Spectral triples for subshifts

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Spectral triple, $(\mathcal{H}, A, D)$:

$$\begin{array}{cl} \mathcal{H} & \text{Hilbert space} \\ A \subseteq \mathcal{B}(\mathcal{H}) & \text{unital } *\text{-algebra} \\ D : \mathcal{H} \to \mathcal{H} & \text{unbounded self-adjoint} \\ (1 + D^2)^{-1/2} \in \mathcal{K}(\mathcal{H}) & \\ [D, a] \in \mathcal{B}(\mathcal{H}) & \text{for all } a \in A \end{array}$$

Prototype:

$$\begin{array}{ccl} \mathcal{H} & = & L^2(\mathbb{S}^1) \\ A & = & C^\infty(\mathbb{S}^1) \\ D(\xi) & = & (2\pi i)^{-1}\xi', \xi \in L^2(\mathbb{S}^1), \\ [D, f] & = & (2\pi i)^{-1} f'. \end{array}$$

Despite the obvious connections with differential topology/geometry, there has been extensive interest in the case $A = C(X)$ or $C(X) \times G$, where X is compact, metrizable, totally disconnected with no isolated points; i.e. a Cantor set. There are important connections with aperiodic order.

Connes, Pearson-Bellissard, Savinien, Kellendonk, Julien, Falconer.

Let $\mathcal{A}$ be a finite set and consider $\mathcal{A}^{\mathbb{Z}}$ (which we regard as sequences), which is a Cantor set. It is also a dynamical system

$$\sigma(x)_n = x_{n+1}, n \in \mathbb{Z}, x \in \mathcal{A}^{\mathbb{Z}}.$$

Definition 1. *A* subshift *is a non-empty subset* $X \subseteq \mathcal{A}^{\mathbb{Z}}$ *which is closed and satisfies* $\sigma(X) = X$*.*

We will fix a σ-invariant measure μ on X which we assume has full support.

Our Hilbert space is $\mathcal{H} = L^2(X, \mu)$.

For $n \geq 1$, a *word of length* n *in* X is a finite sequence $x_1 x_2 \cdots x_n$, $x \in \mathcal{A}^{\mathbb{Z}}$.

X_n denotes all words of length n.

If w is a word of length $n = 2k$, we define

$$U(w) = \{x \in X \mid x_{1-k} \cdots x_k = w\},$$

while for $n = 2k + 1$,

$$U(w) = \{x \in X \mid x_{-k} \cdots x_k = w\},$$

Define

$$C_n = span\{\chi_{U(w)} \mid w \in X_n\}.$$

so that

$$\mathbb{C} = C_0 \subseteq C_1 \subseteq C_2 \subseteq \cdots$$

are finite-dimensional subspaces of $\mathcal{H}$, or subalgebras of $C(X)$, with union $C_\infty(X)$, the locally constant functions, which are dense.

We follow Christensen-Ivan, who considered AF-algebras and defined:

$$D|(C_n \cap C_{n-1}^{\perp}) = \alpha_n, n \geq 1,$$

where $\alpha_n, n \geq 1$ is a sequence of non-negative reals tending to infinity. (Observe D is not just self-adjoint, but positive.)

The two main differences for us are first that our sequence of subspaces/algebras C_n is canonical from $X \subseteq \mathcal{A}^{\mathbb{Z}}$.

Secondly, we have
Lemma 2. *Letting* $u\xi = \xi \circ \sigma^{-1}$, $[D, u]$ *extends to a bounded operator if and only if* $\alpha_n - \alpha_{n-1}$ *is bounded.*

Define D_X using $\alpha_n = n$. So $(L^2(X, \mu), A, D_X)$ is a spectral triple for either $A = C_\infty(X)$ or $A = C_\infty(X) \times \mathbb{Z}$.

Summability

The function $n \to \#X_n$ is well-studied in dynamics; it is called the *complexity* of the subshift.

Theorem 3. *1. If $s > h(X,\sigma)$ (the entropy of (X,σ)) then $Tr(e^{-sD_X}) < \infty$.*

2. If $Tr(e^{-sD_X}) < \infty$, then $s \geq h(X,\sigma)$.

Theorem 4. *1. If there are constants M, s_0 such that $\#X_n \leq Mn^{s_0}$, for all $n \geq 1$, then for $s > s_0$, $Tr((D_X^2 + 1)^{-s/2}) < \infty$.*

2. If $Tr((D_X^2 + 1)^{-s/2}) < \infty$, then there is a constant M such that $\#X_n \leq Mn^s$, for all $n \geq 1$.

Proof: the dimension of the eigenspace of D_X for eigenvalue n is $\#X_n - \#X_{n-1}$.

The Connes metric

If $(\mathcal{H}, A, D)$ is a spectral triple, the formula

$$d(\phi, \psi) = \sup\{|\phi(a) - \psi(a)| \mid \|[D, a]\| \leq 1\}$$

may define a metric on the state space of A and its topology *may* coincide with the weak-* topology. If this occurs we call $(\mathcal{H}, A, D)$ a *quantum metric space* (Rieffel).

This now depends on the subshift in quite a subtle way.

Two quantities play key parts:

1. For x in X, if the set of n where $n = 2k+1$ with

$$U(x_{-k}\cdots x_k) \subsetneq U(x_{1-k}\cdots x_k)$$

or $n = 2k$ with

$$U(x_{1-k}\cdots x_k) \subsetneq U(x_{1-k}\cdots x_{k-1})$$

is sparse, then this helps the Connes metric.

2. If the numbers

$$\frac{\mu(U(x_{1-k}\cdots x_k))}{\mu(U(x_{-k}\cdots x_k))}, \frac{\mu(U(x_{1-k}\cdots x_{k-1}))}{\mu(U(x_{1-k}\cdots x_k))}$$

can be bounded, then this helps the Connes metric.

Shifts of finite type

A *shift of finite type* is a subshift of the following form: let $G = (G^0, G^1, i, t)$ be a finite directed graph.

$$X_G = \{x \in (G^1)^{\mathbb{Z}} \mid t(x_n) = i(x_{n+1}), n \in \mathbb{Z}\}.$$

Theorem 5. *If G is an irreducible, finite, directed graph (there is a path from any vertex to any other), then the Connes metric associated to $(L^2(X_G), C_\infty(X_G), D_{X_G})$ is infinite.*

Substitutions

An example is the Thue-Morse substitution: $0 \to 01, 1 \to 10$ can be iterated to produce infinite sequences:

$$0 \to 01 \to 0110 \to 01101001 \to \cdots$$

or, by substituting twice and making it symmetric about the middle,

$$0.1 \to 0110.1001 \to \cdots .$$

Call the limiting sequence x and let X be the closure of $\{\sigma^n(x) \mid n \in \mathbb{Z}\}$.

Theorem 6. *If X is a primitive substitution subshift (or more generally a linearly recurrent subshift), then $(\mathcal{H}, C_\infty(X), D_X)$ a quantum metric space. (Connes metric is finite and induces weak-* topology.)*

Sturmian subshifts

Begin with an irrational number $0 < \theta < 1$. Draw a line in the plane of slope θ which does not meet any point of $\mathbb{Z}^2$.

Put a 0 at a point on the line if its x-coordinate is an integer and a 1 when the y-coordinate is an integer. This produces a sequence in $\{0, 1\}^{\mathbb{Z}}$. The closure of all such sequences is X_θ, a Sturmian subshift.

Theorem 7. *Let* $a_1, a_2, \ldots$ *be the continued fraction expansion of* θ. *If there exist constants* M, s *such that* $a_n \leq M n^s$, *for all* $n \geq 1$, *then* $(L^2(X_\theta), C_\infty(X_\theta), D_{X_\theta})$ *is a quantum metric space.*

Corollary 8. *For almost all* θ *in* $[0,1]$, $(L^2(X_\theta), C_\infty(X_\theta), D_{X_\theta})$ *is a quantum metric space.*

Theorem 9. *There exists* θ *in* $[0,1]$ *such that Connes metric associated with* $(L^2(X_\theta), C_\infty(X_\theta), D_{X_\theta})$ *is infinite.*