The ordered K-theory of C^*-algebras associated with substitution tilings

Ian F. Putnam*,
Department of Mathematics and Statistics,
University of Victoria,
Victoria, B.C., Canada

May 2, 2000

Abstract

We consider the C^*-algebra, A_T, constructed from a substitution tiling system which is primitive, aperiodic and satisfies the finite pattern condition. Such a C^*-algebra has a unique trace. We show that this trace completely determines the order structure on the group $K_0(A_T)$; a non-zero element in $K_0(A_T)$ is positive if and only if its image under the map induced from the trace is positive.

1 Introduction and statement of the main result

We begin by introducing some of the terminology and notation. All of these things are developed more fully in the survey article [KP]. We have included other references to more original sources where appropriate.

A *substitution tiling system* in $\mathbb{R}^d$ consists of a finite collection of bounded, regular closed sets $p_1, \dots, p_N$ in $\mathbb{R}^d$ called *prototiles*. We also have a constant $\lambda > 1$ and, for each $i = 1, \dots, N$, $\omega(p_i)$ which is a finite collection of subsets of $\mathbb{R}^d$ with pairwise disjoint interiors; each is a translate of one of the prototiles

*Supported in part by a grant from NSERC, Canada

and their union is $\lambda p_i = \{\lambda x \mid x \in p_i\}$. In general, we call a translate of one of the prototiles a *tile*. Several one and two dimensional examples, including the Penrose tiles, are given in [AP]. As a generalization of the above, one can also have a finite set called *labels*. A *labelled prototile* is then a bounded, closed regular subset together with a label. (The idea being that we now have a way of distinguishing two prototiles which may be exactly the same geometric object.) It is clear how to extend the remainder of the definitions to this situation. All of our results apply equally well to the situation of labelled prototiles.

Collections of tiles with pairwise disjoint interiors are called *partial tilings*. The union of such a set of tiles is called the *support* of the partial tiling and a partial tiling whose support is $\mathbb{R}^d$ is called a *tiling*. If T denotes a tiling (or even a partial tiling), then for any x in $\mathbb{R}^d$, $T + x$ denotes the tiling (or partial tiling) obtained by translating all tiles in T by x.

Notice that we can extend our definition of ω to tiles by $\omega(p + x) = \omega(p) + \lambda x$, for any prototile p and vector x. We can further extend this definition to partial tilings by $\omega(T) = \{\omega(t) \mid t \in T\}$. This also means that we can iterate ω, and for any prototile p, we may construct $\omega^n(p)$, for any $n = 1, 2, \ldots$, which is a partial tiling with support $\lambda^n p$.

We will assume here that all of our prototiles contain the origin in their interior. (This loses no generality.) We define the *puncture* of any prototile p to be the origin and for any vector x, we define the puncture of $t = p + x$, denoted $x(t)$, to be x. So each tile has a distinguished point in its interior.

We say that the substitution is *primitive* if there is a positive integer k such that, for every ordered pair of prototiles p, p', a translate of p' appears inside $\omega^k(p)$.

We construct a tiling as follows. There exists a prototile p, a vector x, and a positive integer k so that so that the sequence of partial tilings $\omega^{nk}(p + x)$, $n = 1, 2, \ldots$ is coherent in the sense that the nth one contains all the earlier ones. Moreover, these grow to cover $\mathbb{R}^d$. We will not prove this here, although the proof is not difficult. We let T denote the union of these partial tilings which is a tiling. We look at all translations of T and put a metric d on this set as in [RW, Rud, So1]). The completion of this set of translations of T is denoted Ω. It is also worth noting that the elements of this completion can be viewed as tilings with the same tiles. This space is actually independent of the choice of T as above. From now on, we revert to using T to denote an arbitrary element of Ω. Under a hypothesis called the finite pattern condition, [RW], it is a compact metric space. The map

$$\omega : \Omega \to \Omega$$

then becomes a continuous surjection. We will focus our attention on the case when this map is also injective, and hence, a homeomorphism. This is usually referred to as the unique composition property or that the substitution is locally invertible. However, Solomyak [So2] has shown that, with the hypothesis of the finite pattern condition, this is equivalent to the set Ω containing no periodic tilings. That is, if T in Ω and x in $\mathbb{R}^d$ satisfy $T + x = T$, then $x = 0$. (In the terminology of [So2], Ω is the local isomorphism class of any of its elements.) Therefore, we will say that the substitution system is *aperiodic* if the map ω is injective. We will also assume that our substitution *forces its border* [Kel1]. As discussed in [KP], this loses no generality, provided we allow labelled tiles.

We define Ω_{punc} to be the set of all tilings in Ω which have a puncture on the origin. This set is compact and totally disconnected. We want to describe a base for its topology consisting of clopen sets. Fix some finite partial tiling P inside $\omega^k(p)$, where p is a prototile and k is a positive integer, and let t be any tile in P. We let

$$U(P, t) = \{T' \mid P - x(t) \subset T'\}.$$

That is, we translate the patch P back by vector $x(t)$, so that t now covers the origin, with its puncture exactly on the origin. Then we look at all tilings containing this patch. This set is closed and open in Ω_{punc} and such sets form a base for the topology.

We are interested in the equivalence relation on Ω_{punc} which is simply translational equivalence. That is, we define

$$R_{punc} = \{(T, T + x) \mid T, T + x \in \Omega_{punc}\}.$$

This set is also given a topology which is easiest to describe as follows. Let P be a patch as before and let t, t' be two tiles in P. The map sending T in $U(P, t)$ to $T + x(t) - x(t')$ is a homeomorphism onto $U(P, t')$. Its graph is contained in R_{punc} and is denoted $U(P, t, t')$. These sets form a base for the topology of R_{punc}. Indeed, they are actually clopen sets and R_{punc} is totally disconnected.

This makes R_{punc} into a locally compact, Hausdorff, σ-compact, r-discrete, principal groupoid with counting measure as a Haar system. (See [Ren] as a general reference on the subject of groupoids, or [Put2] for a leisurely treatment.) We use r, s to denote the range and source maps from R_{punc} to Ω_{punc}. That is, $r(T, T') = T$ and $s(T, T') = T'$.

We let A_T denote its C^*-algebra of R_{punc}. We refer the reader to [KP] or to [Ren] as the main source for the construction of C^*-algebras from groupoids. For general references to C^*-algebras, we suggest [Fi, Da, Pe]. (We should note that this really doesn't depend on T. The notational confusion comes because this is a special case of a more general construction [KP]. It would probably be preferable to use the notation A_ω, but we will stay with this for historical reasons.) This C^*-algebra is the completion in a certain norm of the *-algebra of continuous compactly supported functions on R_{punc}, denoted $C_c(R_{punc})$.

Let us mention some properties of this C^*-algebra. The key point is that the space Ω with map ω can be viewed as a *Smale space* [AP]. Then the space Ω_{punc} can be viewed as an abstract transversal to the relation of unstable equivalence. The reduction of this groupoid on Ω_{punc} is exactly R_{punc}. Hence A_T is strongly Morita equivalent to $U(\Omega, \omega)$ and the results of [PS] apply. In particular, the equivalence relation R_{punc} is minimal in the sense that every equivalence class is dense in Ω_{punc} and also amenable in the sense of Renault [Ren].

We are interested in the computation of the K-theory of A_T, and especially its K-zero group. We refer the reader to [Bl, W-O] as general references for K-theory for operator algebras and [Be1, Kel1] for further information and motivation for this problem in physics. Methods have been given in [AP, Be2, BCL, Kel1, Kel2] for the computation of the K-theory of A_T. In some cases, these included the order structure on K-zero. Here, we will prove a more general result.

The space Ω_{punc} possesses a natural measure μ. It is most easily described as follows. The mixing Smale space, (Ω, ω), has a measure of maximum entropy which is a product measure with respect to the canonical stable and unstable coordinates. The entropy is $d \log(\lambda)$. The set Ω_{punc} is contained in a finite collection of local stable sets and the measure μ is simply the restriction of the stable component of the measure of maximal entropy. Its key properties are that it is finite and R_{punc}-invariant. This means that it is preserved under the local homeomorphisms whose graphs make up our topology base above. This measure has full support. This is because the equivalence relation R_{punc} is minimal and, since μ is R_{punc}-invariant, its support is also. This measure is also unique. We will have more to say about this later.

This measure μ defines a trace, τ, on the C^*-algebra A_T. For an element f which lies in the dense sub-algebra $C_c(R_{punc})$, its trace is given by

$$\tau(f) = \int_{\Omega_{punc}} f(x,x)d\mu(x).$$

This is a positive bounded linear functional of norm one. It is also faithful since the measure has full support.

Such a trace induces a positive group homomorphism on the K-zero group of A_T [Bl, W-O]

$$\hat{\tau} : K_0(A_T) \to \mathbb{R}.$$

It is our goal here to show that, under a very mild hypothesis regarding the topology of the prototiles, this homomorphism completely determines the order structure on $K_0(A_T)$ [Bl, W-O].

Theorem 1.1. *Let $p_1, \ldots, p_N, \omega$ be a substitution tiling system (or labelled substitution tiling system) in $\mathbb{R}^d$ which is primitive, aperiodic and satisfies the finite pattern condition. Suppose that for each prototile, the capacity or box-counting dimension of its boundary is strictly less than d. Then the order on $K_0(A_T)$ is determined by the trace. That is, for any element a in $K_0(A_T)$, a is in $K_0(A_T)^+$ if and only if $a = 0$ or $\hat{\tau}(a) > 0$.*

Notice that the hypothesis regarding dimension is satisfied by any polyhedra, where the boundaries are made up of lower dimensional hypersurfaces.

Our proof will be presented in the last section. It will make use of a canonical C^*-subalgebra of A_T, denoted AF_T [Kel1, Kel2]. This sub-algebra is reasonably large inside A_T, but also has the advantage of being AF or *approximately finite dimensional*. (Again, AF_ω might be more appropriate notation.) The structure of this C^*-algebra is fairly well-understood. It is one of the AF-algebras constructed by Cuntz and Krieger from a mixing topological Markov chain, and the analogue of our main result above is known for such C^*-algebras. Our proof will make use of this. The rest of the argument is to show how we may interpolate between projections in A_T with projections in AF_T. The details appear in section 3.

Let us give a description of AF_T. For each prototile p and each positive integer n, let $Punc(p, n)$ denote the set of all punctures in the tiles of the partial tiling $\omega^n(p)$. Suppose that x is in some $Punc(p, n)$, and T is any tiling in Ω_{punc} such that p is in T. Recall that the puncture in p is the origin.

Then the tiling $\omega^n(T) - x$ is again in Ω_{punc}. We let $W(p, n, x)$ denote the set of all tilings of this form. It is not difficult to check that, for a fixed value of n, the collection of sets

$$\{W(p, n, x) \mid p \text{ a prototile, } x \in Punc(p, n)\},$$

is a partition of Ω_{punc} into clopen sets. Since we assume the substitution forces its border, then as n varies, these generate the topology on Ω_{punc}.

Suppose that p is a prototile and n is a positive integer. If we have x and y in $Punc(p, n)$, then the map sending T to $T + x - y$ is a homeomorphism from $W(p, n, x)$ onto $W(p, n, y)$. The graph of this map is denoted by $W(p, n, x, y)$. It is a clopen subset of R_{punc}. We define R_{AF} to be the union of these sets, which is then an open subgroupoid of R_{punc}. Then the C^*-algebra of R_{AF} is denoted by AF_T and the obvious inclusion of

$$C_c(R_{AF}) \subset C_c(R_{punc})$$

extends to an inclusion $AF_T \subset A_T$.

To see that AF_T is approximately finite dimensional, it suffices to notice that if we take A_n to be the linear span of the characteristic functions of the sets $W(p, n, x, y)$, where p is a prototile and x, y are in $Punc(p, n)$, then in fact, this is actually a finite dimensional C^*-subalgebra. The details are given in [KP]. It is also shown there that the matrix which describes the embedding of $A_n \subset A_{n+1}$ is the same for every n and is equal to the $N \times N$ matrix whose i, j entry is the number of different translates of p_i appearing in $\omega(p_j)$ for all i, j. Since the substitution is primitive, so is this matrix in the sense that some power has no zero entries [LM]. Our trace, τ, restricts to a trace on AF_T. By the results of [Ha], such a C^*-algebra has a unique trace. This implies the uniqueness of our R_{punc}-invariant measure μ since any other measure would give rise to another trace. These would be distinct on $C(\Omega_{punc})$ which is contained in AF_T.

While our main theorem gives a complete answer to the question of the order on $K_0(A_T)$, there is one important question which we leave unanswered. That is to compute the range of the map $\hat{\tau}$. There is a natural conjecture, namely

$$\begin{aligned} \hat{\tau}(K_0(A_T)) &= \hat{\tau}(K_0(AF_T)) \\ &= \{\mu(E) \mid E \subset \Omega_{punc} \text{ clopen } \} + \mathbb{Z}. \end{aligned}$$

One inclusion in the first equality is obvious. In some special situations in low dimensions ($d \leq 3$), equality is known [vE]. As well, our result in the next section, Theorem 2.1, suggests that this will be true under the same hypothesis as Theorem 1.1. Furthermore, the set $\hat{\tau}(K_0(AF_T))$ is known to be the subgroup of $\mathbb{R}$ generated by numbers of the form $\lambda^{-nd}\xi_i$, where n is a positive integer and ξ_i is an entry of the left Perron eigenvector of the primitive matrix of the last paragraph.

I would like to thank Johannes Kellendonk, Chris Bose, Florin Diacu and Rua Murray for helpful conversations.

2 A technical result

In this section we will prove a technical result which we will need in the proof of the main theorem. We include it as a separate section because it may be of some independent interest, as we will try to explain below.

As we discussed in the introduction, we have two equivalence relations (or principal groupoids), $R_{AF} \subset R_{punc}$, on the space Ω_{punc}. The second has a topology in which it is locally compact, Hausdorff, metrizable, r-discrete σ-compact and for which counting measure is a Haar system. The first is an open subgroupoid. Roughly speaking, the structure of the subgroupoid is fairly well-understood and the difficulty in analyzing the C^*-algebra A_T usually involves $R_{punc} - R_{AF}$. Our main technical result here is to show that, at the level of measure theory, the difference is negligible. Specifically, we will prove the following.

Theorem 2.1. *Let $p_1, \ldots, p_N, \omega$ be a substitution tiling system in $\mathbb{R}^d$ which is primitive, aperiodic and satisfies the finite pattern condition. Let R_{punc} and R_{AF} be the associated principal groupoids and let μ be the unique R_{punc}-invariant probability measure on Ω_{punc}. If the boundary of each p_i has capacity or box-counting dimension strictly less than d, then we have*

$$\mu(r(R_{punc} - R_{AF})) = 0.$$

The proof will be broken into a series of lemmas and we will introduce some new notation.

Recall that $Punc(p, n)$ denotes the set of all punctures in the tiles of $\omega^n(p)$. For any x in $Punc(p, n)$, we define $\partial(x)$ to be the Euclidean distance from x to the boundary of $\lambda^n p$, which is the support of $\omega^n(p)$; that is,

$$\partial(x) = \inf\{|x - y| \mid y \in \mathbb{R}^d - \lambda^n p\},$$

where || denotes the usual Euclidean norm on $\mathbb{R}^d$. We fix $b > 0$ so that $B(0, b) \subset p$, for all prototiles p. Here, $B(x, r)$ denotes the open ball in $\mathbb{R}^d$ centred at x and with radius r. Notice that this means for all tiles t, $B(x(t), b) \subset t$.

We let $\#X$ to denote the number of elements of any finite set X.

Lemma 2.2. *For each prototile p, there is a positive constant a_p such that*

$$\#Punc(p, n) \geq a_p \lambda^{dn}.$$

for all positive integers n.

Proof. We define the $N \times N$ substitution matrix B as follows. The i, j entry of B is the number of occurrences of translates of p_j in $\omega(p_i)$. The fact that the substitution is primitive is equivalent to the fact this non-negative matrix is primitive [LM]. Let v be the vector in $\mathbb{R}^d$ whose ith entry is the volume of p_i. It is easy to calculate that v is a (right) eigenvector of B with eigenvalue λ^d. Since this eigenvector is clearly positive, this is the Perron eigenvector for B and λ^d is the Perron eigenvalue. (See section 4.2 of [LM].) Since B is primitive, we may apply 4.5.12 of [LM] to conclude that, for every pair i, j, we may find a positive constant $a_{i,j}$ such that

$$\lim_{n\to\infty} |(B^n)_{i,j} - a_{i,j}\lambda^{dn}| = 0.$$

But $\#Punc(p, n)$ is simply the number of different tiles in $\omega^n(p)$ which is sum over all i of $(B^n)_{i,j}$, where $p_j = p$. The result follows easily from this. □

Lemma 2.3. *Let $p_1, \ldots, p_N, \omega$ be a primitive substitution tiling system in $\mathbb{R}^d$ such that, for each prototile p, boundary of p, ∂p, has box-counting dimension strictly less than d. Then, for any $R > 0$ and prototile p, we have*

$$\lim_{n\to\infty} \frac{\#\{x \in Punc(p, n) \mid \partial(x) \leq R\}}{\#Punc(p, n)} = 0.$$

Proof. We let δ be the maximum box-counting dimension of the boundaries of the prototiles. So our hypothesis is that $\delta < d$. This means that there is a constant K and a function $m(\epsilon) \leq K\epsilon^{-\delta}$ such that, for any prototile p, we may cover its boundary with $m(\epsilon)$ balls of radius ϵ, for any $\epsilon > 0$. Fix a prototile p and a positive integer n, let $\epsilon = R\lambda^{-n}$. Choose an open cover of ∂p with ϵ-balls as above and denote their centres by $x_i, i = 1, \ldots, m(\epsilon)$.

Now if x is in $Punc(p,n)$ and $\partial(x) \leq R$, then for some y in $\partial(\lambda^n p)$, we have $|x-y| \leq R$. Then we have $|\lambda^{-n}x - \lambda^{-n}y| \leq R\lambda^{-n} = \epsilon$, $\lambda^{-n}y$ is in ∂p and hence $|x\lambda^{-n} - x_i| < 2\epsilon$, for some i.

So each point x of $Punc(p,n)$ within R of the boundary of $\lambda^n p$, is contained in some $\lambda^n B(x_i, 2\epsilon)$. Notice that

$$\begin{aligned} \lambda^n B(x_i, 2\epsilon) &= B(\lambda^n x_i, \lambda^n 2\epsilon) \\ &= B(\lambda^n x_i, 2R) \end{aligned}$$

We next want an upper bound on the number of such x, for a fixed i. Let

$$k_i = \#(Punc(p,n) \cap B(\lambda^n x_i, 2R)).$$

We will use the fact that $B(x(t), b) \subset t$, for any tile t. This means that the balls $B(x,b)$, for x in $Punc(p,n)$, are pairwise disjoint. And if x is also in $B(\lambda^n x_i, 2R)$, then $B(x,b)$ is contained in $B(\lambda^n x_i, 2R+b)$. This means that

$$Vol(B(\lambda^n x_i, 2R+b)) \geq \sum Vol(B(x,b))$$

where the sum is taken over all x in $Punc(p,n) \cap B(\lambda^n x_i, 2R)$. There is a positive constant, V_d, so that for all positive r, $Vol(B(x,r)) = V_d r^d$. So we have

$$V_d(2R+b)^d \geq k_i V_d b^d$$

which in turn gives us

$$k_i \leq (1 + 2R/b)^d. \quad (1)$$

Now if we sum over i, we obtain

$$\begin{aligned} \#\{x \in Punc(p,n) \mid \partial(x) \leq R\} &\leq \sum_{i=1}^{m(\epsilon)} k_i \\ &\leq \sum_{i=1}^{m(\epsilon)} (1+2R/b)^d \\ &\leq m(\epsilon)(1+2R/b)^d \\ &\leq K\epsilon^{-\delta}(1+2R/b)^d \\ &= K(R\lambda^{-n})^{-\delta}(1+2R/b)^d \\ &= K(1+2R/b)^d R^{-\delta}\lambda^{n\delta} \\ &= K'\lambda^{n\delta}, \end{aligned}$$

where $K' = K(1+2R/b)^d R^{-\delta}$ is independent of n. Now we combine this estimate with Lemma 2.2 to obtain

$$\begin{aligned}\lim_{n\to\infty} \frac{\#\{x \in Punc(p,n) \mid \partial(x) \leq R\}}{\#Punc(p,n)} &\leq \lim_{n\to\infty} \frac{K'\lambda^{n\delta}}{(a_p\lambda^{dn})} \\ &= \lim_{n\to\infty} (K'/a_p)\lambda^{n(\delta-d)} \\ &= 0,\end{aligned}$$

since $\delta < d$. □

Definition 2.4. *For any tiling T in $r(R_{punc} - R_{AF})$, there is a vector x in $\mathbb{R}^d$ such that $T - x$ is in Ω_{punc}, but $(T, T-x)$ is not in R_{AF}. For such T we define*

$$\rho(T) = \inf\{|x| \mid (T, T-x) \in R_{punc} - R_{AF}\}.$$

Lemma 2.5. *Let p be a prototile and n be a positive integer. Suppose that x is in $Punc(p,n)$ and that T is in $r(R_{punc} - R_{AF}) \cap W(p,n,x)$. Then we have*

$$\rho(T) \geq \partial(x).$$

Proof. The hypothesis means that we can write $T = \omega^n(T') - x$, where T' contains the tile p.

Suppose that y is any vector with $|y| < \partial(x)$. We claim that if $T - y$ is in Ω_{punc}, then $(T, T-y)$ is in R_{AF}. From this it follows that if $(T, T-y)$ is to be in $R_{punc} - R_{AF}$, we must have $|y| \geq \partial(x)$ and the conclusion follows from the definition of ρ.

As for the claim, we begin by noting that if $|y| < \partial(x)$, then $x + y$ is in the interior of $\lambda^n p$. If, in addition, $T - y$ is in Ω_{punc}, then the $T - y = \omega^n(T') - x - y = \omega^n(T') - (x+y)$ and so $x + y$ is in $Punc(p,n)$. The graph of the map sending $\omega^n(T') - x$ to $\omega^n(T') - (x+y)$ is contained in R_{AF}. In particular, the pair $(T, T-y)$ is in R_{AF}. This completes the proof of the claim. □

Recall that we are trying to prove that $\mu(r(R_{punc} - R_{AF})) = 0$. It is easy to check that for fixed $R > 0$, the set

$$\{(T, T-x) \mid |x| \leq R\} \cap (R_{punc} - R_{AF})$$

is compact in R_{punc}. It follows that for any $R > 0$,

$$r(R_{punc} - R_{AF}) \cap \rho^{-1}[0, R]$$

is compact in Ω_{punc}. To prove our result, it suffices to show that the μ-measure of this set is zero, for any R.

We now fix $R_0 > 0$ and, for convenience, we denote $\{T \in r(R_{punc} - R_{AF}) \mid \rho(T) \leq R_0\}$ by Y_0.

We will construct a sequence of positive constants, $R_0 < R_1 < R_2 < \dots$, and a sequence of locally defined maps $\gamma_1, \gamma_2, \dots$ with the following properties. Each γ_m is a local homeomorphism whose graph is a clopen set in R_{AF} and whose domain contains Y_0. Moreover, for all T in Y_0, we will have

$$R_{m-1} < \rho(\gamma_m(T)) \leq R_m - 1.$$

We may conclude from this last equation that the sets $\gamma_m(Y_0)$ are pairwise disjoint. Moreover, the maps γ_m all preserve the measure μ. So each of these sets has the same measure as Y_0 and since the measure is finite, we conclude that $\mu(Y_0) = 0$, as desired.

We begin by setting

$$R_0 = sup\{\rho(T) \mid T \in Y_0\}.$$

Assume that, for some $m \geq 1$, we have R_{m-1} defined with the property that $\rho(T) < R_{m-1}$, for all T in Y_0. We define γ_m as follows. We apply Lemma 2.3 using the value $R = R_{m-1} + 1$. We may find a n sufficiently large so that the ratio in the limit is less than $1/2$, for all prototiles p. This means that, for this value of n,

$$\begin{aligned}\#\{x \in Punc(p, n) \quad &| \quad \partial(x) \leq R_{m-1}\} \\ &\leq \quad \#\{x \in Punc(p, n) \mid \partial(x) \geq R_{m-1} + 1\},\end{aligned}$$

for each prototile p. Now for each prototile, p, we may define an injection, η, from the first set above to the second. (Of course, there is a different η for each p, but we will suppress this in out notation.)

The domain of the map γ_m will be the union of all sets $W(p, n, x)$ where p is any prototile and x is in $Punc(p, n)$ with $\partial(x) \leq R_{m-1}$. For T in $W(p, n, x)$, we define

$$\gamma_m(T) = T + x - \eta(x).$$

It is easy to check that γ_m is a homeomorphism on its domain. Also, for every T in Y_R, we know that T is in some set $W(p, n, x)$. It follows from Lemma 2.5 that

$$R_{m-1} \geq \rho(T) \geq \partial(x)$$

and so $W(p, n, x)$, and hence T, is in the domain of γ_m. It is also clear from the definition and Lemma 2.5 that

$$\rho(\gamma_m(T)) \geq \partial(\eta(x)) \geq R_{m-1} - 1.$$

Therefore γ_m has all the required properties. To complete the induction, we choose R_m to be

$$R_m = \sup \rho(\gamma_m(Y_0)) + 1.$$

This completes the proof of Theorem 2.1.

3 Proof of the main result

We begin a proof of the main result Theorem 1.1. The key ingredient is the following.

Lemma 3.1. *Let p be a non-zero projection in A_T and suppose that $0 < \epsilon < \tau(p)$. Then there is a projection q in AF_T satisfying*

$$[q] \leq [p] \text{ in } K_0(A_T)$$

and

$$|\tau(p) - \tau(q)| < \epsilon.$$

The proof will take some time and involve several lemmas. Begin by choosing $0 < \delta \leq \epsilon/20$ and so that $\delta < 1/400$. We use the facts that

$$C_c(R_{punc}) \subset A_T$$

is dense and that R_{punc} is totally disconnected to find a function f in $C_c(R_{punc})$ which is locally constant (i.e. f has finite range) and so that

$$\|p - f\| < \delta. \tag{2}$$

By replacing f by f^*f if necessary, we may assume that f is positive in A_T. We may also assume that $\|f\| \leq 1$. It follows from equation 2 that

$$\|f^2 - f\| < 3\delta \tag{3}$$

Note that when we write f^2, we mean the product in A_T, which is the convolution product on R_{punc}, not the pointwise product.

Let

$$K = r(supp(f) \cap (R_{punc} - R_{AF}))$$

which is a compact subset of Ω_{punc} and has $\mu(K) = 0$, by Theorem 2.1. We may choose a clopen set $F \supset K$ such that

$$\int_F f^2(x,x) d\mu(x) < \delta. \tag{4}$$

We define a function e on R_{punc} by

$$e(T, T') = \begin{cases} 1 & \text{if } T = T' \notin F \\ 0 & \text{otherwise} \end{cases}$$

Notice that e is a projection in AF_T.

Lemma 3.2. *The element ef (product in A_T) is a locally constant function on R_{punc} and ef is in $C_c(R_{AF})$. Finally, we have*

$$|\tau(f) - \tau(f^2 ef)| < 7\delta. \tag{5}$$

Proof. The first statement is obvious since both e and f have the same property. As for the second, we only need to see that ef is zero on $R_{punc} - R_{AF}$. We have

$$ef(T, T') = e(T, T)f(T, T'),$$

for any (T, T') in R_{punc}. If (T, T') is in $R_{punc} - R_{AF}$ and f is not zero on this point, then $T = r(T, T')$ is in K and so $e(T, T) = 0$.

For the last inequality, we have

$$\begin{aligned}
|\tau(f) - \tau(f^2 e f)| &\leq |\tau(f) - \tau(f^2)| + |\tau(f^2) - \tau(fef)| \\
&\quad + |\tau((f - f^2)ef)| \\
&\leq \|f - f^2\| + |\tau(f^2) - \tau(ef^2)| \\
&\quad + \|f - f^2\| \\
&\leq 6\delta + \tau((1-e)f^2) \\
&= 6\delta + \int_F f^2(x,x)\,d\mu(x) \\
&< 7\delta,
\end{aligned}$$

by equation 4. □

We now know that the element $fef = (ef)^*(ef)$ is self-adjoint and lies in AF_T. Since it is a locally constant function on R_{AF} it will actually lie in one of the canonical approximating finite-dimensional C^*-algebras, denoted by A_N in [KP]. This means that its spectrum is finite and we may write

$$fef = \sum_{i=1}^{m} \lambda_i e_i \tag{6}$$

where the λ_i are positive constants less than or equal to 1 and the e_i are projections in AF_T satisfying

$$\begin{aligned}
\sum_{i=1}^{m} e_i &= 1, \\
e_i e_j &= 0, \text{ for } i \neq j.
\end{aligned}$$

By re-arranging the order of the terms, we may assume that

$$\begin{aligned}
\lambda_i &\leq 1/2, \text{ for } i = 1, \ldots, k, \\
\lambda_i &\geq 1/2, \text{ for } i = k+1, \ldots, m,
\end{aligned}$$

for some fixed k.

We now define

$$q = \sum_{i=k+1}^{m} e_i.$$

Notice immediately that q is a self-adjoint projection and lies in AF_T.

Lemma 3.3. *1.*

$$\|pq - q\| < 4\delta.$$

2.

$$\|pqp - q\| < 8\delta.$$

Proof. We use the definition of q and equation 6:

$$\begin{aligned}
\|pq - q\| &= \|(p-1)\sum_{i>k} e_i\| \\
&= \|(p-1)\sum_{i>k}\left(\sum_{j=1}^{m}\lambda_j e_j\right)\lambda_i^{-1}e_i\| \\
&= \|(p-1)\sum_{i>k} fef\lambda_i^{-1}e_i\| \\
&\leq \|(p-1)f\|\|e\|\|f\|\|\sum_{i>k}\lambda_i^{-1}e_i\| \\
&\leq (\|p(f-p)\| + \|p-f\|)\sup_{i>k}\{\lambda_i^{-1}\} \\
&< (2\delta)2 = 4\delta.
\end{aligned}$$

The second inequality follows at once from the first. We omit the details. □

Since $\delta < 1/400$, we obtain $\|(pqp)^2 - pqp\| < 24\delta < 1/16$, so the spectrum of pqp is contained in $[-1/8, 1/8] \cup [7/8, 9/8]$ and so we may apply functional calculus and obtain $q' = \chi_{(1/2,\infty)}(pqp)$. Then q' is a self-adjoint projection in A_T within $1/8$ of pqp and hence within distance $1/2$ of q. Therefore, $[q'] = [q]$, by 4.3.2 of [Bl] or 5.2.6 of [W-O]. Also, the element q' can be obtained as a limit of polynomial functions with zero constant term applied to pqp. From this we see that $pq' = q' = q'p$, or $q' \leq p$. We have all the properties we desired from q and q', except the estimate on the trace of q.

Lemma 3.4.

$$|\tau(p) - \tau(q)| < \epsilon.$$

Proof. First, we want to estimate $|\tau(f) - \tau(fq)|$. Recall that the sum of the e_i's was the identity. So that, we have

$$\begin{aligned}|\tau(f) - \tau(fq)| &= |\sum_{i=1}^{m} \tau(fe_i) - \sum_{i>k} \tau(fe_i)| \\ &= |\sum_{i=1}^{k} \tau(fe_i)|.\end{aligned}$$

Now we use the fact that, for $i \leq k$, we have $\lambda_i \leq 1/2$. So we may continue

$$\begin{aligned}|\tau(f) - \tau(fq)| &\leq |\sum_{i=1}^{k} 2(1-\lambda_i)\tau(fe_i)| \\ &\leq |\sum_{i=1}^{n} 2(1-\lambda_i)\tau(fe_i)| \\ &= 2|\sum_{i=1}^{n} \tau(fe_i) - \tau(f\lambda_i e_i)| \\ &= 2|\tau(f) - \tau(f^2 ef)| \\ &< 14\delta\end{aligned}$$

by Lemma 3.2.

Now we are ready to compute

$$\begin{aligned}|\tau(p) - \tau(q)| &\leq |\tau(p) - \tau(f)| + |\tau(f) - \tau(fq)| \\ &\quad +|\tau(fq) - \tau(pq)| + |\tau(pq) - \tau(q)| \\ &< \delta + 14\delta + \|f - p\| + \|pq - q\| \\ &< 20\delta < \epsilon,\end{aligned}$$

using Lemma 3.3 for the last term. □

We have now completed the proof of Lemma 3.1 and we are now ready to give a proof of Theorem 1.1. First, we consider the "only if" direction. If a is any positive element in $K_0(A_T)$, then by definition, $a = [p]$ for some projection p in some $M_n(A_T)$. By applying II.4.2 of [Ren], we may view p as a matrix of functions on R_{punc}. Since $p = p^*p$, the diagonal elements of the matrix p are non-negative on the diagonal in R_{punc}. This means that $\tau(p)$ is non-negative. Moreover, if it is zero, then since μ has full support, then each diagonal entry of p is zero. This in turn implies that $p = 0$.

Now we turn to the "if" direction of the proof. That is, suppose that $a = [p] - [q]$ is in $K_0(A_T)$ and has $\hat{\tau}(a) = \tau(p) - \tau(q) > 0$. We will show that $[p] \geq [q]$ in $K_0(A_T$. We will first consider the case that the projections actually lie in the algebra A_T, rather than in matrices over A_T.

Begin with two projections p_1 and p_2 in A_T and suppose that $\tau(p_1) > \tau(p_2)$. let $\epsilon = (\tau(p_1) - \tau(p_2))/3$. We apply Lemma 3.1 to the projection p_1 and $\epsilon > 0$ to obtain q_1 in AF_T with $[p_1] \geq [q_1]$ and $|\tau(p_1) - \tau(q_1)| < \epsilon$. We apply the same result to the projection $1 - p_2$ and the same ϵ to obtain a projection in AF_T. We let q_2 be its orthogonal complement. So we have $[q_2] \geq [p_2]$ and $|\tau(p_2) - \tau(q_2)| < \epsilon$. Then by a simple application of the triangle inequality, we have

$$\tau(q_1) - \tau(q_2) \geq \epsilon > 0.$$

The C^*-algebra AF_T has a unique trace and it is a simple AF-algebra. For simple AF-algebras, the order on their K-zero groups is completely determined by the traces [EHS]. We know then that $[q_1] \geq [q_2]$ and hence $[p_1] \geq [p_2]$ in $K_0(A_T)$ as desired.

In the case that the projections lie in $M_n(A_T)$, we can use the same argument by replacing the groupoids R_{punc} and R_{AF} by their products with the trivial groupoid $\{1, \ldots, n\} \times \{1, \ldots, n\}$. All of the essential features of the groupoids remain and the effect at the level of C^*-algebras is to tensor on $C^*(\{1, \ldots, n\} \times \{1, \ldots, n\}) \cong M_n$, the C^*-algebra of $n \times n$ matrices. We omit the details.

References

[AP] J.E. Anderson and I.F. Putnam, Topological invariants for substitution tilings and their associated C^*-algebras, *Ergodic Theory and Dynamical Systems* 18(1998), 509-537.

[Be1] J. Bellissard, K-theory of C^*-algebras in solid state physics, in *Statistical Mechanics and Field Theory: Mathematical Aspects*, ed. T.C. Dorlas, N.M. Hugenholtz and M. Winnik, Lecture Notes in Physics 257, Springer-Verlag, 1986, 99-156.

[Be2] J. Bellissard, Gap labelling theorems for Schrödinger's Operators, in *From Number Theory to Physics*, ed. M. Waldschmidt, P. Moussa, J.M. Luck and C. Itzykson, Springer-Verlag, 1992, pp 538-630.

[BCL] J. Bellissard, E. Contensou, A Legrand, K-théorie des quasicristaux, image par la trace, le cas du ré octaggonal, *C. R. Acad. Sci. Paris*, t.326, Série I, (1998), 197-200.

[Bl] B. Blackadar, *K-theory for Operator Algebras*, MSRI Publications 5, Springer-Verlag, Berlin-Heidelberg-New York, 1986.

[Co2] A. Connes, *Non-commutative Geometry*, Academic Press, San Diego, 1994.

[Da] K.R. Davidson, *C^*-algebras by example*, Amer. Math. Soc., Providence, R.I., 1996.

[Ef] E.G. Effros, *Dimensions and C^*-algebras*, CBMS Regional Conf. Ser. no. 46, Amer. Math. Soc., Providence, R.I., 1981.

[EHS] E.G. Effros, D.E. Handelman and C.-L. Shen, Dimension groups and their affine representations, *Amer. J. Math.* 102 (1980), 385-407.

[vE] A. van Elst, Gap-labelling theorems for Schrödinger operators on the square and cubic lattice, *Rev. Math. Phys.* 6 (1994), 319-342.

[Fi] P.A. Fillmore, *A user's guide to operator algebras*, Wiley, New York, 1996.

[GS] B. Grünbaum and G.C. Shephard, *Tilings and Patterns*, Freeman and Co., New York, 1987.

[Ha] D. Handelman, Positive matrices and dimension groups associated with topological Markov chains, *J. Operator Th.* 6(1981), 55-74.

[Kel1] J. Kellendonk, Non-commutative geometry of tilings and gap labelling, *Rev. Math. Phs.* 7(1995), 1133-1180

[Kel2] J. Kellendonk, The local structure of tilings and their integer group of coinvariants,*Comm. Math. Phys.* 187(1997), 115-157.

[KP] J. Kellendonk and I.F. Putnam, Tilings, C^*-algebras and K-theory, preprint.

[LM] D. Lind and B. Marcus, *An introduction to symbolic dynamics and coding*, Cambridge University Press, Cambridge, 1995.

[Pe] G.K. Pedersen, *C^*-algebras and their automorphism groups*, Academic Press, London, 1979.

[Put1] I.F. Putnam, C^*-algebras from Smale spaces, *Canad. J. Math.* 48 (1996), 175-195.

[Put2] I.F. Putnam, *Hyperbolic dynamical systems and generalized Cuntz-Krieger algebras*, Lecture Notes from the summer school in operator algebras, Odense, 1996.

[PS] I.F. Putnam and J. Spielberg, The structure of C^*-algebras associated with hyperbolic dynamical sustems, *J. Func. Anal.* 163 (1999), 279-299.

[RW] C. Radin, and M. Wolff, Space tilings and local isomorphism, *Geom. Ded.* 42(1992), 355-360.

[Ren] J.N. Renault, *A groupoid approach to C^*-algebras*, Lecture Notes in Math. 793, Springer-Verlag, 1980.

[Rud] D.J. Rudolph, Markov tilings of $\mathbb{R}^d$ and representations of $\mathbb{R}^d$ actions, *Contemp. Math.* 94(1989), 271-289.

[So1] B. Solomyak, Dynamics of self-similar tilings, *Ergodic Theory and Dynamical Systems* 17(1997), 695-738.

[So2] B. Solomyak, Non-periodicity implies unique composition for self-similar translationally-finite tilings, *Disc. Comp. Geom.* 20 (1998), 265-279.

[W-O] N.E. Wegge-Olsen, *K-theory and C^*-algebras*, Oxford University Press, Oxford, 1993.