C^*-algebras for projection method tilings

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- introduction to aperiodic order
- projection method tilings
- C^*-algebras
- K-theory and KK-theory

References:

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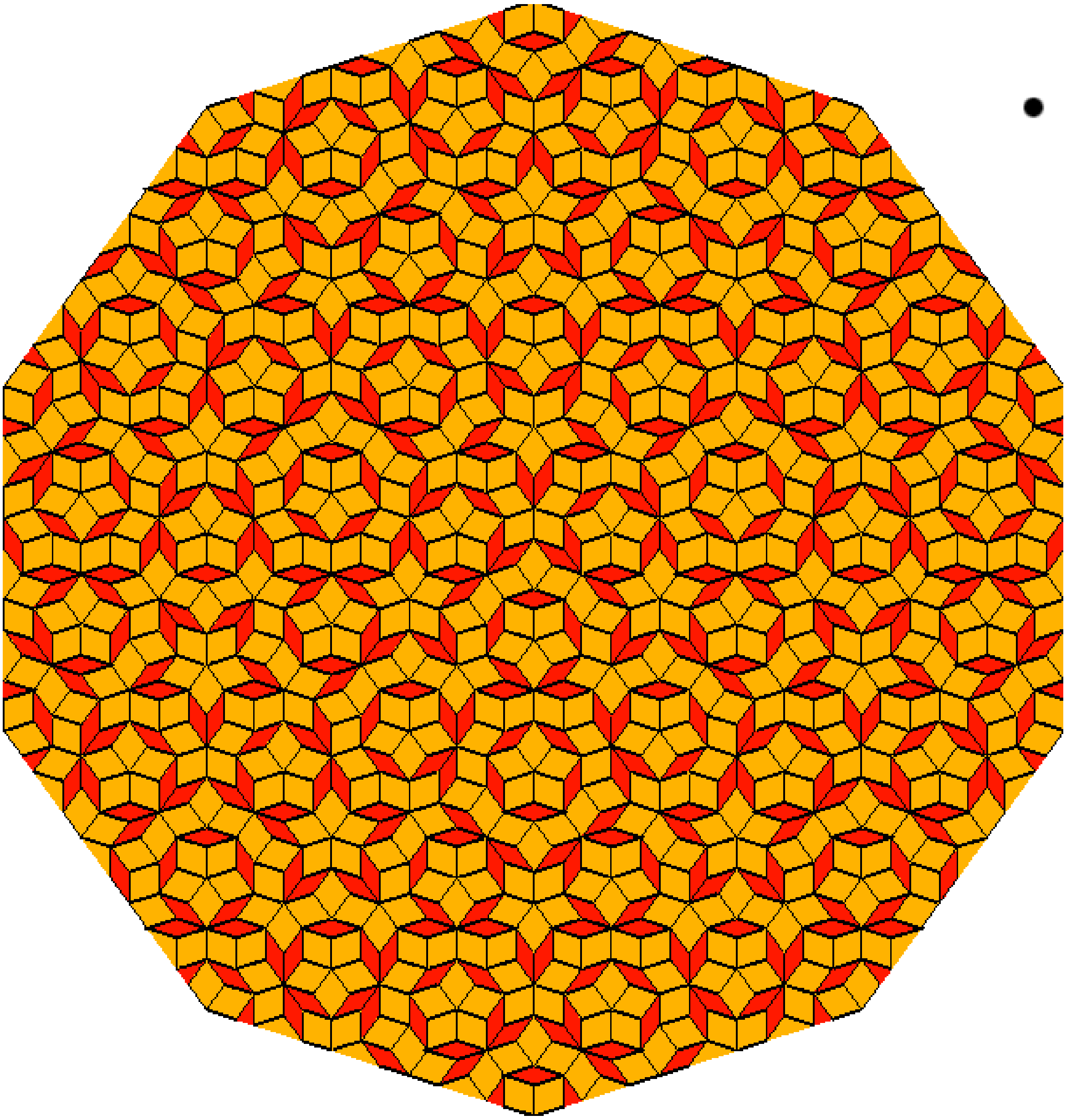
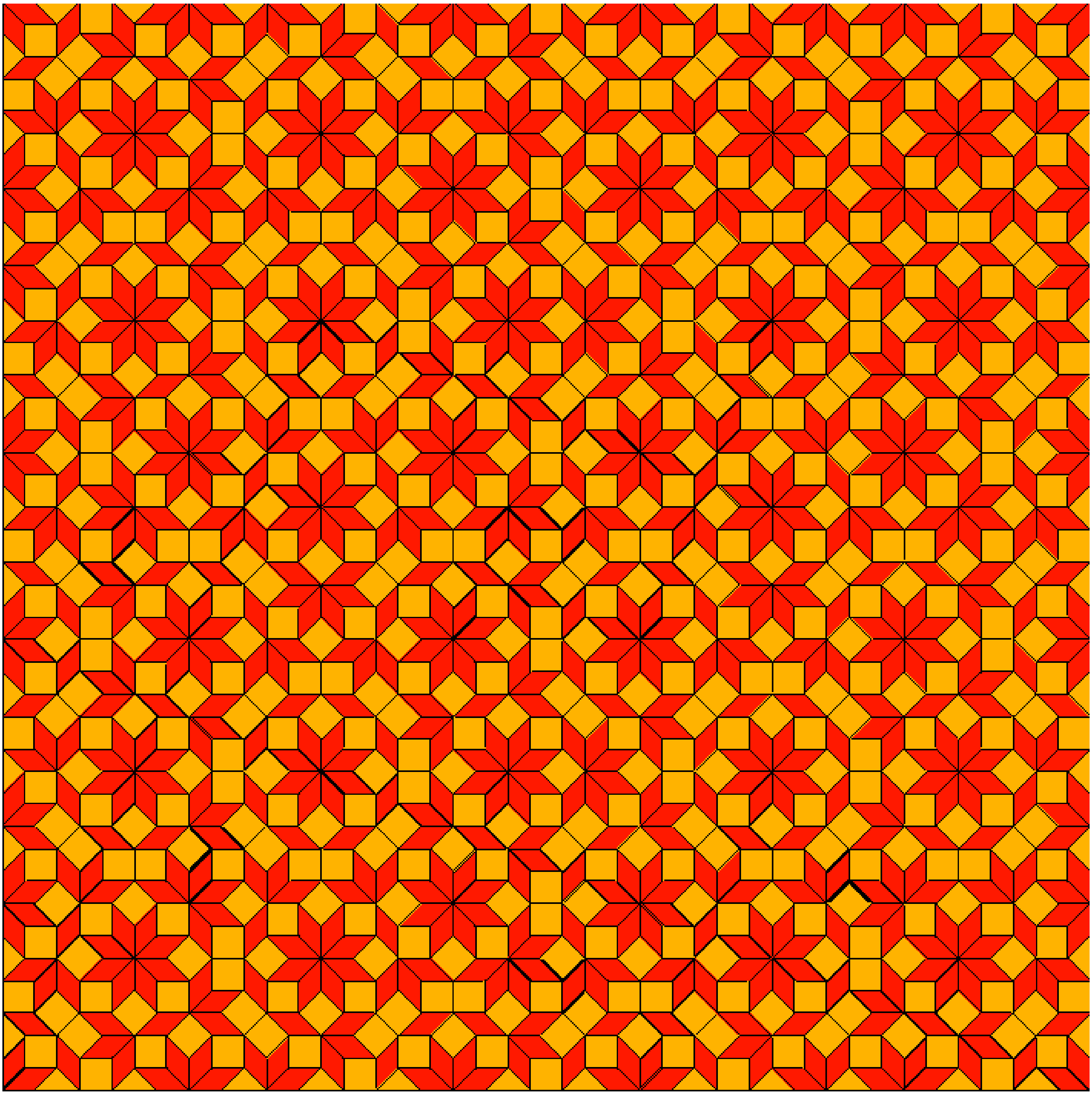
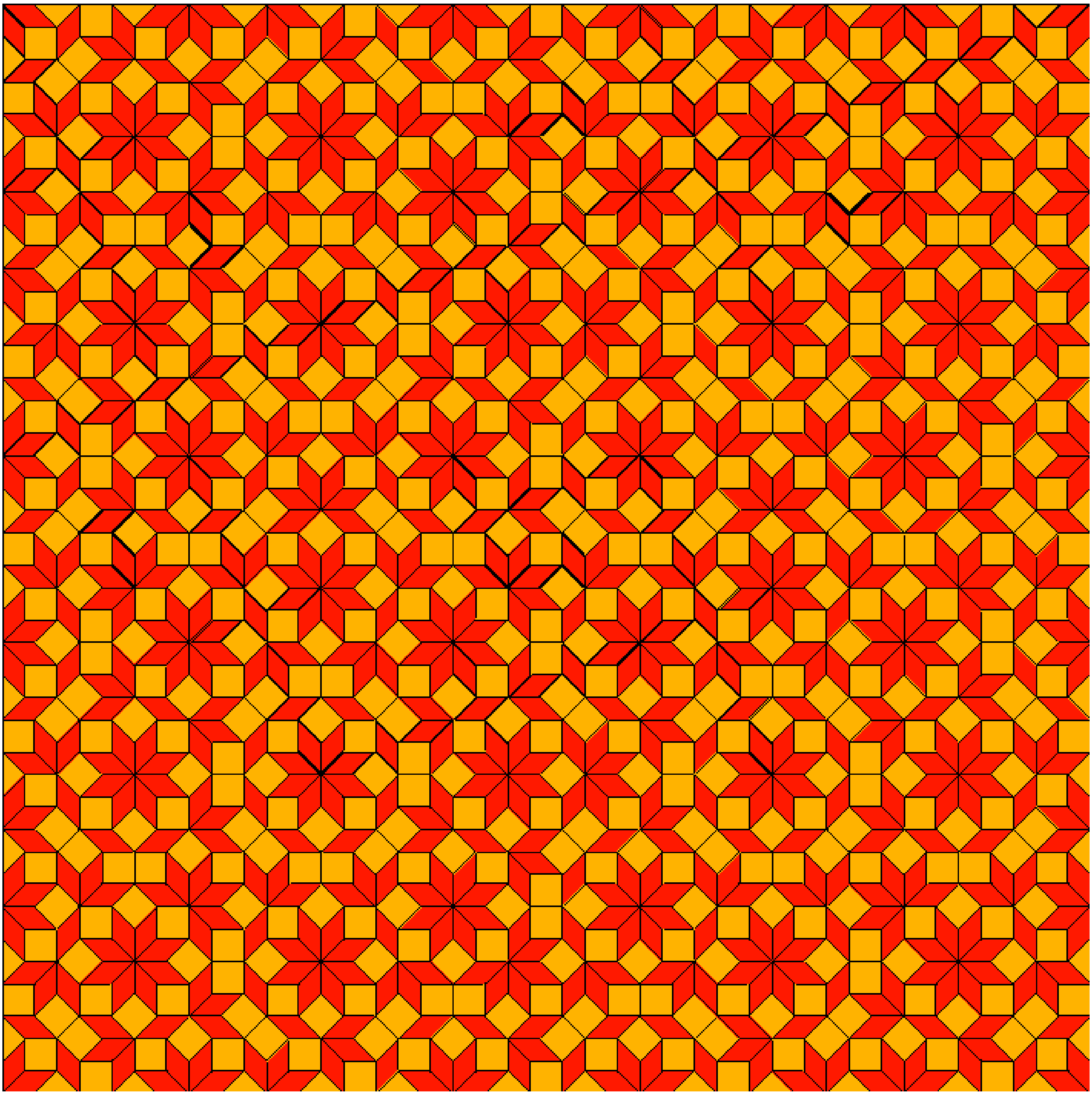


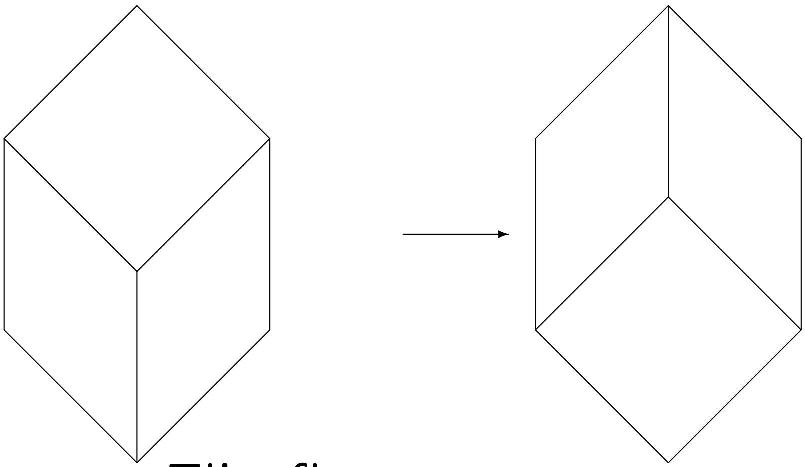
A Penrose tiling



An Octagonal or Ammann-Beenker tiling



Another Octagonal tiling



Flip-flop

Aperiodic Order

In the 1960's and 70's, various geometric patterns in Euclidean space were discovered which displayed a high degree of regularity, but not periodicity.

The most famous are Roger Penrose' tilings. There are an uncountable number of such tilings (even after ignoring translations), but they all have the same highly regular local structure.

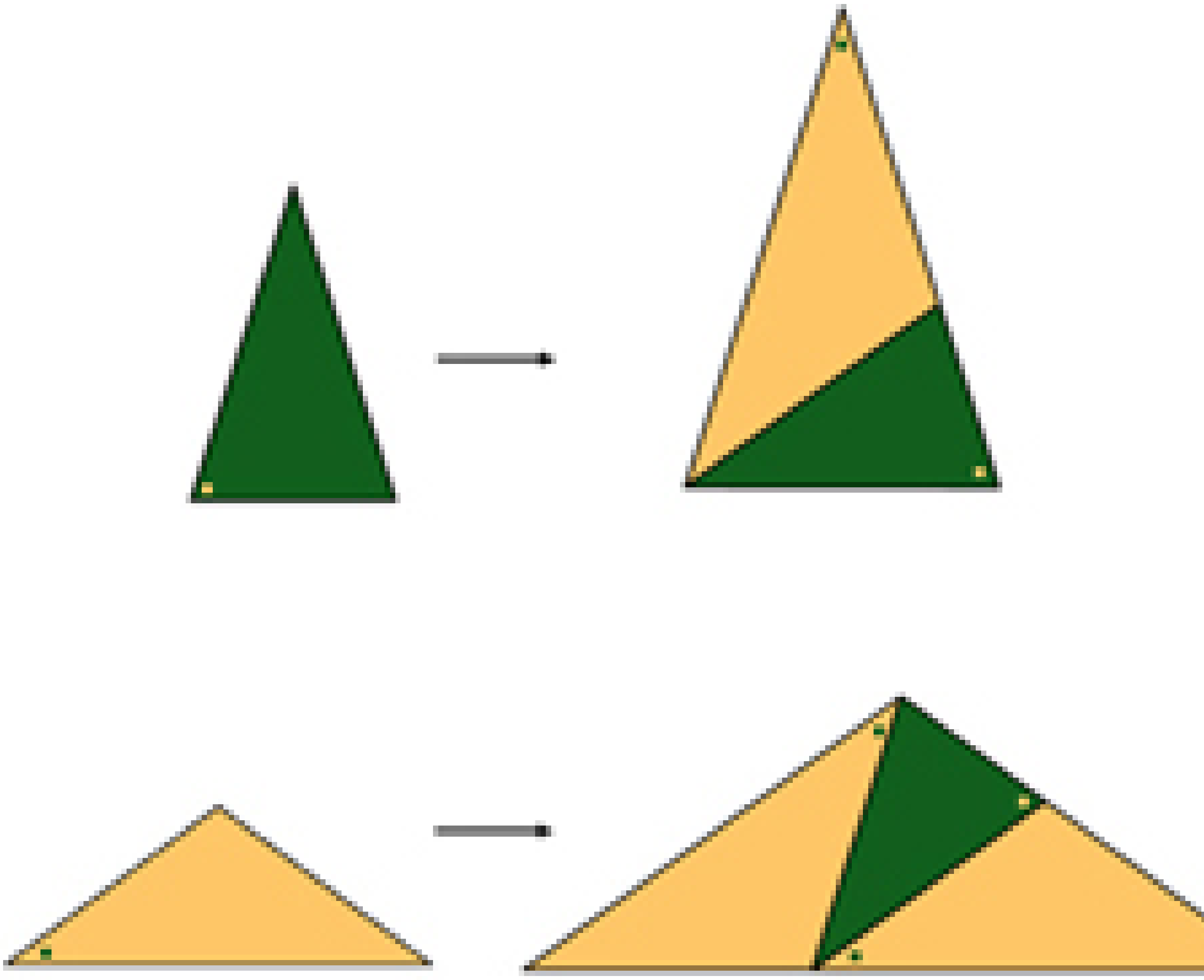
For example: given any finite patch in any Penrose tiling, there is a constant R such that the same patch will appear in any ball of radius R in any other Penrose tiling.

In 1980, Shechtman et. al. discovered physical materials (now called quasicrystals) which display the same kind of aperiodic order: pure-point diffraction patterns, but five-fold rotational symmetry. There are by now several hundred such materials known.

There is no definition of 'Aperiodic order', per se, but there are methods for producing examples. The two most common are the Substitution Method and the Projection Method.

Many famous tilings, such as the Penrose tiling and the Ammann-Beenker or octagonal tiling may be constructed by both methods.

The Penrose substitution:



The projection method

We begin with the data:

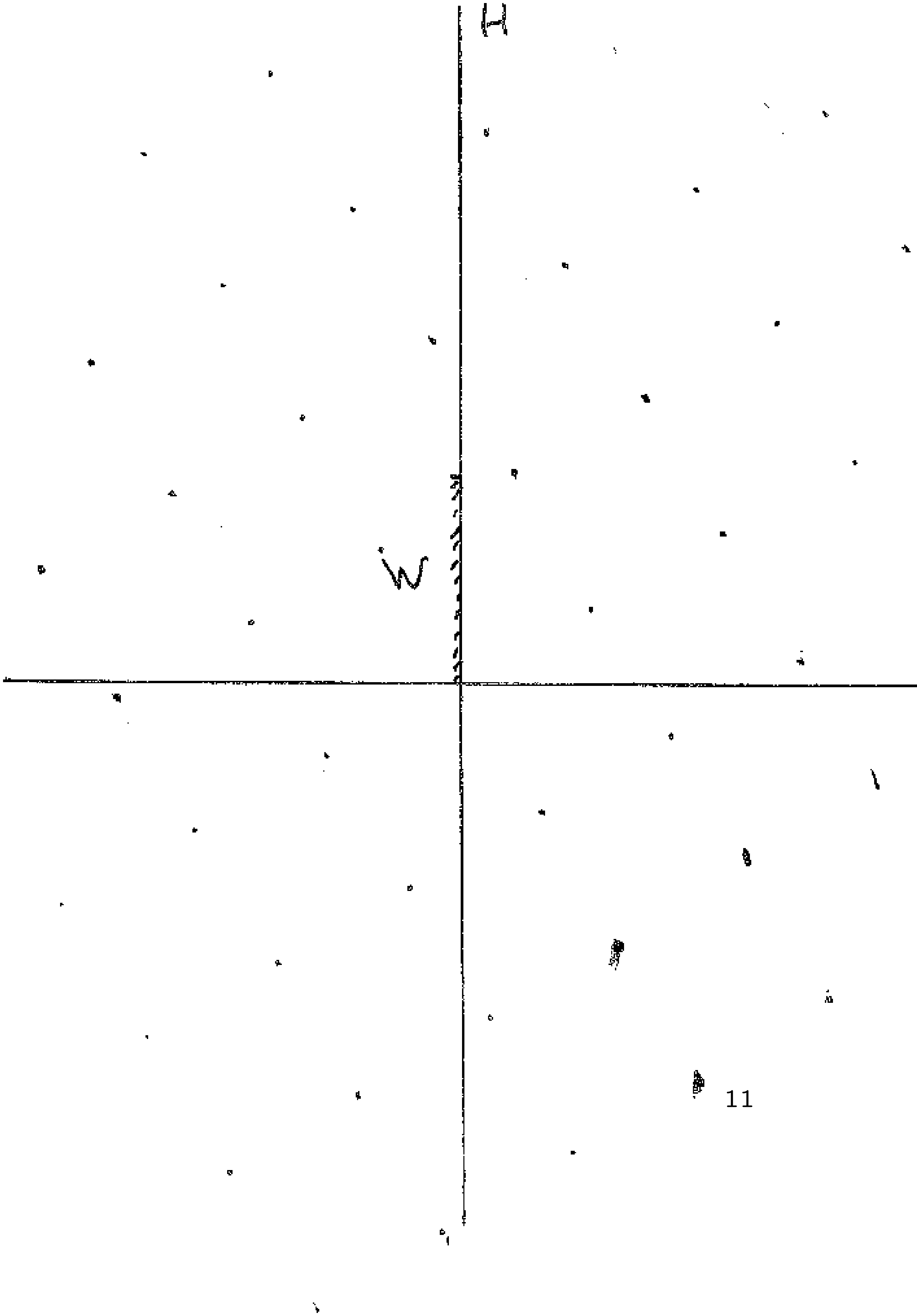
Physical space	$G \cong \mathbb{R}^d$
Internal space	$H \cong \mathbb{R}^N$
Projections	$\pi : G \times H \to G$
	$\pi^\perp : G \times H \to H$
Lattice	$\mathcal{L} \subset G \times H$
Window	$W \subset H$

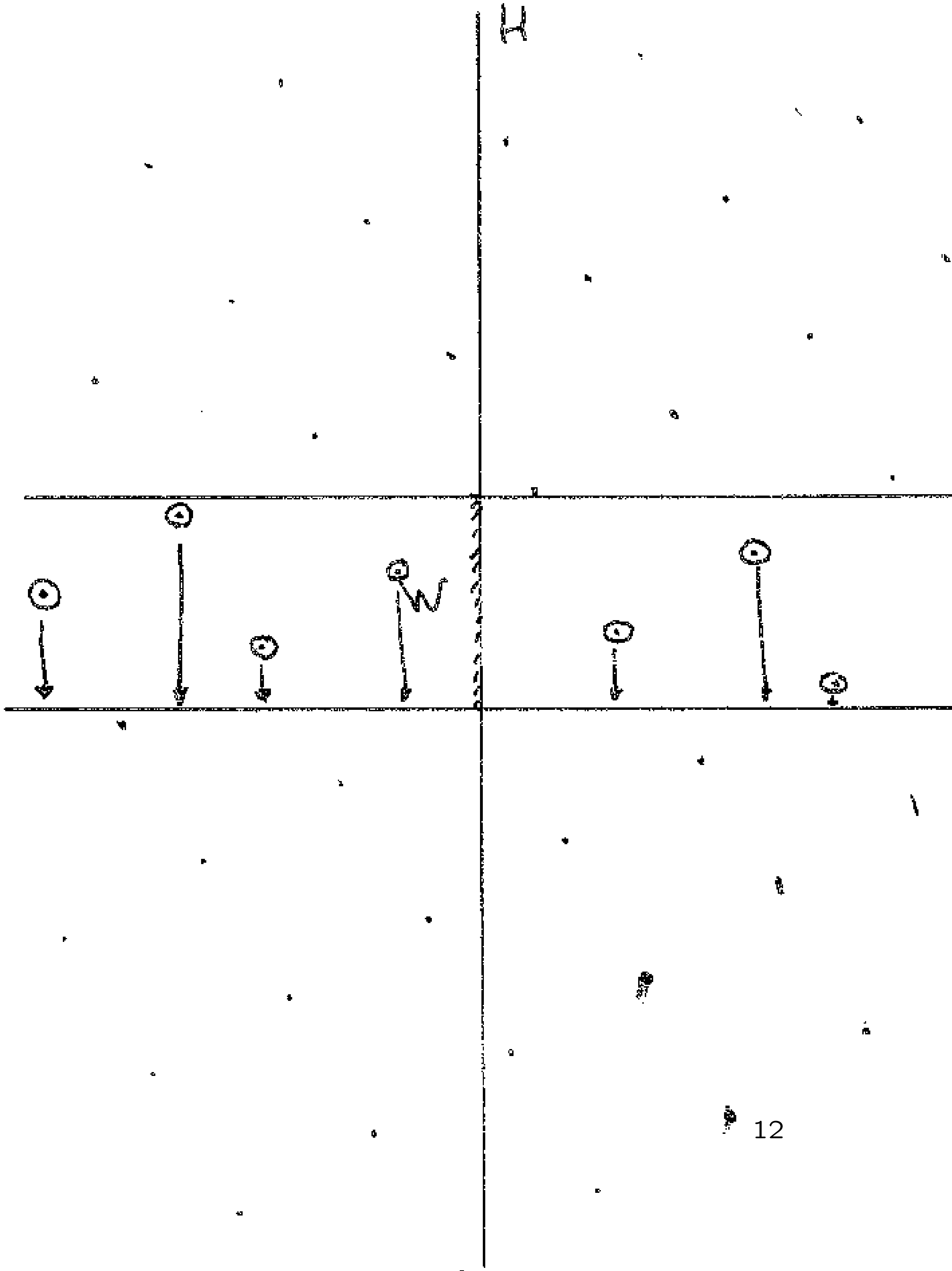
We require

$\pi \mid \mathcal{L}$	injective
$L = \pi^\perp(\mathcal{L}) \subset H$	dense
W	a polyhedron

For x in $G \times H$, define

$$G \supset \Lambda_x = \pi((x + \mathcal{L}) \cap (G \times W)).$$


H
W


H

For x in $G \times H$, define

$$\Lambda_x = \pi((x + \mathcal{L}) \cap (G \times W)).$$

for x *non-singular*: $(x + \mathcal{L}) \cap (G \times \partial W) = \emptyset$. $\mathcal{N} \subset G \times H$ denotes the non-singular points.

The result is a *model set*. It is uniformly dense and uniformly discrete (Delone set). It is aperiodic.

Observe:

$$x + \mathcal{L} = y + \mathcal{L} \Rightarrow \Lambda_x = \Lambda_y$$

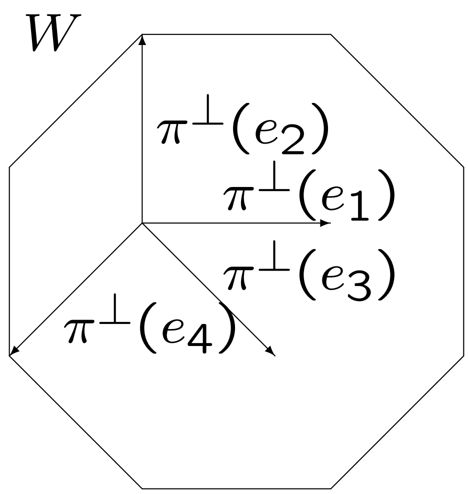
Parameter space: $\mathcal{N}/\mathcal{L} \subset G \times H/\mathcal{L}$.

$$\Lambda_{x+s} = \Lambda_x + s \text{ if } s \in G.$$

Dynamical system with G acting.

Octagonal tilings:

The internal space	$H = \mathbb{R}^2$,
Lattice	$\mathcal{L} = \mathbb{Z}\{e_1, e_2, e_3, e_4\}$,
Projected lattice	$L = \pi^{\perp}(\mathcal{L})$,
Window	W.



We define a metric on $\{\Lambda_x \mid x \in \mathcal{N}\}$:

$$d(\Lambda, \Lambda') < \epsilon$$

if there exist $y, y' \in G$, $|y|, |y'| < \epsilon$ with

$$(\Lambda - y) \cap B(0, \epsilon^{-1}) = (\Lambda' - y') \cap B(0, \epsilon^{-1})$$

The *hull*, Ω, is the completion of $\{\Lambda_x \mid x \in \mathcal{N}\}$ in this metric.

We note:

Ω	is compact,
$\Lambda \in \Omega$	may be seen as a subset, $\Lambda \subset G$
G	acts by translation, minimal and free

We are interested in the crossed product C^*-algebra

$$C(\Omega) \times G.$$

Example:

$G = \mathbb{R}^1 = H$, $\mathcal{L} = R_\theta(\mathbb{Z}^2)$ as before, $W = [0, 1]$.

The origin is singular as it sits on the boundary of W. Consider $\Lambda_{(0,t)}$ for small values of t.

For $t > 0$, $(0, t) \in (0, t) + \mathcal{L}$ is *in* the strip $G \times W$ and hence 0 is in $\Lambda_{(0,t)}$.

For $t < 0$, $(0, t)$ is *not* in the strip and 0 is *not* in $\Lambda_{(0,t)}$.

As t tends to zero from below,

$$\Lambda_{(0,t)} \to \Lambda,$$

where Λ does not containing 0. As t tends to zero from above,

$$\Lambda_{(0,t)} \to \Lambda \cup \{0\}.$$

Consider the C^*-algebra

$$\{f \cdot \chi_W + g \cdot \chi_{H \setminus W} \mid f, g \in C_0(H)\}$$

(as multiplications operators on $L^2(H)$).

The spectrum is just H, with ∂W removed, replaced by two copies, one is a limit of the interior points of W, the other of the exterior points. (Castle with moat.)

Instead, consider

$$C^*\{f, \chi_{W+a} \mid f \in C_0(H), a \in L\}.$$

(Caution: L is a dense subgroup of H!) Let $\tilde{H}(W)$ denote its spectrum. That the C^*-algebra contains $C_0(H)$ means $\tilde{H}(W)$ maps onto H.

Theorem 1. *The space $G \times \tilde{H}(W)$ has continuous actions of $\mathcal{L}$ and G. moreover, we have*

$$(\Omega, G) \cong (G \times \tilde{H}(W)/\mathcal{L}, G),$$

as dynamical systems.

Theorem 2. *The C^*-algebra $C(\Omega) \times G$ is strongly Morita equivalent to $C_0(\tilde{H}(W)) \times L$.*

How to improve on $\tilde{H}(W)$?

Let $\mathcal{P}$ be the set of all affine, codmiension one hyperplanes which form the boundaries of W, along with all their translates by L. For P in $\mathcal{P}$, let P^+ and P^- denote the two half-spaces obtained as $H \setminus P$.

Consider

$$C^*\{f \cdot \chi_{P^\pm} \mid f \in C_0(H), P \in \mathcal{P}\}.$$

Let $\tilde{H}(\mathcal{P})$ denote it spectrum. Each hyperplane in $\mathcal{P}$ is replaced by two.

Proposition 3 (Forrest, Hunton, Kellendonk)**.** *Assume W is the canonical window, namely the projection of the unit cube in $\mathcal{L}$ to H. Then*

$$\tilde{H}(W) = \tilde{H}(\mathcal{P}).$$

Octagonal tiling: $\mathcal{P}$ consists of the L-translates of four lines, the two axes and the two lines through the origin with slopes ± 1. Let $\mathcal{P}_i$ denote the translations under L of the first i of them, $1 \leq i \leq 3$.

We have:

$$\begin{aligned} C_0(H) \times L &\subset C_0(\tilde{H}(\mathcal{P}_1)) \times L \\ &\subset C_0(\tilde{H}(\mathcal{P}_2)) \times L \\ &\subset C_0(\tilde{H}(\mathcal{P}_3)) \times L \\ &\subset C_0(\tilde{H}(\mathcal{P})) \times L. \end{aligned}$$

This allows us to compute the K-theory of these C^*-algebras, working up the chain.

$C_0(H) \times L \subset C_0(\tilde{H}(\mathcal{P}_1)) \times L.$

Let P_1 be the x-axis, $L_1 = L \cap P_1 \cong \mathbb{Z}^2$ and $L = L_1 \oplus L_1'$. So $P_1 + L_1 = P_1$ and $\{P_1 + L_1'\} = \mathcal{P}_1$.

Let i_+, i_- be the two inclusions of P_1 into $\tilde{H}(\mathcal{P}_1)$ as the boundaries of P_1^+ and P_1^-.

For f in $C_0(\tilde{H}(\mathcal{P}_1))$, define $\pi_\pm(f)$ in $C^b(P_1 \times L_1')$ by

$$\pi_\pm(f)(x,s) = f(i_\pm(x) + s).$$

Observe that for f in $C_0(H)$:

$$\pi_+(f\chi_{P_1^+})(x,s) - \pi_-(f\chi_{P_1^+})(x,s) = f(x)\delta_0(s)$$

which is in $C_0(P_1 \times L_1')$.

If we include L-actions, we have

$$\begin{aligned} C_0(P_1 \times L_1') \times L &= C_0(P_1 \times L_1') \times L_1 \times L_1' \\ &\cong C_0(P_1) \times L_1 \otimes \mathcal{K}(l^2(L_1')). \end{aligned}$$

Each map $\pi_\pm$ may be extended to include the action of L and we obtain

$$\begin{array}{rcl}\pi_\pm : C_0(\tilde{H}(\mathcal{P}_1)) \times L & \to & M(C_0(P_1) \times L_1 \otimes \mathcal{K}) \\ Range(\pi_+ - \pi_-) & \subset & C_0(P_1) \times L_1 \otimes \mathcal{K}.\end{array}$$

This means that

$$[\pi_+, \pi_-] \in KK(C_0(\tilde{H}(\mathcal{P}_1)) \times L, C_0(P_1) \times L_1).$$

This map becomes part of a six-term exact sequence

$$\begin{array}{c} K_*(C_0(H) \times L) \\ K_*(C_0(\tilde{H}(\mathcal{P}_1)) \times L) \\ K_*(C_0(P_1) \times L_1) \end{array}$$

The first is (Morita equivalent to) a noncommutative 4-torus and the last to a noncommutative 2-torus.

A similar exact sequence may be used on the inclusion

$$C_0(\tilde{H}(\mathcal{P}_1)) \times L \subset C_0(\tilde{H}(\mathcal{P}_2)) \times L.$$

Continuing this reproduces a spectral sequence first obtained by Forrest-Hunton-Kellendonk in cohomology.

What is special to the situation in the octagonal tiling here is that P_1 is the internal space and L_1 the projected lattice for a one-dimensional projection method tiling: these are the horizontal patterns in the flip-flop.