The Pareto, Zipf and other power laws.

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Abstract.

Many empirical size distributions in economics and elsewhere exhibit power-law behaviour in the upper tail. This article contains a simple explanation for this. It also predicts lower-tail power-law behaviour, which is verified empirically for income and city-size data.

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Many empirical distributions encountered in economics and other realms of inquiry exhibit power-law behaviour. In economics prime examples are the distributions of incomes (Pareto's law) and city sizes (Zipf's law or the rank-size property), as well as the standardized price returns on individual stocks or stock indices. Elsewhere empirical size distributions for which power-law behaviour has been claimed include those of sand particle sizes; of meteor impacts on the moon; of numbers of species per genus in flowering plants; of frequencies of words in long sequences of text and of areas burnt in forest fires *etc*. This widespread observed regularity has been explained in many ways. It continues to fascinate both natural scientists, who have recently proposed explanations based on current ideas such as self-organized criticality and highly optimized tolerance (*e.g*. Newman, 2000), as well as economists, as recent papers by Gabaix,

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(1999,) and Brakman *et al*., (1999) testify. While it seems unlikely that there is a single general theory that could explain all instances of power-law behaviour, there is I claim, a simple, plausible explanation which has apparently been overlooked and which can explain many examples in economics (including the Pareto and Zipf laws) and other areas. This is outlined below.

The temporal evolution of many phenomena exhibiting power-law behaviour, is often considered to involve a varying, but size independent proportional growth rate, which mathematically can be modelled by geometric Brownian motion (GBM),

$$dX = \mu X\,dt + \sigma X\,dw \qquad (1)$$

where dw is white noise or the increment of a Wiener process. Thus for GBM the proportional increment in X in time dt comprises a systematic component μdt and a random component $\sigma\, dw$ and GBM can be seen to be a stochastic version of simple exponential growth. Phenomena frequently modelled by GBM include the evolution of stock prices (e.g. for the Black-Scholes theory of option pricing); firm sizes, city sizes and individual incomes (all coming under the rubric of Gibrat's law of proportional effects). It is well known that the state of a GBM after a fixed time T follows a lognormal distribution, which does not exhibit power-law behaviour. Why then should power-law behaviour occur for phenomena evolving as GBM?

I claim that the answer lies in the fact that the time of observation, T, should itself be regarded as a random variable, often with a distribution close to an exponential distribution. Consider for example a census or sample survey of incomes. Even though each individual income may follow GBM, the time during which it has been so evolving will vary from individual to individual. If recruitment to the workforce has been growing at a more or less constant rate, the distribution of time in the workforce of any individual will follow an exponential distribution (truncated at the retirement age). Thus, provided all income earners had the same starting income (numeraire one say), the current distribution of incomes should be that of a GBM observed after an exponentially distributed time T. This distribution is what I call a *double Pareto distribution*, with a density proportional to x^{-a-1} for $x>1$ and proportional to x^{b-1} for $x<1$ (for details and proof see the Appendix). More realistically individual starting incomes will also vary and evolve over time, say as another GBM. In this case current incomes can be represented as the product of a double Pareto random variable with an independent lognormal random variable, and will exhibit power law behaviour in both tails (for details see Reed, 2000 a). Thus, not only does this simple model offer a plausible explanation of the Pareto Law of Incomes (upper tail), it also predicts power-law behaviour in the lower tail. Does such behaviour occur? The figure below (left-hand panels, both axes logarithmic) for data on all U.S. male income earners in 1998 (U.S. Census Bureau, 1999) suggests that power law behaviour occurs in both tails. Indeed lower-tail power-law behaviour has been identified before (Champernowne, 1953) but is not apparently widely recognized.

Consider now the distribution of human settlement sizes. If sizes evolve as GBM (Gabaix, 1999) and the foundation of settlements follows a random process in which, at any time, any existing settlement can create a satellite settlement, with the same probability per unit time - a Yule process (Yule, 1924), first proposed as a model of speciation - then it can be shown that, provided that the time since the original 'Ur' settlement is large, that the time since foundation of any settlement currently in existence will be approximately exponentially distributed. Thus the numeraire size of current settlements should follow the double Pareto distribution. Allowing for variation in initial sizes will modify this somewhat (Reed, 2000 b) but one would still expect power law behaviour in both tails. Power-law behaviour in the upper tail is well documented, but apparently again not for the lower tail. But it does indeed occur, as the right-hand panels, of the figure below testify. These show the cumulative frequencies of the largest 5000 and smallest 5000 settlements of over 38,000 listed for the U.S.A in 1998[2]

Another example, outside of economics, for which a similar explanation might hold, concerns the body-size distribution of animal species (*e.g.* May, 1988). Here it would be assumed that the body mass of any individual species evolved through natural selection following GBM, while speciations occurred in a Yule process. Power-law behaviour in the lower tail of particle-size distributions and in the upper tail of forest-fire size distributions could also be explained in a similar way. In the former case repeated random fractures indicate a form of random geometric decay, while in the latter case the area burnt might follow random proportional growth through time until stopped at random (*e.g.* by the onset of suitably heavy rainfall).

No doubt there are many other examples fitting within this paradigm, whose essential elements are random proportional (geometric) change and random stopping or observation. For example if new stock issues occurred in a Poisson process and individual stock prices evolved following GBM, one might expect that the distribution of the ratio of current price to issue price over all such stocks would follow a power law in each tail.

In summary, phenomena evolving according to Gibrat's law, which are observed after an exponentially distributed period of time should be expected to exhibit distributions with power-law tail behaviour.

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[2] Data from U.S. Census Bureau website http://www.census.gov/population/www/estimates

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Appendix. Derivation of the double Pareto distribution.

For the GBM (1) above, with initial state X_0, the state a *fixed* time T later is lognormally distributed with

$$Y(T) = \log(X(T)) \sim N(X_0 + (m - s^2/2)T, s^2 T) \qquad \text{(A1)}$$

which has moment generating function (m.g.f)

$$M_{Y(T)}(s) = E(e^{Y(t)s}) = \exp(X_0 s + [(\mu - \sigma^2/2)s + ½\sigma^2 s^2]T).$$

Now consider T to be a random variable, exponentially distributed with probability density function $f_T(t) = \lambda e^{-\lambda t}, t \geq 0$, and m.g.f $M_T(s) = \dfrac{\lambda}{\lambda - s}$. The state $\bar{X} = X(T)$ after the random time T will no longer be lognormally distributed. Rather its logarithm $\bar{Y}$ will have m.g.f.

$$M_{\bar{Y}}(s) = E(e^{\bar{Y}s}) = E_T(E(e^{Y(T)s} \mid T)) = E_T(M_{Y(T)}(s)) \qquad \text{(A2)}$$

which (from A1) and the expression above for the m.g.f. of T can be written

$$M_{\bar{Y}}(s) = \frac{\lambda e^{X_0 s}}{\lambda + (\mu - ½\sigma^2)s - ½\sigma^2 s^2}$$

Letting α and $-\beta$ $(\alpha, \beta > 0)$ denote the two roots of the characteristic quadratic equation

$$½\sigma^2 z^2 + (\mu - ½\sigma^2)z - \lambda = 0 \qquad \text{(A4)}$$

this can be written

$$M_{\bar{Y}}(s) = e^{X_0 s}\frac{\alpha\beta}{(\alpha - s)(\beta + s)} \qquad \text{(A5)}$$

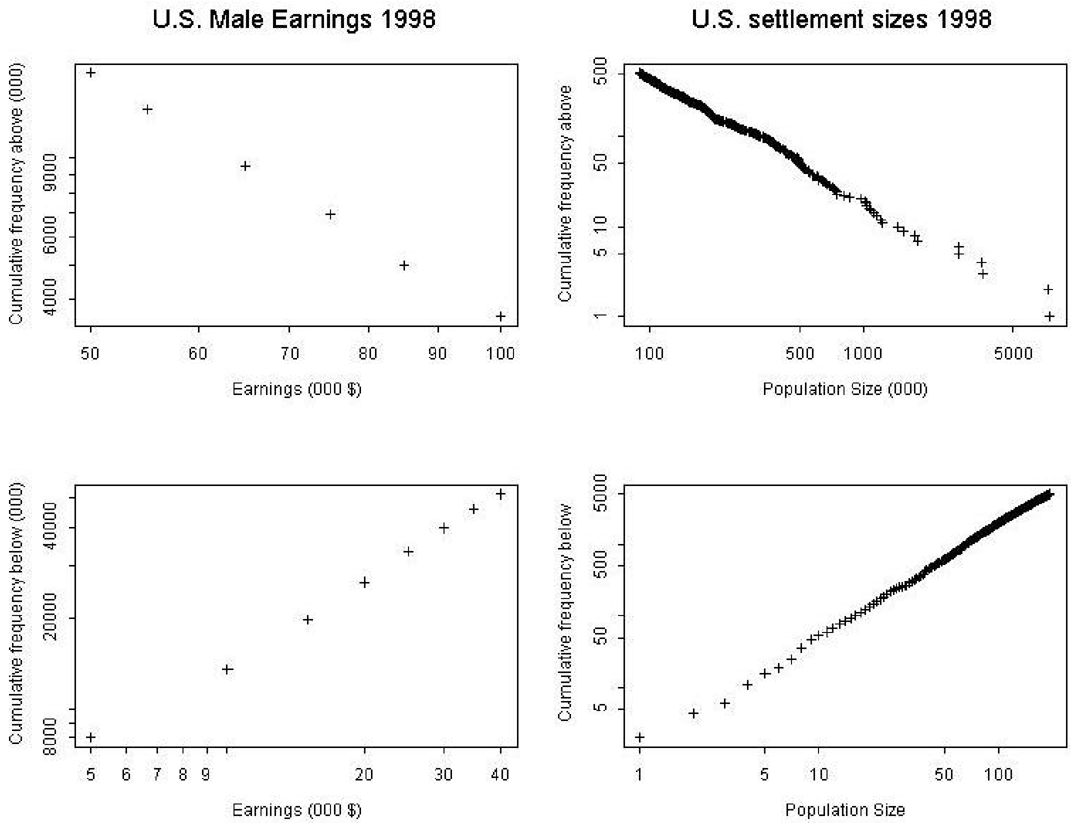
It can be verified (e.g. by direct computation) that (A5) is the m.g.f of the asymmetric Laplace distribution centered on $Y_0 = \log(X_0)$ with probability density function

$$f_{\bar{Y}}(y) = \begin{cases} \dfrac{\alpha\beta}{\alpha+\beta} e^{\beta(y - Y_0)} & , y < Y_0 \\ \dfrac{\alpha\beta}{\alpha+\beta} e^{-\alpha(y - Y_0)} & , y \geq Y_0 \end{cases} \qquad \text{(A6)}$$

From this it follows that the distribution of $\bar{X}$ has probability density function

$$f_{\bar{X}}(x) = \begin{cases} \dfrac{\alpha\beta}{\alpha+\beta}\left(\dfrac{x}{X_0}\right)^{\beta-1} & , x < X_0 \\ \dfrac{\alpha\beta}{\alpha+\beta}\left(\dfrac{x}{X_0}\right)^{-\alpha-1} & , x \geq X_0 \end{cases}$$

which I call the double-Pareto distribution and which exhibits power-law behaviour in both tails. Note that the Pareto exponents a and b depend on the mean drift and diffusion parameters m and s of the GBM along with the parameter l of the exponential distribution of *T*, via the characteristic equation (A4). It is not difficult to show that both a and b decrease with increases in m and l.



Plots revealing power-law behaviour in both tails of 1998 U.S. male earnings distribution (left-hand panels) and 1998 U.S. settlement size distribution (right-hand panels). The top row, with plots of the cumulative frequency (ordinate - log scale) above a given level (abscissa - log scale) demonstrates the upper-tail power-law behaviour, long recognised in the laws of Pareto and Zipf. The bottom row (similar but with cumulative frequency below a given level) illustrates the lower-tail power-law behaviour predicted by the model.