{G'} : H' \to (H')^0$, so we may find $y'_k, k \geq 1$ in $H$ with $r_G(y'_k) = \pi(u_k), k \geq 1$ converging to $\pi(x)$. From our standing hypotheses and Lemma 7.3, for every $k \geq 1$, we may find $y_k$ in $H$ with $\pi(y_k) = y'_k, k \geq 1$ and $r_G(y_k) = u_k, k \geq 1$. The sequence $y_k, k \geq 1$ lies in $\pi^{-1}\{y'_k, \pi(x) \mid k \geq 1\}$, which is compact since $\pi$ is proper (from $d_H$ to $d_{H'}$). By passing to a subsequence, we may assume $y_k, k \geq 1$ converging to some $y$. On one hand, we have

$$\pi(y) = \lim_k \pi(y_k) = \lim_k y'_k = \pi(x)$$

and on the other,

$$r_G(y) = \lim_k r_G(y_k) = \lim_k u_k = r_G(x).$$

From our standing hypotheses, this implies that $x = y$. So for some $k$ sufficiently large, $y_k$ is in $H(x, \epsilon)$ and $u_k = r_G(y_k)$, a contradiction. This completes the proof. □

Our next task is to find a sequence of approximants to $G$, which we will use in constructing our subalgebra $\mathcal{A}$ within $C_c(G)$. For each $n \geq 1$, we define an equivalence relation, $\sim_n$, on $G$ as follows. For $x$ in $G$, we set

$$[x]_n = \begin{cases} \{x\}, & diam_G(\pi^{-1}\{\pi(x)\}) > n^{-1} \\ \pi^{-1}\{x\}, & diam_G(\pi^{-1}\{\pi(x)\}) \leq n^{-1} \end{cases}$$

Observe that each $\sim_n$-equivalence class is compact. We let $G_n$ be the quotient space $G/\sim_n$ which we equip with the quotient topology. We let $q_n : G \to G_n$ denote quotient map and since $[x]_n \supseteq [x]_{n+1}$, we let $p_n : G_n \to G_{n-1}$ be the obvious quotient map, for $n \geq 2$ and $q'_n : G_n \to G'$ be the map sending $[x]_n$ to $\pi(x)$. All of these maps are clearly continuous.

It will also be convenient for us to define

$$H'_n = \{x \in G' \mid diam_G(\pi^{-1}\{x\}) > n^{-1}\} \subseteq H',$$

and

$$H_n = \pi^{-1}(H'_n) \subseteq H,$$

for all $n \geq 1$.

**Lemma 7.11.** (1) *Each space $G_n$ is locally compact.*
(2) *Each space $G_n$ is Hausdorff.*
(3) *The space $G$ is the inverse limit of*

$$G_1 \xleftarrow{p_2} G_2 \xleftarrow{p_3} \cdots$$

(4) *Each set $H_n$ is open in $H$.*
(5) *The closure of $H_n$ in $G$, is contained in $H$.*
(6) *$H_1 \subseteq H_2 \subseteq \cdots$ and the union is $H$.*

*Proof.* The first part follows easily from the following observation: if $U' \subseteq G'$ is open with compact closure, then by the continuity of $q'_n$, $(q'_n)^{-1}(U')$ is open and is contained in $q_n(\pi^{-1}(\overline{U'}))$, which is compact.

The second part follows quite easily from the following fact: if $U \subseteq G$ is an open set, then

$$U_n = \{[z]_n \mid [z]_n \subseteq U\}$$

is open in $G_n$. To prove this, we need to show that

$$(q_n)^{-1}(U_n) = \{z \in G \mid [z]_n \subseteq U\}$$

is open in $G$.

Let $z$ be in $(q_n)^{-1}(U_n)$. As $[z]_n$ is compact, we may find $\epsilon > 0$ such that

$$G([z]_n, \epsilon) \subseteq U.$$

We consider two cases. First, suppose that $[z]_n = \pi^{-1}\{\pi(z)\}$. We appeal to Lemma 7.5 to find an open set $\pi(z) \in U' \subseteq G'$ such that

$$\pi^{-1}(U') \subseteq G(\pi^{-1}\{\pi(z)\}, \epsilon) \subseteq U.$$

Then $z \in \pi^{-1}(U')$ is open in $G$ and, if $x$ is in $\pi^{-1}(U')$, then we have

$$[x]_n \subseteq \pi^{-1}\{\pi(x)\} \subseteq G(\pi^{-1}\{\pi(z)\}, \epsilon) \subseteq U$$

so $\pi^{-1}(U') \subseteq (q_n)^{-1}(U_n)$.

The second case is $[z]_n = \{z\}$. It follows from the definition of $[z]_n$ that $diam_G(\pi^{-1}\{\pi(z)\}) > n^{-1}$. Now, we find an open set $\pi(z) \in U' \subseteq G'$ satisfying the regularity condition at $\pi(z)$ for

$$\epsilon' < 2^{-1}(diam_G(\pi^{-1}\{\pi(z)\}) - n^{-1}).$$

We also require that $\epsilon' < 2^{-1}\epsilon$. Then $z \in \pi^{-1}(U') \cap G(z, \epsilon')$, which is open in $G$. We claim this set is contained in $(q_n)^{-1}(U_n)$. Suppose $x$ is in $G(z, \epsilon')$ with $\pi(x)$ in $U'$. We need to show that $[x]_n \subseteq U$. The case $[x]_n = \{x\}$ is easy as $x$ is in $G(z, \epsilon') \subseteq G(z, \epsilon) \subseteq U$. In the case $[x]_n = \pi^{-1}\{\pi(x)\}$, we then know that $diam_G(\pi^{-1}\{\pi(x)\}) \leq n^{-1}$. From this, we see that

$$\begin{aligned} d_G(\pi^{-1}\{\pi(x)\}, \pi^{-1}\{\pi(z)\}) &\geq 2^{-1}\left(diam_G(\pi^{-1}\{\pi(z)\})\right. \\ &\qquad \left.-diam_G(\pi^{-1}\{\pi(x)\})\right) \\ &> \epsilon'. \end{aligned}$$

It follows from the regularity condition and the fact that $\pi(x)$ is in $U'$ that $diam_G(\pi^{-1}\{\pi(x)\}) \leq \epsilon'$. Then we have

$$d_G(\pi^{-1}\{\pi(x)\}, z) \leq diam_G(\pi^{-1}\{\pi(x)\}) + d_G(x, z) < 2\epsilon' < \epsilon.$$

The desired conclusion follows.

We now prove that $H_n$ is open in $H$. In fact, this follows if we show that $H'_n$ is open in $H'$. Let $x$ be in $H'_n$. Choose $\epsilon > 0$ so that $diam_G(\pi^{-1}\{x\}) > n^{-1} + 3\epsilon$. If $y$ is in $H'$ with $d_h(x, y) < \epsilon$, then $d_G(\pi^{-1}\{x\}, \pi^{-1}\{y\}) < \epsilon$ and it follows that

$$diam_G(\pi^{-1}\{y\}) \geq diam_G(\pi^{-1}\{x\}) - 2\epsilon > n^{-1}.$$

Hence $y$ is in $H'_n$.

Now suppose that $x$ in $G$ is in the closure of $H_n$. We will show that $\pi(x)$ is in $H'$. Every neighbourhood of $\pi(x)$ contains a point $x'$ with $diam_G(\pi^{-1}\{x'\}) > n^{-1}$. On the other hand, if $\pi(x)$ is not in $H'$, then $\pi^{-1}\{x\}$ is a single point. In this case, Lemma 7.5 implies that there is an open set $U'$ containing $\pi(x)$ such that $diam(\pi^{-1}(U'))$ is arbitrarily small. This would be a contradiction.

The last statement is obvious. □

We now begin the task of establishing the conditions needed to apply Theorem 3.4. We will use $B = C_r^*(G)$ and $A = C_r^*(H)$. We will define a Hilbert space with representations of these $C^*$-algebras and a bounded $*$-derivation $\delta$ as in Theorem 3.2. We will show these satisfy the hypotheses on Theorem 3.4 and that, under some conditions, $C_r^*(G) \cap \ker(\delta) = C_r^*(G'), C_r^*(H) \cap \ker(\delta) = C_r^*(H')$. The first part is to find a Hilbert space with actions of both $C_r^*(G)$ and $C_r^*(H)$. We assume throughout that $\pi$ satisfies the standing hypothesis and is regular.

We define

$$G \times_\pi G = \{(x, y) \in G \times G \mid \pi(x) = \pi(y)\},$$

the fibred product of $G$ with itself over $\pi$. This is actually a groupoid in an obvious fashion. We will not need that fact, but it does influence our notation. It receives the relative topology from the product. We denote the map sending $(x, y)$ in $G \times_\pi G$ to $\pi(x) = \pi(y)$ in $G'$ by $\pi$. For $u, v$ in $G^0$ with $\pi(u) = \pi(v)$, we let $(G \times_\pi G)^{(u,v)} = \{(x, y) \in G^u \times G^v \mid \pi(x) = \pi(y)\}$. There is an analogous definition for $(G \times_\pi G)_{(u,v)}$. It follows from the standing hypothesis that $\pi|_{(G\times_\pi G)^{(u,v)}}$ is a homeomorphism to $(G')^{\pi(u)}$. We let $\nu^{(u,v)}$ be the measure $\nu^{\pi(u)}$ pulled back by this map. There is an analogous definition of $\nu_{(u,v)}$. We represent $C_c(G)$ on the Hilbert space $L^2((G \times_\pi G)_{(u,v)}, \nu_{(u,v)})$ by

$$(b\xi)(x, y) = \int_{(r(w),r(z))=(r(x),r(y))} b(w)\xi(w^{-1}x, z^{-1}y)d\nu^{(r(x),r(y))}(w, z),$$

for $b$ in $C_c(G)$, $\xi$ in $L^2((G \times_\pi G)^{(u,v)}, \nu^{(u,v)})$ and $(x, y)$ in $(G \times_\pi G)_{(u,v)}$ It is obvious that this is unitarily equivalent to the left regular representation of $C_c(G)$ on $L^2(G_u, \nu_u)$ and hence extends to all of $C_r^*(G)$. Let us also remark at this point that if $x, y, w, z$ are in $G$ with $r(w) = r(x), r(y) = r(z), \pi(w) = \pi(z), \pi(x) = \pi(y)$, then
$\sigma(w, w^{-1}x) = \sigma'(\pi(w), \pi(w)^{-1}\pi(x)) = \sigma(z, z^{-1}y)$ as a consequence of the last part of our standard hypothesis.

We define our Hilbert space

$$\mathcal{H} = \oplus_{(u,v)\in G^0\times_\pi G^0} L^2((G \times_\pi G)_{(u,v)}, \nu_{(u,v)}).$$

We also observe that all of the preceding discussion applies equally well to the groupoid $H$, provided $u, v$ are in $H$, which is implied by the condition $u \neq v$. If $u, v$ are not in $H$, we simply represent $C_c(H)$ on $L^2((G \times_\pi G)_{(u,v)}, \nu_{(u,v)})$ as the zero representation.

Observe that the operator $(F\xi)(x, y) = \xi(y, x)$ is a unitary from $L^2((G \times_\pi G)_{(u,v)}, \nu_{(u,v)})$ to $L^2((G \times_\pi G)_{(v,u)}, \nu_{(v,u)})$. In the case $u = v$, it is the identity. We denote its extension to $\mathcal{H}$ by $F$ also. Observe that $F^2 = I, F = F^*$. We we define $\delta(a) = i[a, F] = i(aF - Fa)$, for all $a$ in $C_r^*(G) + C_r^*(H)$.

It is probably worth recording the following fact. Its proof is trivial and we omit it.

**Lemma 7.12.** *Let $(u, v)$ be in $G \times_\pi G$, $\xi$ be in $L^2((G \times_\pi G)_{(v,u)}, \nu_{(v,u)})$ and $b$ be in either $C_c(G)$ or $C_c(H)$. We have*

$$\delta(b)\xi(x, y) = i \int (b(w) - b(z))\, \xi(z^{-1}y, w^{-1}x)\sigma(w, w^{-1}x)d\nu^{(r(x),r(y))}(w, z),$$

*for* $(x, y)$ *in* $(G \times_\pi G)_{(v,u)}$*, where the integral is over* $(w, z) \in (G \times_\pi G)^{(r(x),r(y))}$.

**Lemma 7.13.** *If* $b$ *is in* $C_c(G_n) \subseteq C_c(G)$*, for some* $n \geq 1$*, then* $\delta(b) \in \delta(C_c(H))$.

*Proof.* We regard $b$ as a function on $G$, constant on the $\sim_n$-equivalence classes. Let $K$ be a compact set in $G$ such that $b$ is zero off $K$ and define

$$X = \{x \in H \cap \pi^{-1}(\pi(K)) \mid diam_G \pi^{-1}\{\pi(x)\} \geq n^{-1}\}.$$

First, we claim that $X$ is compact in $H$. If $x_k, k \geq 1$ is any sequence in $X$, then, as $K$ is compact and also $\pi^{-1}(\pi(K))$, it has a subsequence which converges to some $x$ in $K$ in the metric $d_G$. It also follows that $diam_G \pi^{-1}\{\pi(x_k)\} \geq n^{-1}$, for every $k$ and so $diam_G \pi^{-1}\{\pi(x)\} \geq n^{-1}$, as well. This implies that $x$ is in $H$. It also follows from the fact that $\pi$ is regular that $\pi^{-1}\{\pi(x_k)\}$ converges to $\pi^{-1}\{\pi(x)\}$ and hence $x_k, k \geq 1$ converges to $x$ in $d_H$.

Let $U'$ be an open set in $G'$, which contains $\pi(X)$ and its closure is compact. We may find $e : H' \to [0, 1]$ in $C_c(H)$ such that $e(x') = 1$, for $x' \in \pi(X)$ and $e(x') = 0$, for $x' \notin U'$. Then for any $x$ in $H$, we define $a(x) = b(x)e(\pi(x))$. This function clearly has compact support. To see it is continuous, it suffices to observe that it is non-zero only on $\pi^{-1}(\overline{U'})$, which is compact in $H$ and as the inclusion map is continuous, it is also compact in $G$ and the two relative topologies agree there.

It remains to prove $\delta(b) = \delta(a)$. From the formula provided by Lemma 7.12, it suffices for us to prove that $b(z) - b(w) = a(z) - a(w)$, for all $(w, z)$ in $G \times_\pi G$ (where we interpret $b$ to be zero off of $H$). If $w, z$ are not in $H$, then $w = z$ and the conclusion holds. Next, let us suppose that $diam_G \pi^{-1}\{\pi(w)\} \leq n^{-1}$. As $b$ is in $C_c(G_n)$, it follows that $b(w) = b(z)$ and then $a(w) = b(w)e(\pi(w)) = b(z)e(\pi(z)) = a(z)$ and the conclusion holds. Now, let us assume that $diam_G \pi^{-1}\{\pi(w)\} > n^{-1}$. If $w$ is not in $\pi^{-1}(\pi(K))$, then $w$ is not in $K$ so $b(w) = 0$. But this also means that $z$ is not in $\pi^{-1}(\pi(K))$, so $b(z) = 0$, as well. Again, we have $a(w) = a(z) = 0$. Finally, we are left with the case that $w$ and $z$ are both in $X$. It follows that

$$\begin{aligned} a(z) - a(w) &= b(z)e(\pi(z)) - b(w)e(\pi(w)) \\ &= b(z) \cdot 1 - b(w) \cdot 1 \\ &= b(z) - b(w), \end{aligned}$$

as $e(x') = 1$, for $x'$ in $\pi(X)$. $\square$

The hypothesis of the next result is not particularly strong. We know that the space $H'$ has been given a topology in which the fibres of the map $\pi$ vary continuously. This hypothesis ensures the existence on measures on these fibres, also varying continuous.

**Proposition 7.14.** *We say that $\pi$ is* measure regular *if there is a continuous function $\mu : (H')^0 \to M(H^0)$, the set of Borel probability measures on $H^0$ with the weak-* topology such that*

(1) *for any $u'$ in $(H')^0$, the support of $\mu(u')$ is contained in $\pi^{-1}\{u'\}$,*
(2) *for any $x'$ in $H'$ and Borel subset $\phi$ of $\pi^{-1}\{x'\}$,*

$$\mu(r_{H'}(x'))(r_H(\phi)) = \mu(s_{H'}(x'))(s_H(\phi)),$$

*In this case, the $*$-algebra $C_c(H)$ satisfies condition C1 of Theorem 3.4. Moreover, we have $\delta(C_r^*(G,\sigma)) \subseteq \delta(C_r^*(H,\sigma))$.*

*Proof.* For simplicity, we will ignore the cocycle. Let $a$ be any element of $C_c(H)$ and define

$$a'(x') = \int_{\pi^{-1}\{r_{H'}(x')\}} a(x) d\mu(r_{H'}(x'))(x),$$

for any $x'$ in $H'$.

It is clearly in $C_c(H')$. Fix a pair of units in $H^0$, $(u,v)$ with $\pi(u) = \pi(v)$. Let $W_v$ denote the canonical unitary between $L^2(G_u, \nu_u)$ and $L^2((G \times_\pi G)_{(u,v)}, \nu_{(u,v)}$ induced by the projection onto the first factor.

It follows from Lemma 7.12 that

$$\int_v W_v^*(a - FaF)W_v d\mu(u)(v) = a - a'.$$

The conclusion then follows from the fact that $\mu(u)$ is a probability measure and

$$\|a - FaF\|_r = \|\delta(a)\|_r.$$

For the last statement, we know that $\delta(C_c(G))$ is contained in $\delta(C_c(H))$. As $\delta$ is continuous, $\delta(C_r^*(G,\sigma))$ is contained in the closure of $\delta(C_c(H))$ which is $\delta(C_r^*(H,\sigma))$ as a consequence of C1. □

**Proposition 7.15.** *If $\pi$ is regular and there is a continuous morphism of groupoids $\mu : H' \to H$ such that $\pi \circ \mu = id_{H'}$, then $\pi$ is measure regular.*

*Proof.* We define a function, also denoted by $\mu$, from $(H')^0$ to $M(H^0)$ by setting $\mu(u')$ to be point mass at $\mu(u')$, for $u'$ in $(H')^0$. The hypotheses of Proposition 7.14 are obviously satisfied. □

A main case of interest is when these fibres are actually finite. One should consider the hypotheses in this case to be analogous to those of the the condition of finite index 6.11 in the subgroupoid case.

**Proposition 7.16.** *If $\pi : G \to G'$ is regular and there is a positive integer $N \geq 2$ such that, for each $x'$ in $H'$, $\#\pi^{-1}\{x'\} = N$, then $\pi$ is measure regular.*

*Proof.* We define a function, also denoted by $\mu$, from $(H')^0$ to $M(H^0)$ by setting

$$\mu(u') = N^{-1} \sum_{u \in \pi^{-1}\{u'\}} \mu_u,$$

where $\mu_u$ denotes point mass at $u$, for $u'$ in $(H')^0$. The hypotheses of Proposition 7.14 are obviously satisfied. □

**Lemma 7.17.** *Suppose that $\pi : G \to G'$ is regular. The $*$-algebra $C_c(H)$ satisfies condition C2 of Theorem 3.4.*

*Proof.* Let $a_1, \ldots, a_I$ be in $C_c(H)$. We may find a compact set $K \subseteq H$ such that all are zero off of $K$. By the last two parts of Lemma 7.11 that there is some $n$ with $K \subseteq H_n$. We may apply the Tietze extension Theorem to each function, $a_i$, restricted to the closure of $H_n$ in $G$ to find $b_i$ in $C_c(G_n)$ such that $b_i|_{H_n} = a_i|_{H_n}$. We also regard these functions as being in $C_c(G)$ as well.

As $H_n$ is open, $r(H_n) \cup s(H_n)$ is open in $H^0$ and contains $r(K) \cup s(K)$ so we may find $e$, a function in $C_c(H^0)$, which is identically 1 on $r(K) \cup s(K)$. and zero outside $r(H_n) \cup s(H_n)$. It is now routine to check these satisfy the properties in C2. □

**Corollary 7.18.** *If $\pi : G \to G'$ is a factor map satisfying the standing hypotheses and is regular and measure regular, then*

$$K_*(\ker(\delta) \cap C_r^*(G, \sigma); C_r^*(G, \sigma)) \cong K_*(C_r^*(H', \sigma); C_r^*(H, \sigma)).$$

We would now like to replace $\ker(\delta) \cap C_r^*(G, \sigma)$ with $C_r^*(G', \sigma)$.

**Theorem 7.19.** *Assume that $\pi$ is both regular and measure regular. If the closure of the sets $H_n \subseteq H, n \geq 1$ in $G$, denoted $Cl(H_n)$, satisfy the $C$-extension property, for some $C \geq 1$, then we have*

$$C_r^*(G, \sigma) \cap \ker(\delta) = C_r^*(G', \sigma).$$

*Proof.* The containment $\supseteq$ is clear from Lemma 7.12 and the fact that $\delta$ is continuous. For the converse, if $b$ is in $C_r^*(G, \sigma)$ and $\delta(b) = 0$, then we may find a sequence $b_n, n \geq 1$ in $C_c(G)$ converging to $b$. In view of part 3 of Lemma 7.11 (and after doing some re-indexing), we may assume that $b_n$ is in $G_n$, for all $n$. it follows from Lemma 7.13 that there exist $a_n, n \geq 1$ in $C_c(H)$ with $\delta(b_n) = \delta(a_n)$. By Proposition 7.14, we know that condition C1 of Theorem 3.4 holds. By adding an element of $C_c(H')$, we may assume that

$$\|a_n\|_r \leq K\|\delta(a_n)\|_r = K\|\delta(b_n)\|_r.$$

First, the condition that $\delta(b_n) = \delta(a_n)$ implies that $b_n|_H = a_n$. Let $K_n$ be a compact subset of $H$ containing the support of $a_n$. The sets $H_k, k \geq 1$, are an open cover of $H$, so we may find $k_n \geq n$ such that $H_{k_n}$ contains $K_n \cup s(K_n)$. In addition, the function sending $x$ in $H$ to $diam_G(\pi^{-1}\{\pi(x)\})$ is continuous and positive, hence it is bounded below on any compact set. It follows then that we may also choose $k_n$ so that $diam_G(\pi^{-1}\{\pi(x)\}) > k_n^{-1}$, for all $x$ in $K_n \cup s(K_n)$.

We use the extension property for $Cl(H_{k_n})$ to find $c$ in $C_c(G)$ such that $c_n|_{H_{k_n}} = a_n|_{k_n}$ and $\|c_n\|_r \leq C\|a_n\|_r$.

As $b_n - c_n$ is in $C_c(G - G_{k_n})$, we may find a compact set $L_n \subseteq H - H_{k_n}$ such that the support of $\delta(b_n - c_n)$ is in $L_n$. We claim that $\pi(s(L_n))$ is disjoint from $\pi(s(K_n))$. Recall that $k_n$ was chosen so that $diam_G(\pi^{-1}\{\pi(x)\}) > k_n^{-1}$, for all $x$ in $s(K_n)$. If $x$ is in $L_n$, then $diam_G(\pi^{-1}\{\pi(x)\}) \leq l_n^{-1} < k_n^{-1}$. The fact that $s$ is a contraction implies that the same conclusion holds for $s(x)$. This establishes the claim.

We let $h_n : (G')^0 \to [0,1]$ be continuous, compactly supported function which is identically one on $\pi(s(K_n))$ and identically zero on $\pi(s(L_n))$. We next claim that $b_n - c_n h_n$ is in $C_c(G')$. It suffices to check that

$$0 = \delta(b_n - c_n h_n) = \delta(b_n) - \delta(c_n)h_n,$$

as $h_n$ is in $C_c((G')^0)$. The function on the right is clearly supported in $K_n \cup L_n$. As $h_n = 1$ on $\pi(s(K_n))$, we have

$$(\delta(b_n) - \delta(c_n)h_n)\,|_{K_n} = \delta(b_n)|_{K_n} - \delta(c_n)|_{K_n} = \delta(b_n - c_n)|_{K_n} = 0$$

as $K_n \subseteq H_{k_n}$ where $b_n$ and $c_n$ agree. On the other hand, we also know that $h_n$ is zero on $\pi(s(L_n))$ so

$$(\delta(b_n) - \delta(c_n)h_n)\,|_{L_n} = \delta(b_n)|_{L_n} - 0 = 0$$

since $L_n \subseteq H_{l_n} - H_{k_n}$.

Finally, we have

$$\|c_n h_n\|_r \leq \|c_n\|_r \leq C\|a_n\|_r \leq CK\|\delta(b_n)\|_r$$

which tends to zero. Hence $b_n - c_n h_n$ is in $C_c(G')$ and also converges to $b$. $\square$

We will finish by giving some special examples of factor maps where all of our hypotheses are satisfied.

We assume that $X$ is a locally compact metric space with an action by the locally compact metric group $\Gamma$ by homeomorphisms. We suppose that $Y$ is a $\Gamma$-semi-invariant set (in the sense of 5.8) and that $\Gamma_Y \backslash \Gamma$ is discrete. Let us also suppose that, for any $\epsilon > 0$, the set

$$\{\Gamma_Y \gamma \in \Gamma_Y \backslash \Gamma \mid diam_X(Y\gamma) > \epsilon\}.$$

is finite.

We define the space $X'$ as the quotient of $X$ which identifies each set $Y\gamma, \gamma \in \Gamma$ to a single point. We let $\pi$ denote the quotient map, $\pi : X \to X'$. From the hypotheses on the set $Y$ and on $\Gamma_Y$, $X'$ is locally compact and Hausdorff. There is an obvious action of $\Gamma$ on $X'$ by homeomorphisms and we have a factor map, which we also denote by $\pi$ from $G = X \rtimes \Gamma$ to $G' = X' \rtimes \Gamma$.

**Theorem 7.20.** *Let $X, \Gamma, Y$ be as above and satisfy the hypotheses there. Let $H, H'$ be the groupoids of of 7.6. The following hold.*

(1) *We have*

$$C_r^*(H') \cong C_0(\Gamma_Y \backslash \Gamma) \rtimes_r \Gamma$$

*and hence is Morita equivalent to $C_r^*(\Gamma_Y)$ while*

$$C_r^*(H) \cong C_0(\cup_{\Gamma_Y \gamma \in \Gamma_Y \backslash \Gamma} Y\gamma) \rtimes_r \Gamma$$

*and hence is Morita equivalent to $C_0(Y) \rtimes_r \Gamma_Y$.*

(2) *The factor map $\pi$ is regular 7.7.*

(3) *If the action of $\Gamma_Y$ on $Y$ admits an invariant probability measure, then $\pi$ is measure regular.*

(4) *If there is a short exact sequence*

$$0 \to C_0(X - Y) \rtimes_r \Gamma_Y \to C_0(X) \rtimes_r \Gamma_Y \to C_0(Y) \rtimes_r \Gamma_Y \to 0$$

*then each of the closed sets $Cl(H_n) \subseteq H$ has the $C$-extension property, for any $C \geq 1$.*

*Proof.* The descriptions of $H' \cong \Gamma_Y \backslash \Gamma \rtimes \Gamma$ and $H = C_0(Y \times \Gamma_Y \backslash \Gamma) \rtimes \Gamma$ are immediate from the definitions. The other parts of the first part follow from example 2.4 of [16].

To prove the factor map is regular, we use the sum metric

$$d_G((x, \gamma), (x', \gamma')) = d_X(x, x') + d_\Gamma(\gamma, \gamma')$$

where $d_X, d_\Gamma$ are metrics on $X$ and $\Gamma$, respectively. Let us verify regularity. Let $\epsilon > 0$ and $(\pi(Y\gamma_1), \gamma_2)$ in $H'$ be given. There are only finite many sets of the form $Y\gamma$ with diameter less than $\epsilon$, so we may choose $\pi(Y\gamma_1) \subseteq U$, an open set in $X'$ which is disjoint from the images of these other than $\pi(Y\gamma_1)$, which is a finite set in $X'$. Letting $U' = U \times \Gamma_Y \gamma_2$, there is only point in $U'$ with diameter of the pre-image greater than $\epsilon$, which is the single point $\pi(Y\gamma_1) \times \Gamma_Y \gamma_2$ and the desired conclusion is trivial.

Let $\mu$ be a $\Gamma_Y$-invariant measure on $Y$. It is a simple matter to check that

$$\mu_(Y\gamma)(E) = \mu(E\gamma^{-1}),$$

for $\gamma \in \Gamma, E \subseteq Y\gamma$, is a well-defined function from $(H')^0$ to $M(H^0)$ satisfying he conditions of Proposition 7.14, so $\pi$ is measure regular.

The proof of the last statement is very similar to that of Theorem 6.18 and we omit the details. □

**Corollary 7.21.** *Let $X, \Gamma, Y$ be as above and satisfy the hypotheses there. Assume that the action of $\Gamma_Y$ on $Y$ admits an invariant probability measure and that there is a short exact sequence*

$$0 \to C_0(X - Y) \rtimes_r \Gamma_Y \to C_0(X) \rtimes_r \Gamma_Y \to C_0(Y) \rtimes_r \Gamma_Y, \to 0$$

*then*

$$K_*(C_0(X') \rtimes_r \Gamma; C_0(X) \rtimes_r \Gamma) \cong K_*(C_r^*(\Gamma_Y); C_0(Y) \rtimes_r \Gamma_Y).$$

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