A brief survey of tiling cohomology

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Tiling cohomology means: how various types of cohomology theories from algebraic topology may be fruitfully used in the study of aperiodic order.

This talk is:

1. Not for experts.

2. Very informal.

3. Not very precise.

How did topology get into tilings?

Periodicity, aperiodicity and almost periodicity of tilings involves translations in some sense. Instead of looking at a single tiling, the dynamics person wants an ensemble of tilings and each translation gives a self-map of this collection.

Success comes when this ensemble can be made into a **finite** measure space or **compact** topological space.

In the case of tilings, this happens in many of cases.

1. Take a single tiling, $\mathcal{T}$, all translates of it, put a metric on them and complete, $\Omega_{\mathcal{T}}$.

2. Take all tilings which are constructed from the same substitution rule, local matching rule, etc, and find a metric on them all, Ω.

The result is called the hull.

What is the cohomology of a space X?

(Be prepared not to like it.)

1. Take a finite open cover $\mathcal{U}$ of X.

2. Associated to $\mathcal{U}$ is a simplicial complex: vertices are the elements of $\mathcal{U}$, edges are non-empty intersections of two elements of $\mathcal{U}$, ...

3. Take the cohomology of the simplicial complex.

4. Refine the open cover, get an inductive system of cohomologies and take the limit.

Can it be done for a hull Ω? What will it tell us?

Is it really that bad?

If we have some polygons, attached to each other along their edges with resulting space X, the computation gets a little easier:

$$\begin{aligned} C^0 &= \{f : \text{vertices} \to \mathbb{Z}\} \\ C^1 &= \{f : \text{edges} \to \mathbb{Z}\} \\ C^2 &= \{f : \text{faces} \to \mathbb{Z}\} \end{aligned}$$

There are maps $\partial_i : C^i \to C^{i+1}$. For $f : vertices \to \mathbb{Z}$,

$$\partial_0(f)(E) = f(t(E)) - f(i(E)),$$

where $i(E)$ and $t(E)$ are the start and end of the edge E. For $f : edges \to \mathbb{Z}$:

$$\partial_1(f)(F) = \sum_{E \text{ an edge of } F} \pm f(E).$$

$H^i(X) \cong \ker(\partial_i)/Im(\partial_{i-1})$.

Can we compute $H^*(\Omega)$?

One very nice property of cohomology: if the space X is an inverse limit:

$$X = \lim X_0 \overset{f_0}{\leftarrow} X_1 \overset{f_1}{\leftarrow} X_2 \overset{f_2}{\leftarrow}$$

then

$$H^*(X) = \lim H^*(X_0) \overset{f_0^*}{\rightarrow} H^*(X_1) \overset{f_1^*}{\rightarrow} H^*(X_2) \overset{f_2^*}{\rightarrow}$$

This helps! Tiling spaces are inverse limits: Anderson-Putnam (Substitutions), Bellissard-Benedetti-Gambaudo, Gähler-Sadun.

For substitutions, the computations can be done! X_n is the same for all n: take all the tiles, attach one to another if they ever appear attached in that way in a tiling and f_n is just the substitution map. (With border forcing.)

Penrose: $H^0(\Omega) \cong \mathbb{Z}, H^1(\Omega) \cong \mathbb{Z}^5, H^2(\Omega) \cong \mathbb{Z}^8$.

Computing cut-and-project examples

A machine for computing cohomology for cut-and-project systems was developed by Forrest, Hunton and Kellendonk. The key new data is the torus parameterization:

$$\pi : \Omega_{\mathcal{T}} \to \mathbb{T}^{d+N}.$$

Works effectively for the standard window, need the information of where the faces of the $d+N$ cube intersect. The answer is given in terms of a spectral sequence.

For both substitutions and cut-and-project systems, Franz Gähler has produced very impression computer calculations ($H^k(\Omega_{\mathcal{T}}) \cong \mathbb{Z}^{1200}$!).

Why compute $H^*(\Omega)$?

Short answer: $H^*(\Omega)$ is (alleged to be) a quantitative measure of aperiodicity.

Homology vs. cohomology and the periodic case

Suppose that $\mathcal{T}$ a completely periodic tiling of $\mathbb{R}^d$. Let

$$Per(\mathcal{T}) = \{x \in \mathbb{R}^d \mid \mathcal{T} - x = \mathcal{T}\}.$$

$\Omega_{\mathcal{T}}$ is all translations of $\mathcal{T}$ and is $\mathbb{R}^d / Per(\mathcal{T})$.

$H_1(\Omega_{\mathcal{T}})$ consists of loops in $\Omega_{\mathcal{T}}$. How do you find a loop of tilings? Suppose x is in $Per(\mathcal{T})$. Then

$$\mathcal{T}^x(t) = \mathcal{T} - tx, 0 \leq t \leq 1,$$

is a loop of tilings since $\mathcal{T}^x(0) = \mathcal{T}^x(1)$. In fact,

$$x \in Per(\mathcal{T}) \to \mathcal{T}^x \in H_1(\Omega)$$

is an isomorphism.

What happens if $\mathcal{T}$ is aperiodic? $H_1(\Omega) =$???, but $H^*(\Omega)$ is still interesting.

A De Rham theorem

Let $\mathcal{T}$ be a tiling of $\mathbb{R}^N$. A function $f : \mathbb{R}^N \to A$ is $\mathcal{T}$-equivariant if, there is a constant $R > 0$ such that, for any x, y in $\mathbb{R}^N$,

$$\begin{array}{rrcl} & (\mathcal{T} - x) \cap B(0, R) & = & (\mathcal{T} - y) \cap B(0, R) \\ \Rightarrow & f(x) & = & f(y). \end{array}$$

Let $C^k_{\mathcal{T}}$ denote the set of all smooth differential forms of degree k on $\mathbb{R}^N$ which are $\mathcal{T}$-equivariant.

$$\begin{array}{rcl} C^0_{\mathcal{T}}(\mathbb{R}^2) & = & \{f(x, y), \mathcal{T} - \text{equivariant}\} \\ C^1_{\mathcal{T}}(\mathbb{R}^2) & = & \{P(x, y)dx + Q(x, y)dy, \mathcal{T} - \text{equiv.}\} \\ C^2_{\mathcal{T}}(\mathbb{R}^2) & = & \{g(x, y)dxdy, \mathcal{T} - \text{equivariant}\} \end{array}$$

Notice $d : C^k_{\mathcal{T}} \to C^{k+1}_{\mathcal{T}}$. Let

$$H^k_{\mathcal{T}}(\mathbb{R}^N) = \ker(d)/Im(d).$$

J. Kellendonk -P.:

$$H^*_{\mathcal{T}}(\mathbb{R}^N) \cong H^*(\Omega_{\mathcal{T}}, \mathbb{R}).$$

Shouldn't these invariants be geometric?

For the Penrose tilings, $H^1(\Omega) \cong \mathbb{Z}^5$; doesn't look like a quantitative measure of aperiodicity.

If ω is in $C^k_{\mathcal{T}}$, we can take

$$\tau(\omega) = \lim_R vol(R)^{-1} \int_{|x| \leq R} \omega(x) dx \in \Lambda^k(\mathbb{R}^N)$$

We get, in particular,

$$H^1(\Omega_{\mathcal{T}}) \to H^1(\Omega_{\mathcal{T}}, \mathbb{R}) \cong H^1_{\mathcal{T}} \xrightarrow{\tau} (\mathbb{R}^N)^* \cong \mathbb{R}^N.$$

In the Penrose case, the image is generated by the fifth-roots of 1. (This subgroup of $\mathbb{R}^2$ is rank 4, so the map has $\mathbb{Z}$ as a kernel.)

If $\mathcal{T}$ is completely periodic, then the image of $H^1(\Omega_{\mathcal{T}})$ is the dual lattice of $Per(\mathcal{T})$.

Periodic $\Rightarrow$ lattice. Aperiodic $\Rightarrow$ dense in $\mathbb{R}^N$?

Clark & Sadun: Look at $H^1(\Omega, \mathbb{R}^d)$.

Recall Ω is an inverse limit: X_0 assembled from the polyhedra in the tiling; it codes the combinatorics, but not the geometry.

Recall $C^1 = \{f : Edges \to \mathbb{R}^d\}$. The tiling itself does this! It is the geometry of the tiles. What does $\partial_1 f = 0$ mean? At every face F,

$$0 = \partial_1 f(F) = \sum_{E \subset F} \pm f(E).$$

The edges sum to zero just means that these vectors form the boundary of a tile.

Small elements of $\ker(\partial_1) \subset C^1$ determine a deformation of the tiling $\mathcal{T}$. The new tiling is mutually locally derivable with the original if and only if the element is a co-boundary; i.e. it is zero in H^1.