C^*-algebras and Tilings, Aperiodic Order, CIRM, Luminy

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September 10, 2007

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Part 1 : C^*-algebra basics

Definition 1. *A C^*-algebra is a set A:*

- *A is an algebra over $\mathbb{C}$, the complex numbers (Not nec. commutative or unital)*

- *there is an involution $a \to a^*$, $a \in A$*

- *A has a norm, $\| \, \|$,*

such that

- $(a + \lambda b)^* = a^* + \overline{\lambda} b^*$, $a, b \in A$,

- $(ab)^* = b^* a^*$, $a, b \in A$,

- *A is complete in $\| \, \|$,*

- $\| a^* a \| = \| a \|^2$, $a \in A$.

Examples:

- $\mathbb{C}$, the complex numbers,

- For $n \geq 1$, $M_n(\mathbb{C})$, $n \times n$ complex matrices. $*$=conjugate transpose.

- For $\mathcal{H}$ a complex Hilbert space, $\mathcal{B}(\mathcal{H})$, the bounded linear operators on $\mathcal{H}$. $*$= adjoint.

- Any $A \subset \mathcal{B}(\mathcal{H})$ which is an algebra, closed under $*$, closed in the norm topology.

Let X be a compact, Hausdorff space.

$$C(X) = \{f : X \to \mathbb{C} \mid f \text{ continuous }\}.$$

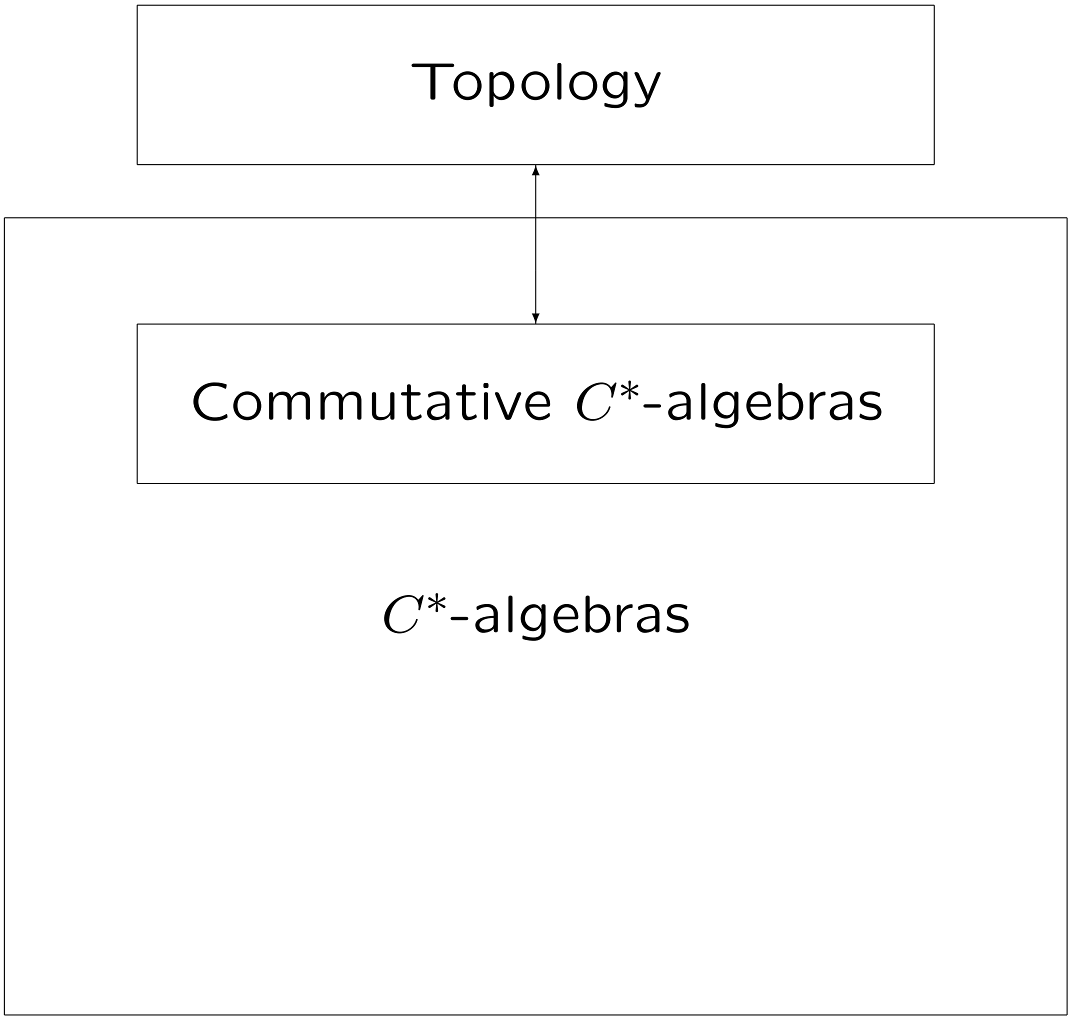
It is a C^*-algebra with pointwise algebraic operations, $*=$ pointwise complex conjugation, $\| \, \|$ is the supremum norm.

We can generalize: if the space X is *locally* compact, replace $C(X)$ with $C_0(X)$, the continuous complex functions which vanish at infinity. This is unital if and only if X is compact.

These are both *commutative*.

Gelfand-Naimark Theorem: Every commutative C^*-algebra arises in this way. $C_0(X)$ and $C_0(Y)$ are isomorphic if and only if X and Y are homeomorphic.

Theorem 2. *The functor $X \to C_0(X)$ is an equivalence of categories between locally compact, Hausdorff spaces and commutative C^*-algebras.*



- Can we extend standard topological notions to C^*-algebras?

- Are the some geometric constructions of non-commutative C^*-algebras?

Gelfand-Naimark dictionary:

Topology	Commutative C^*-alg's
closed set	closed ideal
$Y \subset X$	$I = \{f \in C(X) \mid f\|Y = 0\}$ is a closed ideal in $C_0(X)$
Borel measure	functional
μ	$\varphi_\mu(f) = \int_X f d\mu$ $\varphi_\mu : C_0(X) \to \mathbb{C}$
K-theory	K-theory

An application: Hilbert space $L^2[0,1]$.

Let $Cut = \{p2^{-k} \mid p,k \in \mathbb{Z}\} \cap [0,1]$

For each $a < b$ in Cut, let $\chi_{[a,b)}$ denote the characteristic function of $[a,b)$, which we regard as an operator on $L^2[0,1]$ by pointwise multplication.

Let A be the closed linear span of $\{\chi_{[a,b)} \mid a < b, a, b \in Cut\}$ in $\mathcal{B}(L^2[0,1])$.

This is a commutative, unital C^*-algebra. Hence, $A \cong C(X)$, for some X. What is X?

It should be a space where our functions $\chi_{[a,b)}$ are continuous: from $[0,1]$, remove each point a in Cut and replace it with two points a^-, a^+. Topologically, imagine a^- as a left endpoint of $[0,a]$ and a^+ as a right endpoint for $[a,1]$, separated by a gap. This is X and it is a Cantor set.

Part 2: C^*-algebras from dynamics

Situation 1: Topological equivalence relations

Let X be a compact, Hausdorff space.

R an equivalence relation on X.

$r, s : R \to X$ are the projections:

$$r(x, y) = x, s(x, y) = y, (x, y) \in R.$$

Assume R has an étale topology: r, s are open and local homeomorphisms.

Idea: if (x, y) is in R, there are open sets $x \in U$, $y \in V$ and a (unique) homeomorphism $\rho : U \to V$ such that

$$\begin{aligned} \rho(x) &= y, \\ \{(u, \rho(u)) \mid u \in U\} &\subset R. \end{aligned}$$

$C^*(R)$:

First look at $C_c(R)$, the continuous, complex-valued functions of compact support on R. It is a linear space in an obvious way. Define a product and involution:

$$\begin{aligned}(f \cdot g)(x,y) &= \sum_{(x,z)\in R} f(x,z)g(z,y), \\ f^*(x,y) &= \overline{f(y,x)}.\end{aligned}$$

Complete in a norm to get a C^*-algebra, $C^*(R)$.

Example: $X = \{1, 2, \ldots, N\}, R = X \times X$.

$$C^*(R) = M_N(\mathbb{C}).$$

Start with $C(X) = \mathbb{C}^N = span\{\chi_1, \ldots, \chi_N\}$ and add $e_{i,j}$ such that

$$\begin{aligned}e_{i,j}^* e_{i,j} &= \chi_j, \\ e_{i,j} e_{i,j}^* &= \chi_i,\end{aligned}$$

The last example illustrates a general property:

$$f \in C(X) \to \delta(f)(x,y) = \begin{cases} f(x) & x = y \\ 0 & x \neq y \end{cases}$$

embeds $C(X)$ as a unital subalgebra of $C^*(R)$.

Assume $U, V \subset X$ are clopen, $\rho : U \to V$ as before, let $w(x,y) = 1, x \in U, y = \rho(x), w(x,y) = 0$, otherwise.

$$\begin{aligned} w^*w &= \delta(\chi_U), \\ ww^* &= \delta(\chi_V), \\ w\delta(f)w^* &= \delta(f \circ \rho) \end{aligned}$$

if f is supported in U.

Example: X locally compact, $R ==$ (equality).

$$C^*(R) = C_0(X).$$

Example (Kellendonk): $\mathcal{P} = \{p_1, \ldots, p_N\}$, a finite set of prototiles in $\mathbb{R}^d$. Each has a distinguished interior point $x(p_i)$ called a puncture.

Translate: $x(p_i + y) = x(p_i) + y, y \in \mathbb{R}^d$

Suppose Ω a compact, translation invariant collection of tilings which are made from translates of $\mathcal{P}$.

$$\Omega_{punc} = \{T \in \Omega \mid x(t) = 0, \text{ for some } t \in T\}.$$

$$R_{punc} = \{(T, T + x) \mid T, T + x \in \Omega_{punc}, x \in R^d\}$$

is an étale groupoid.

Let $T \in \Omega$, $t_1, t_2 \in T$:

$$\begin{aligned} U &= \{T' \mid t_1 - x(t_1), t_2 - x(t_1) \in T'\} \\ V &= \{T' \mid t_1 - x(t_2), t_2 - x(t_2) \in T'\} \\ \rho(T') &= T' + x(t_1) - x(t_2). \end{aligned}$$

Situation 2: Actions of countable groups

G a countable abelian (for notation) group, X a loc. cmpct Hausdorff space, φ an action of G on X:

$$s \in G, \varphi^s : X \to X,$$

is a homeomorphism.

Action is free if $\varphi^s(x) = x \Rightarrow s = 0$.

$C_0(X) \times_\varphi G$: Generators: $C_0(X)$, $u_s, s \in G$, Relations:

$$\begin{aligned} u_0 &= 1, \\ u_s u_t &= u_{s+t}, \\ u_s^* &= u_{-s}, \\ u_s f u_s^* &= f \circ \varphi^{-s} \\ u_s f &= (f \circ \varphi^{-s}) u_s \end{aligned}$$

$s, t \in G$, $f \in C_0(X)$.

Consider all formal sums

$$\sum_{s \in G} f_s u_s$$

where only finitely many $f_s \in C(X)$ are non-zero. The rules above define product and involution. We give this a norm and then complete.

Idea: Each s in G defines an automorphism of $C_0(X)$: $f \to f \circ \varphi^{-s}$. Here $\delta(f) = f u_0$ and $C_0(X) \subset C_0(X) \times_\varphi G$ and all these automorphisms become inner. u_s is a unitary. (Caution: u_s is in $C_0(X) \times_\varphi G$ only if X is compact.)

Example: $X = \{1, \ldots, N\}, G = \mathbb{Z}_N$, φ is addition, mod N. $C(X) \times G \cong M_N$.

Gelfand-Naimark dictionary (for free actions):

Dynamics (X, G, φ)	C^*-alg. $C_0(X) \times_\varphi G$
closed invariant set	two-sided closed ideal
$Y \subset X$	$I = \{\sum_s f_s u_s \mid f_s \vert Y = 0\}^-$ is a closed two-sided ideal in $C_0(X) \times_\varphi G$
invariant measure	trace
μ	$\tau_\mu(\sum_s f_s u_s) = \int_X f_0 d\mu$ $\tau_\mu(ab) = \tau_\mu(ba)$

Comparison of topological equivalence relations and actions of countable groups.

Start with (X, G, φ).

Let

$$R_\varphi = \{(x, \varphi^s(x)) \mid x \in X, s \in G\},$$

is an equivalence relation. The classes are the orbits.

If G acts *freely* ($\varphi^s(x) = x$ only if $s = e$), this can be given an étale topology. The local homeomorphisms are $\varphi^s, s \in G$.

$$C(X) \times_\varphi G \cong C^*(R_\varphi).$$

Situation 3: Continuous group actions

G a locally compact abelian group, X a locally compact Hausdorff space, φ an action of G on X:

$$s \in G, \varphi^s : X \to X,$$

is a homeomorphism.

$C_c(X \times G)$ is a linear space and is given a product and involution:

$$\begin{aligned} (f \cdot g)(x, s) &= \int_G f(x,t) g(\varphi^t(x), s-t) d\lambda(t), \\ f^*(x,s) &= f(\varphi^{-s}(x), s), \end{aligned}$$

f, g in $C_c(X \times G)$, x in X, s in G,

λ is Haar measure on G.

G discrete: $u_s(x,t) = \begin{cases} 1 & t = s \\ 0 & t \neq s \end{cases}$

Part 4: Morita equivalence for C^*-algebras
(Rieffel, Muhly-Renault-Williams)

"Morita equivalence is more natural than isomorphism" - A. Connes.

If A and B are Morita equivalent ($A \sim B$), then

- A and B have isomorphic lattices of closed two-sided ideals

- there is a bijection between classes of representations as operators on Hilbert space

- A and B have isomorphic K-theory

What is *not* preserved:

- linear dimension
- commutativity

Example 1: $M_m(\mathbb{C}) \sim M_n(\mathbb{C})$ are Morita equivalent for all $m, n \geq 1$.

Example 2: φ a free, wandering action of G on X. $q : X \to X/R_\varphi$ is the quotient map. Wandering implies that the space of orbits X/R_φ is Hausdorff in the quotient topology.

$A = C_0(X) \times_\varphi G \sim B = C_0(X/R_\varphi)$ are Morita equivalent.

e.g. $C_0(\mathbb{R}) \times \mathbb{Z} \sim C(S^1)$.

Moral: if the quotient X/R_φ is a bad space (there is some recurrence in φ), then $C_0(X) \times_\varphi G$ is its non-commutative replacement.

Example 3: X locally compact, Hausdorff, φ an action of G, ψ an action of H,

$$\varphi^s \circ \psi^t = \psi^t \circ \varphi^s, s \in G, t \in H.$$

If the actions φ and ψ are both wandering, then

$$\begin{aligned} A &= C_0(X/R_\varphi) \times_\psi H \\ B &= C_0(X/R_\psi) \times_\varphi G \\ C &= C_0(X) \times_{\varphi \times \psi} (G \times H) \end{aligned}$$

are all Morita equivalent.

Example 4: If φ is an $\mathbb{R}$-action on X and has a transversal T, let ψ be the Poincaré first return map on T. Under mild conditions,

$$C_0(X) \times_\varphi \mathbb{R} \sim C_0(T) \times_\psi \mathbb{Z}.$$

Example 5: Let Ω be a continuous hull. It has an action of $\mathbb{R}^d$ and we consider the C^*-algebra $C(\Omega) \times \mathbb{R}^d$.

Recall

$$\Omega_{punc} = \{T \in \Omega \mid x(t) = 0, \text{ some } t \in T\}$$

and

$$R_{punc} = \{(T, T + x) \mid, T, T + x \in \Omega_{punc}$$

and the C^*-algebra $C^*(R_{punc})$.

- Ω_{punc} is a transverse to the $\mathbb{R}^d$-action,

- restricting the $\mathbb{R}^d$-orbits to Ω_{punc} gives R_{punc} which is étale

- every $\mathbb{R}^d$ orbit in Ω meets Ω_{punc}.

$C^*(R_{punc})$ and $C(\Omega) \times \mathbb{R}^d$ are Morita equivalent.

Part 5: C^*-algebras for projection method tilings (Forrest-Hunton-Kellendonk)

Data:

- $\mathbb{R}^d$, physical space (to be tiled),
- H, internal space, locally cpct ab. group,
- $\pi : \mathbb{R}^d \times H \to \mathbb{R}^d, \pi^\perp : \mathbb{R}^d \times H \to H$,
- $\mathcal{L} \subset \mathbb{R}^d \times H$, discrete, co-compact (lattice),
- $\pi|\mathcal{L}, \pi^\perp|\mathcal{L}$ one-to-one, $L = \pi^\perp(\mathcal{L})$ dense in H.
- $W \subset H$, compact, regular, $\lambda(\partial W) = 0$.

A point x in $\mathbb{R}^d \times H$ is *non-singular* if

$$\pi^{\perp}(x + \mathcal{L}) \cap \partial W = \emptyset.$$

$\mathcal{N}$ is the set of non-singular points.

$$\Lambda_x = \pi\{y \in x + \mathcal{L} \mid \pi^{\perp}(y) \in W\}$$

is a Delone set, called a regular model set.

The hull Ω is the completion of

$$\{\Lambda_x \mid x \in \mathcal{N}\}.$$

Comments:

- $\mathcal{N}$ is invariant under the actions of $\mathbb{R}^d$ and $\mathcal{L}$,

- $\Lambda_{x+s} = \Lambda_x$, if $s \in \mathcal{L}$,

- $\Lambda_{x+u} = \Lambda_x + u$, if $u \in \mathbb{R}^d$.

Lemma 3. *Suppose* $x_n \in \mathcal{N}$ *converges to* $x \in \mathbb{R}^d \times H$*.* Λ_{x_n} *converges in* Ω *(i.e. is Cauchy in the tiling metric) if and only if, for every* $s \in L$*, the sequence* $\pi^{\perp}(x_n)$ *is eventually either in* $W + s$ *or in its complement.*

Theorem 4. *For* $s \in L$*,*

$$\Lambda_x \to \chi_{W+s}(x), x \in \mathcal{N} \cap H$$

extends to a continuous function on Ω*.*

Definition 5. *Consider* A*, the* C^**-algebra of operators on* $L^2(H, \lambda)$ *generated by* $C_0(H)$ *and* $\chi_{W+s}, s \in L$*. Let* $\widehat{H}$ *be its spectrum; i.e.* $A \cong C_0(\widehat{H})$*.*

The action of L on E extends to $\widehat{H}$. $L \subset H$ is dense implies that $\widehat{H}$ is totally disconnected.

Theorem 6. *The hull Ω is homeomorphic to*

$$\mathbb{R}^d \times \hat{H}/\mathcal{L}$$

The actions of $\mathbb{R}^d$ and $\mathcal{L}$ on $\mathbb{R}^d \times \hat{H}$ are commuting, free and wandering:

Theorem 7. $C_0(\mathbb{R}^d \times \hat{H}/\mathcal{L}) \times \mathbb{R}^d$ *is Morita equivalent to*

$$C_0(\hat{H}) \times L.$$

The actions of $G = \mathbb{R}^d$ and $\mathcal{L} \cong L$ on $\mathbb{R}^d \times \hat{H}$ are commuting and wandering:

$$\mathbb{R}^d \times \hat{H}/\mathbb{R}^d \cong \hat{H}.$$

Further reductions:

Assume $H = \mathbb{R}^N$. So $L \cong \mathcal{L} \cong \mathbb{Z}^{d+N}$, as an abstract group: $C_0(\widehat{H}) \times \mathbb{Z}^{d+N}$. The action is by translation by the vectors L, which is a dense subgroup of $\mathbb{R}^N$.

$\widehat{H}$ is $\mathbb{R}^N$ disconnected along the boundaries of W and its translates by L. In many cases, this can be done in other ways, e.g. by lines.

Example: Fibonacci: $d = 1$, $N = 1$, $L = \mathbb{Z} + \alpha\mathbb{Z}$. $W = [a, b]$. $\widehat{H}$ is $\mathbb{R}^1$ disconnected along the $\mathbb{Z} + \alpha\mathbb{Z}$-orbits of a and b (one orbit or two?).

Example: Penrose: $d = 2$, $N = 2$, L is the subgroup of the plane generated by $exp(2\pi ij/5), j = 0, 1, 2, 3, 4$. $\widehat{H}$ is the plane disconnected along the 5 lines through the origin and $exp(2\pi ij/5), j = 0, 1, 2, 3, 4$, and all translates of them by L.

Example: TTT (Tübingen triangle tiling) Same is the Penrose, but rotate the 5 original lines by $\pi/10$.

Example: Octagonal tiling: $d = 2$, $N = 2$, L is the subgroup generated by $exp(\pi ij/4), j = 0, 1, 2, 3$. $\widehat{H}$ is the plane disconnected along the 4 lines through the origin and $exp(\pi ij/4), j = 0, 1, 2, 3$, and all translates by L.

One more reduction (still with $H = \mathbb{R}^N$). List a set of generators of L: $s_1, \ldots, s_{d+N}$. Act on a disconnected $H = \mathbb{R}^N$. The action of the first N of them is free and wandering: let $\widehat{H}_0$ denote the quotient, which is a Cantor set. It is really a disconnected N-torus. Our C^*-algebra is Morita equivalent to

$$C(\widehat{H}) \times \mathbb{Z}^{d+N} = C(\widehat{H}_0) \times \mathbb{Z}^d.$$

Part 6: K-theory for C^*-algebras

To a C^*-algebra, A, there are associated two abelian groups, $K_0(A)$ and $K_1(A)$. These are based on

projections	$p^2 = p = p^*$
unitaries	$u^* = u^{-1}$,

respectively, in A. It is a recepticle for such data and also an invariant for A. There is (by now) quite a lot of machinery for computing it.

$K_0(A)$: Assume A with unit.

p is a projection if $p^2 = p = p^*$.

Equivalence of projections:

Murray-von Neumann	$p \sim q$	$\exists v, v^*v = p, vv^* = q,$
similarity	$p \sim_s q$	$\exists v, vpv^{-1} = q$
unitary eq.	$p \sim_u q$	$\exists v^* = v^{-1}, vpv^{-1} = q$
homotopy	$p \sim_h q$	$\exists t \to p_t, p_0 = p, p_1 = q$

Note that v above must be in A.

Addition of projections: if p, q are orthogonal ($pq = 0$), then $p + q$ is a projection.

$M_n(A)$ is the set of $n \times n$ matrices with entries from A. It is a C^*-algebra. Its unit is 1_n. For $a \in M_n(A), b \in M_m(A)$,

$$a \oplus b = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in M_{m+n}(A).$$

$P_n(A)$, projections in $M_n(A)$.

$$P_1(A) \subset P_2(A) \subset P_3(A) \subset$$

by identifying p and $p \oplus 0$. Let $P(A) = \cup_n P_n(A)$.

Equivalence: In $P(A)$, we have $\sim = \sim_s = \sim_u = \sim_h$.
Problem: $p + p_0 \sim q + p_0 \not\Rightarrow p \sim q$.

Define $p \approx q$ if and only if $p \oplus 1_n \sim q \oplus 1_n$, for some n. $[p]$ is the class modulo $\approx$.

Addition: $p, q \in P(A)$, $p = p \oplus 0$, $q \sim 0 \oplus q$, which are orthogonal, and so

$$[p] + [q] = [p \oplus q]$$

is a well-defined addition.

$P(A)/\approx$ is a semi-group with identity, $[0]$. $K_0(A)$ is its Grothendieck group, i.e. formal differences of classes of $P(A)$:

$$K_0(A) = \{[p] - [q] \mid p, q \in P(A)\}.$$

It has a natural positive cone:

$$K_0(A)^+ = \{[p] - [0] \mid p \in P(A)\}.$$

Example: $\mathbb{C}$

Consider matrices over $\mathbb{C}$:

Lemma 8. *Two projections p and q in $M_n(\mathbb{C})$ are similar if and only if $rank(p) = rank(q)$.*

Rank is not going to generalize easily to other C^*-algebras, but recall, for a projection $rank(p) = Trace(p)$.

Proposition 9. *The map $Tr : K_0(\mathbb{C}) \to \mathbb{Z}$*

$$Tr([p] - [q]) = Trace(p) - Trace(q)$$

is an isomorphism. Under this, $K_0(\mathbb{C})^+ = \{0, 1, 2, 3, \ldots\} = \mathbb{Z}^+$.

Example: $C(S^2)$

If $p \in M_n(C(S^2))$, then $Trace(p(x))$ is continuous in x. If p is also a projection, its value is integral.

$[p]-[q] \in K_0(C(S^2)) \to Trace(p(x))-Trace(q(x))$

is a homomorphism, but is not injective. There is a projection $p \in M_2(C(S^2))$ such that at every point $p(x)$ is similar to $1 \oplus 0$, but this similarity cannot be made continuous over S^2.

Proposition 10. *If X is totally disconnected, let $C(X,\mathbb{Z})$ be the group of continuous integer-valued functions on X. The function Tr : $K_0(C(X)) \to C(X,\mathbb{Z})$ defined by*

$$Tr([p]-[q])(x) = Trace(p(x)) - Trace(q(x))$$

is an isomorphism. Under this, $K_0(C(X))^+ = C(X,\mathbb{Z}^+)$.

$U \subset X$ clopen, χ_U is a projection in $C(X)$ and also in $C(X,\mathbb{Z})$. The map takes $[\chi_U]-[0]$ to χ_U.

What about dynamics on $C(X)$? $G = \mathbb{Z}$: Pimsner-Voiculescu six-term exact sequences for K-theory of integer actions.

Proposition 11. *For a minimal action of* $\mathbb{Z}$ *on a Cantor set* X, $K_0(C(X) \times_\varphi \mathbb{Z})$ *is isomorphic to*

$$C(X, \mathbb{Z})/\{f - f \circ \varphi \mid f \in C(X, \mathbb{Z})\}$$

and $K_0(C(X) \times_\varphi \mathbb{Z})^+$ *is the image of* $C(X, \mathbb{Z}^+)$.

Inclusion $C(X) \subset C(X) \times \mathbb{Z}$ gives $K_0(C(X)) \cong C(X, \mathbb{Z}) \to K_0(C(X) \times \mathbb{Z})$.

Surjectivity: every projection in $C(X) \times \mathbb{Z}$ is similar to one in $C(X)$.

Let $U \subset X$ be clopen. χ_U is a projection in $C(X)$, but

$$\chi_U \sim_u u_1 \chi_U u_1^* = \chi_U \circ \varphi^{-1} = \chi_{\varphi(U)}.$$

If one replaces $\mathbb{Z}$ by $\mathbb{Z}^d$, $d > 1$, more sophisticated methods (spectral sequences) are needed.

Recall, every φ-invariant measure μ gives a trace τ_μ on $C(X) \times \mathbb{Z}$. This yields a map

$$\hat{\tau}_\mu : K_0(C(X) \times \mathbb{Z}) \to \mathbb{R}.$$

If U is clopen, $\hat{\tau}_\mu[\chi_U] = \mu(U)$.

Theorem 12. *a in $K_0(C(X) \times \mathbb{Z})$ is in $K_0(C(X) \times \mathbb{Z})^+$ if and only if $a = 0$ or $\hat{\tau}_\mu(a) > 0$, for all μ.*

For $d > 1$, the inclusion $C(X) \subset C(X) \times \mathbb{Z}^d$ induces $C(X, \mathbb{Z}) \to K_0(C(X) \times \mathbb{Z}^d)$ which is *not* onto.

Theorem 13 (Gap labelling: B-B-G, B-OO, K-P)**.**

$$\begin{aligned} \hat{\tau}_\mu(K_0(C(X) \times \mathbb{Z}^d)) &= \hat{\tau}_\mu(C(X, \mathbb{Z})) \\ &= \{\mu(U) \mid U \textit{clopen}\} + \mathbb{Z}. \end{aligned}$$

There are some very sophisticated machinery for computing this.

Connes' analogue of the Thom isomorphism:

$$K_i(C(X) \times \mathbb{R}^d) \cong K_{i+d}(C(X)).$$

Can be used in the case $X = \Omega$, the continuous hull. $K_i(C(X))$ is closely related (especially in low dimensions) to the cohomology of X.

However, this isomorphism does *not* respect the order structure on K_0.