On Vlasov-Fokker-Planck Type Kinetic Models for Multilane Traffic Flow

Reinhard Illner*, Axel Klar† and Thorsten Materne†

*Department of Mathematics and Statistics, University of Victoria, P.O. Box 3045 STN CSC, Victoria, B.C. V8W 3P4
†FB Mathematik, TU Darmstadt, Schlossgartenstr. 7, 64289 Darmstadt, Germany

Abstract. We discuss rationales for kinetic descriptions of traffic dynamics and present a class of new models of Vlasov-Fokker-Planck type. These models incorporate (nonlocal and time-delayed) braking and acceleration terms which are consistent with realistic time scales. Correlation assumptions are made such that braking and acceleration terms depend only on macroscopic densities and the relative speeds with respect to the average speed; the braking term includes lane change probabilities, and reasonable assumptions on the dependencies of these probabilities on the traffic situation lead to multivalued fundamental diagrams, consistent with traffic observations.

INTRODUCTION

The use of kinetic models for traffic dynamics is controversial because of the arising scales and the interpretative difficulties. It is clear that kinetic predictions in traffic flow can only be interpreted statistically: how else can one reasonably talk about “the number of cars at location x moving at speeds within $[v_1, v_2]$?” For traffic on one-lane roads, where passing is impossible, it may well be true that microscopic models (each car is described by a delay-differential equation, with a coupling term) or macroscopic equations of conservation type are sufficient for realistic predictions.

We give two reasons for the use of kinetic models in multi-lane scenarios: 1) They offer great flexibility and convenience in the incorporation of hypotheses regarding driver behaviour, and 2) They are natural stepping stones in the derivation of macroscopic models. As a possible third reason one can point out that for traffic flow kinetic models are also feasible for numerical purposes, because position and speed are just one-dimensional variables.

The first kinetic model for traffic flow was suggested by Prigogine and Herman [1], followed several years later by a model proposed by Paveri-Fontana [2]. More recently, Klar and Wegener [3, 4, 5, 6] have introduced kinetic equations of Enskog type for multilane traffic. In the present article we report about a very recent approach to simulate multilane traffic by coupled equations of Vlasov-Fokker-Planck type. These models were introduced in order to resolve scaling difficulties present in Enskog- and other Boltzmann-type models discussed below. In Section 2 we present the new models, formulated for two-lane traffic. The equations include braking and acceleration terms with appropriate nonlocalities and time delays; the braking terms include the lane change probabilities which should depend on the traffic state ahead of the driver (on the same lane) and on the neighbouring lane.

Mathematical models of multilane traffic flow should have a number of properties, some intrinsic in the modeling process, some displayed in the solution patterns predicted by the model. We suggest the following list of basic properties.

1. The scales used in the model for independent and dependent variables, such as position, speed, acceleration, vehicle density and flux, should be realistic and be at realistic ratios.
2. The model should allow trivial equilibria $f(v) = \rho\delta_u(v)$ (all cars travel at the same speed u). These equilibria exist from a fundamental point of view; in certain density-flow regimes they are effectively observed [7, 8, 9, 10].
3. (a) The model should allow the calculation of equilibrium fundamental diagrams $\rho \to u^e(\rho)$ or $\rho \to j^e(\rho) = \rho \cdot u^e(\rho)$.
   (b) At least on multilane highways and within certain density regimes these fundamental diagrams should be multivalued (suggested by traffic observations). It is speculated that bifurcations of this type are linked to

the lane-changing/braking behavior of the drivers.

4. The model should give rise to phase transitions/stop-and-go traffic (presumably related to the bifurcations mentioned in 3(b)).

This list is probably incomplete. Property 1 is basic but of central importance. Property 2 is also basic; it also is consistent with the observation (see [7, 8, 9, 10]) that within certain density regions traffic tends to “synchronize,” i.e., settles at a trivial equilibria. For higher density values these synchronized equilibria give way to moving traffic jams, consistent with property 4. Finally, the fundamental diagrams postulated by property 3(a), and their multi-valued character given by 3(b), are delivered at the simplest level by traffic data collection. They should therefore be computable from the model.

A brief review of other models

Basic “follow-the-leader” models (systems of coupled ordinary delay-differential equations in which each car is represented by its own equation) can certainly have properties 1 and 2; at the simplest level equilibrium fundamental diagrams can be computed with trivial equilibria in mind.

For macroscopic models, i.e., systems of (usually 2) first order partial equations, one for density (continuity equation) and one for flux or mean speed, property 1 can be enforced, but properties 2 and 3 are problematical. In fact, one usually needs an equilibrium fundamental diagram $\rho \to u^e(\rho)$, as well as relaxation times $T^e(\rho)$, from the outset, as the right-hand-side of the speed equation is $\frac{1}{T^e(\rho)}(u^e(\rho) - u)$. The properties of such models (as the most recent we mention the model introduced by Aw and Rascle [11]) depend then largely on the choices of $u^e(\rho)$, $T^e(\rho)$, and on other functions entering the model parametrically.

Kinetic models allow the computation of traffic patterns (such as fundamental diagrams) from basic assumptions about driver behaviour. The most recent models of kinetic type are the Enskog-type models introduced and discussed by Klar, Wegener et al. in ([4, 5]). They are of the type

$$\partial_t f + v \cdot \partial_x f = C^+(f), \tag{1}$$

where C^+ denotes an interaction operator of Boltzmann-Enskog type, with appropriate nonlocalities. As shown in [12] these models may be refined to have properties 2-4. Unfortunately, as pointed out recently in [13], they violate property 1: In every model of type (1) where $C^+(f)$ is a “collection” (interaction) operator of Boltzmann, Enskog or neutron transport type, one implicitly assumes that the interaction itself, and the velocity adjustment of the following car are instantaneous. This assumption is acceptable for microscopic particles like atoms or electrons, but it is not true for vehicles, where a braking or acceleration time in typical density is of the same order of magnitude as the “free driving time” in between interactions.

It has therefore been suggested in [13] that kinetic traffic models which possess property 1 should be of Vlasov-Fokker-Planck type; in such equations, the “force” terms are scalable and speed-dependent. Uncertainty in driver behaviour should give rise to diffusion terms.

THE VLASOV-FOKKER-PLANCK MODELS

We consider 2-lane traffic, where the traffic lanes are labeled 1 or 2, respectively. The kinetic traffic density on lane i at location x and time t will be $f_i(x,v,t)$. Macroscopic density $\rho_i(x,t)$, flux $j_i(x,t)$ and average speed $u_i(x,t)$ are then given by

$$\rho_i(x,t) = \int_0^{v_{\max}} f_i(x,v,t)\,dv, \qquad j_i(x,t) = \int_0^{v_{\max}} v f_i(x,v,t)\,dv,$$
$$u_i(x,t) = (j_i/\rho_i)(x,t).$$

Sometimes f_i is factored as $f_i = \rho_i(x,t) \cdot F_i(x,v,t)$, where F_i is the probability density in v of case at (x,t).

Real observations suggest that there are (at least) 3 different kinds of reaction times to be considered: The average individual reaction time $\tau > 0$, and the reaction times T_B and T_A used for the estimation of braking and acceleration thresholds. This means that a driver at x with speed v will brake in reaction to slower traffic seen at $x + H_B(v)$, where

$H_B(v) = H_0 + T_B v$ and H_0 is the average car length; moreover, the brake reaction will occur with delay τ, i.e., the reference point in (x,t)-space is $(x + H_B(v), t - \tau)$. Similar considerations apply to acceleration, and observations suggest that

$$0 < \tau < T_B < T_A < \infty.$$

For $i = 1, 2$ and $k = 3 - i$ we now write the *general* evolution equation for the traffic density f_i as

$$\begin{aligned}\partial_t f_i + v\partial_x f_i + \partial_v \quad & (B_i(\ldots) f_i - D_i(\ldots)\partial_v f_i) \\ & = p_k(\ldots) f_k - p_i(\ldots) f_i \end{aligned} \tag{2}$$

Here, B_i, D_i, and p_i stand for the braking/acceleration force, the diffusion coefficient and the lane changing rate(s), respectively. The details of the model are contained in the dependencies which we will assume for these quantities.

First we remark that it would be desirable to produce a (more) rigorous derivation from first principles, i.e., a derivation of (2) from an N-car Liouville equation and hierarchy equations. The problem with this is that one needs to postulate interaction rules between drivers (cars) in any case, and empirical input is unavoidable at that level. Secondly, any attempt of deriving (2) via hierarchy equations will at some point reach the difficulty of expressing many-car distribution densities in terms of the one-car density f_i, and at that point one has to make "reasonable" correlation assumptions; it is clear that the usual hypothesis of molecular ("vehicular") chaos cannot be used in the present case and must be corrected with correlation factors (see [4, 12] for examples on this), which will again incorporate empirical observations.

In view of these difficulties we base our assumptions on B_i, D_i and p_i on empirical guidelines. We let ourselves be guided by the properties 1-4 listed in the introduction. When there is no danger of confusion we will sometimes omit the index i.

Lane change probabilities and passing rates

Dimensional considerations suggest taking $p_i = P_i(\ldots) \cdot j_i$, where j_i is the flux in the ith lane and $P_i(\ldots)$ is a (dimensionless) lane changing probability. The dependencies of P_i will be discussed below.

Our fundamental assumptions regarding the braking/acceleration/diffusion /lane-changing behaviours of an individual driver moving with speed $v \in [0, v_{\max}]$ is that the *main* dependency is on the relative speed $v - u_i$, where u_i is the mean traffic speed at a relevant threshold distance *ahead* of the driver. To this end we find it convenient to introduce abbreviations for $u_i(x + H_B, t - \tau)$ as follows:

$$\begin{aligned} u_i^B &:= u_i(x + H_B, t - \tau), \qquad & u_i^A &:= u_i(x + H_A, t - \tau) \\ \rho_i^B &:= \rho_i(x + H_B, t - \tau), \qquad & \rho_i^A &:= \rho_i(x + H_A, t - \tau). \end{aligned}$$

A driver at (x,t) will be in a braking scenario if $v - u_i^B > 0$, in an acceleration scenario if $v - u_i^B < 0$ *and* $v - u_i^A < 0$ (note that the two conditions are not equivalent because $H_A > H_B$ if $v > 0$). It is possible that neither scenario applies.

In the braking scenario the driver will be motivated for change lanes, and we will assign a lane change probability P_i in this case. For acceleration scenarios we shall set $P_i \equiv 0$.

After these preparations we are finally ready to suggest dependencies for the braking/acceleration term.

The braking/acceleration terms

Definition.

$$B_i(x,v,t) := \begin{cases} -c_B\left(v - u_i^B\right)^2 \rho_i^B \cdot (1 - P_i(\ldots)) & \text{if} \quad v > u_i^B \\ c_A\left(v - u_i^A\right)^2 \left(\rho_{\max} - \rho_i^A\right) & \text{if} \quad v \le u_i^B \quad \text{and} \quad v < u_i^A \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Remarks. First we observe that B_i vanishes of quadratic order in $(v - u)$ if $u_i^B = u_i^A$. This is consistent with property 2 listed earlier. In fact, all that is required for this property is that B vanishes as $v \to u$; the choice of making B *quadratic* in

$(v-u)$ and proportional to density differences has the added advantage of making the constants c_B, c_A dimensionless. Braking should be *stronger* in denser traffic, hence the simple linear dependence on ρ_i^B, and acceleration should be weaker in denser traffic, hence the given dependence on $\rho_{\max}-\rho_i^A$ (where $\rho_{\max}=1/H_0$ is the (maximal) standing traffic density). As u_i^B and u_i^A depend on v, the conditions $v>u_i^B$, $v<u_i^A$ etc. are implicit in v.

The asymmetry in (3) is even stronger than perceived as one should take $c_A<c_B$: braking is forced, but acceleration is by choice. Needless to say, the form prescribed by (3) is just the simplest of many possible guesses.

The (crucial) dependencies of P_i are discussed below. If, as stated, P_i is the lane changing probability, then $1-P_i$ is the braking probability.

Diffusion

The diffusion coefficient D will also be set to depend on the relative speeds of a driver, the macroscopic density and the average speed. Specifically, we assume that there is a smooth function $\sigma(\rho,u)$ such that

$$D(\ldots)=D(\rho,u,v-u)=\sigma(\rho,u)\cdot(v-u)^{\gamma},$$

where $\gamma>0$ is a parameter. In equations (2), we take

$$D_i(\ldots)=\begin{cases}\sigma\left(\rho_i^B,u_i^B\right)\cdot\left|v-u_i^B\right|^{\gamma} & \text{if}\quad v>u_i^B\\ \sigma\left(\rho_i^A,u_i^A\right)\cdot\left|v-u_i^A\right|^{\gamma} & \text{if}\quad v\leq u_i^B\quad\text{and}\quad v<u_i^A\\ 0 & \text{otherwise.}\end{cases}\tag{4}$$

The definition implies that D_i is zero if $v=u_i^B=u_i^A$; if $\gamma>1$, this will guarantee property 2. In fact, it is enough for this property to assume that D vanishes sufficiently fast as $v\to u_i^B$.

Before we discuss assumptions on $\sigma(\rho,u)$ we attempt a heuristic justification of the ansatz (4). The key idea is that D_i is "generated" by the inability (or uncertainty) of a driver to estimate relative speeds with accuracy, except when such relative speeds are 0. As a first guess we set the diffusion coefficient proportional to a power of the relative speed.

As for $\sigma(\rho,u)$, a simple first guess would be to choose it constant; this, however, is not realistic at the extremal values $\rho=\rho_{\max}$ (standing traffic expected), $\rho=0$ (free driving expected, no reason for diffusion), $u=0$ (standing traffic expected). What happens at $u=u_{\max}$ must depend on ρ.

Therefore, it seems reasonable to choose $\sigma(\rho,u)$ as a function which vanishes rapidly enough as $\rho\to 0$ or $\rho\to\rho_{\max}$, and as $u\to 0$. In [13], we chose

$$\sigma(\rho,u)=\sigma_c\rho_{\max}u_{\max}^2 m_1\left(\frac{\rho}{\rho_{\max}}\right)m_2\left(\frac{u}{u_{\max}}\right)$$

with $m_2(s)=s(1-s)$ (note that this implies that σ also vanishes at $u=u_{\max}$; a convenient but probably unrealistic *and* unnecessary assumption), and $m_1(r)$ a function which consists of two linked pieces of Gaussian distributions such that $\max_r m_1=m_1(0.3)$ and $m_1(0)=m_1(1)=0$. σ_c is a free parameter. Reasonable choices of σ at the boundaries of the domain $(\rho,u)\in[0,\rho_{\max}]\times[0,v_{\max}]$ are important for the computation of realistic fundamental diagrams near the endpoints $\rho=0$ and $\rho=\rho_{\max}$ (the expected average speeds there are $v_{\max}=u_{\max}$ and $u=0$, respectively).

Lane change probabilities

The probabilities P_i, are only meaningful on multi-lane highways, but their impact on the fundamental diagram is significant. As stated before, we set $P_i=0$ in acceleration scenarios. In braking scenarios $v>u_i^B$, i.e., v satisfies the implicit relationship $v-u_i(x+H_0+T_Bv,t-\tau)>0$, P_i should depend on $v-u_i^B$, on the density ρ_i^B, and on the density ρ_k and average speed u_k $(k=3-i)$ on the neighboring lane, probably with appropriate nonlocalities. For example, P_i should display a monotone dependence on $u_k(x,t)$ and decrease monotonically in $\rho_k(x,t)$ (higher density on the next lane makes lane changing difficult, but higher average speed there motivates a lane change). It is clear that these dependencies are really quite complex: drivers are forced to check their rear view mirror to ensure sufficiently low density on the next lane, but they will only change lanes if the average speed in that lane is in the right window (high enough to motivate a lane change, but low enough to avoid unnecessary risk).

We will refrain from spelling out these possible dependencies in terms of formulas, though this has to be done for full traffic simulation. In a (fictitious) spatially homogeneous situation where the traffic flow is lane- and space-independent, P will depend only on $v-u$, u, and ρ. A simple example for this was considered in [13]:

$$P = \left(\frac{v-u}{v_{\max}-u}\right)^{\delta} \qquad v > 0. \tag{5}$$

Here, P is taken as a simple power $(\delta > 0)$ of the scaled relative velocity. The scaling introduces an explicit dependence on u.

(5) is far from being realistic, but it catches the essence of a higher motivation for lane-changing if $v-u$ is large. A somewhat more realistic ansatz for P would be

$$P = h(\rho)\left(\frac{v-u}{v_{\max}-u}\right)^{\delta} \tag{6}$$

where $h(0) = 1$, h decreases with ρ, and $h(\rho_{\max}) = 0$. In homogeneous scenarios $h(\rho)$ would simply be a constant.

COMPUTING FUNDAMENTAL DIAGRAMS

We first consider the spatially homogeneous case for (2) in which $f_1(v,t) = f_2(v,t)$ for all (v,t). In this situation the spatial nonlocalities disappear. The time lag due to the reaction time τ remains, but we will set $\tau = 0$ for the remainder of this paper (this is *not* justified; we do it only for convenience. The dependence of solutions on τ, in particular the stability of steady solutions as τ varies, is a very interesting question worthy of investigation). We arrive at a nonlinear drift-diffusion equation

$$\partial_t f + \partial_v \left(B(\rho,u,v-u)\, f - D(\rho,u,v-u)\,\partial_v f\right) = 0 \tag{7}$$

where

$$u(t) = \frac{1}{\rho}\int_0^{v_{\max}} v f(v,t)\, dv$$

and we assume the normalization

$$\int_0^{v_{\max}} f(v,t)\, dv = \rho.$$

This is consistent if we assume in addition zero flux boundary conditions

$$B(\ldots)\, f - D(\ldots)\,\partial_v f\Big|_{\substack{v=0\\ v=v_{\max}}} = 0. \tag{8}$$

It is evident that ρ is then an invariant (and it is in general the only invariant of (3.1)).

In the steady case (8) has to hold for all $v \in [0, v_{\max}]$. We can choose a $\rho \in [0, \rho_{\max}]$, a $u \in [0, v_{\max}]$ and then solve the ordinary differential equation(s)

$$B(\rho,u,v-u)\, f = D(\rho,u,v-u)\,\partial_v f \tag{9}$$

subject to the normalization of $\int_0^{v_{\max}} f\, dv = \rho$ (as (9) is homogeneous of first degree in f, this only means an appropriate scaling factor). If $\gamma > 1$ (see (4)), (9) has the "trivial" solution $f(v) = \rho \cdot \delta_u(v)$, consistent with property 2 from the introduction (in the case $\gamma \leq 1$ diffusion near $v = u$ is too strong to keep the trivial equilibrium). Otherwise, (9) must be solved separately for $v < 0$ and $v > 0$, and this can be done explicitly for the B and D introduced earlier. See [13].

As an additional constraint on the solutions of (9) one *may* impose continuity at $v = u$, thus producing a continuous equilibrium "candidate" with P given by (5). Up to a normalizing constant this solution is

$$f(\nu) = \begin{cases} \exp\left(\beta(\rho,u)\,(v-u)^{3-\gamma}\left[\left(\frac{v-u}{v_{\max}-u}\right)^{\delta}\frac{1}{3+\delta-\gamma} - \frac{1}{3-\gamma}\right]\right), & v > u \\[2ex] \exp\left(-\alpha(\rho,u)\,\frac{(u-v)^{3-\gamma}}{3-\gamma}\right), & v \leq u \end{cases} \tag{10}$$

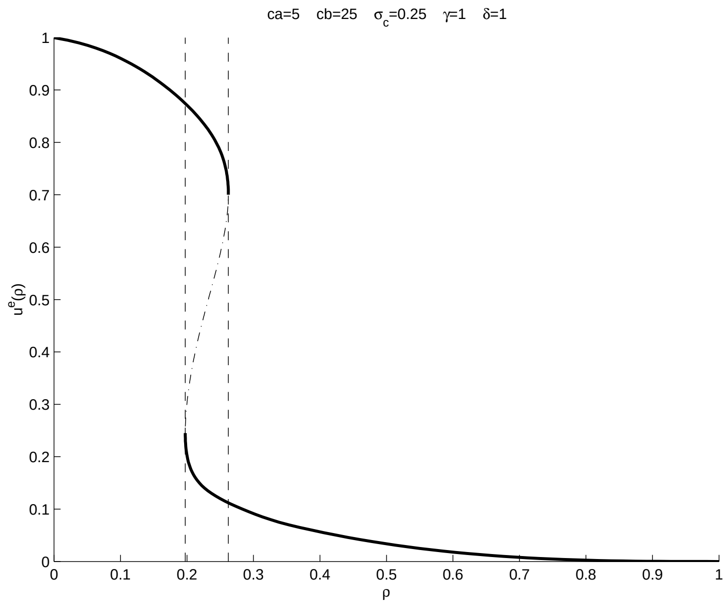

FIGURE 1. A multivalued fundamental diagram

and β, α are given in terms of ρ and u by

$$\beta = c_B \rho / \sigma(r,u), \qquad \alpha = c_A(\rho_{\max} - \rho)/\sigma(\rho, a).$$

Choosing ρ and u therefore produces potential solutions $f(v)$. To find a real equilibrium, the constraint

$$\int_0^{v_{\max}} (v-u) f(v)\, dv = 0 \tag{11}$$

must be satisfied, and for the example given in (10), the integral

$$\int_0^{v_{\max}} (v-u) f(v)\, dv$$

defines a function $R(\rho, u)$, computable explicitly in terms of exponentials and integrals of exponentials (see [13] for explicit formulas). Elementary arguments prove that for any $\rho \in (0, \rho_{\max})$ we have $R(\rho, 0) > 0$, $R(\rho, v_{\max}) < 0$, such that there is always at least *one* $u \in [0, v_{\max}]$ with $R(\rho, u) = 0$. Speeds u with this property determine the fundamental diagram.

Figure 1 shows the fundamental diagram first computed in [13] for the example where $\gamma = 1$, $\delta = 1$, $P = \left(\frac{v-u}{v_{\max}-u}\right)^{\delta}$ (for $v > u$), $c_A = 5$, $c_B = 25$, $\sigma_c = 0.25$ and $\sigma(\rho, u)$ as given before.

The most remarkable fact about this diagram is that it is multivalued in a reasonable density interval; specifically, it is three-valued even for the simplistic passing probability P which we used.

On the complexity of equilibria solutions

We insisted on computing equilibria which are continuous at $v = u$. A possible justification for this could be a “residual” diffusion as drivers assume speed u (after braking *or* accelerating) with a likely “overshoot,” i.e., a driver may brake below u or accelerate above u. We have not included such residual diffusion in the present model; it would eliminate the trivial equilibria.

If the continuity requirement for f at $v = u$ is dropped, many more possibilities arise. Equilibria may suddenly consist of three components: A Dirac delta component supported at $v = u$, a smooth component supported on $v < u$

and a smooth component supported on $v > u$; these smooth components must satisfy (9) on their domains of definition, but the only other constraints are

$$\int_0^{v_{\max}} f\,dv = \rho \qquad \text{and} \qquad \int_0^{v_{\max}} (v-u)\, f\,dv = 0.$$

Clearly, this generality would permit many values $u = u(\rho)$ for the fundamental diagram, more than appear reasonable or realistic. The "residual" diffusion alluded to earlier, no matter how small, therefore becomes appealing, and the ensuing continuity of f becomes compelling.

ZERO PASSING PROBABILITY AND THE MAXWELLIAN EXAMPLE

The cause for the bifurcation leading to multivalued fundamental diagrams is the dependence of P on u *and* $v-u$. To demonstrate this we will study the (much simpler) case where $P = 0$ (no passing), and where

$$\frac{B}{D}(v-u) = \begin{cases} -\rho(v-u) & , \quad v > u \\ -c_0(1-\rho)(v-u), & v < u \end{cases} \tag{12}$$

We call this the "asymmetric Maxwellian" case. c_0 should be thought of as the quotient between C_B and C_A. Clearly (12) determines B and D only up to a common factor g (which may depend on the independent and dependent variables), but the equilibria depend only on $\frac{B}{D}$ and are given by

$$f(v) = c \cdot \begin{cases} e^{-\frac{\rho}{2}(v-u)^2} & , \quad v > u \\ e^{-\frac{c_0(1-\rho)}{2}(v-u)^2}, & v \leq u \end{cases} \tag{13}$$

We have normalized $\rho_{\max} = 1$; we will also set $v_{\max} = 1$. The constant c will be determined by $\int_0^{v_{\max}} f(v)dv = \rho$. The other constraint, $\int_0^1 (v-u) f(v) dv = 0$, reduces after some integrations to

$$\rho e^{-\frac{C_0(1-\rho)}{2}u^2} = \rho - c_0(1-\rho) + c_0(1-\rho)e^{-\frac{\rho}{2}(1-u)^2} \tag{14}$$

Consider the two sides of (14) as functions of $u \in [0,1]$. At $u = 0$, the left-hand side is ρ, the right-hand side is $\rho - c_0(1-\rho)\left(1 - e^{-\frac{\rho}{2}}\right) < \rho$. At $u = 1$, the left hand side is $\rho e^{-\frac{c_0(1-\rho)}{2}}$, less than the right-hand side, which is ρ. Moreover, the left-hand side is strictly decreasing, while the right-hand side is strictly increasing in u. It follows that for each $\rho \in (0,1)$ there is exactly one $u \in [0,1]$ such that equality in (14) holds, so the fundamental diagram is one-valued.

This reasoning generalizes to other cases where $\frac{B}{D}$ depends only on ρ and $v-u$, and demonstrates that passing probabilities and their dependencies are crucial for the multi-valued character of realistic fundamental diagrams.

CONCLUSIONS AND OUTLOOK

We have presented a system of Vlasov-Fokker-Planck type equations with empirical input functions modelling driver acceleration/braking, diffusion based on uncertainty and lane-changing/braking probabilities. Fairly simple assumptions regarding the dependencies of these quantities produce multi-valued fundamental diagrams. We demonstrated that the dependencies of the lane-changing probabilities are crucial for this property.

The new model offers itself to a multitude of analytical and numerical investigations. First, the multi-valued fundamental diagrams and density fluctuations on the highway should conspire to produce stop-and-go traffic, similar to the explanation given in [12]; numerical experiments to this end are in progress. Second, the spatially homogeneous equation

$$\partial_t f + \left(B(\rho,u,v-u)\, f - D(\rho,u,v-u)\, f'\right)' = 0$$

with $u(t) = \frac{1}{\rho}\int v f(v,t) dv$ is a nonlinear drift-diffusion equation which admits (in general) multiple equilibria consistent with the multi-valued fundamental diagram.

This scenario raises questions of stability of equilibria, existence of traffic entropy functionals, and qualitative behavior of time-dependent solutions. Work on all of these questions is in progress.

ACKNOWLEDGEMENTS

This research was supported by grant No. A7847 of the Natural Sciences and Engineering Research Council of Canada. R. Illner would like to acknowledge the hospitality of CEREMADE, Université Paris IX, and Université Paris VII during the month of June 2002.

REFERENCES

1. Prigogine, I., and Herman, R., *Kinetic Theory of Vehicular Traffic*, American Elsevier Publishing Co., New York, 1971.
2. Paveri-Fontana, S., *Transportation Research*, **9**, 225–235 (1975).
3. Klar, A., and Wegener, R., *J. Stat. Phys.*, **87**, 91–114 (1997).
4. Klar, A., and Wegener, R., *SIAM J. Appl. Math.*, **59**, 983–1001 (1998).
5. Klar, A., and Wegener, R., *SIAM J. Appl. Math.*, **59**, i1002–1011 (1998).
6. Klar, A., and Wegener, R., *SIAM J. Appl. Math.*, **60**, 1749–1766 (2000).
7. Kerner, B., *Physical Review Letters*, **81**, 3797–3800 (1998).
8. Kerner, B., *Transp. Res. Rec.*, **1678**, 160–167 (1999).
9. Kerner, B., *Physics World*, **8**, 25–30 (1999).
10. Kerner, B., *J. Phys. A*, **33**, 221–228 (2000).
11. Aw, A., and Rascle, M., *SIAM J. Appl. Math.*, **60**, 916–938 (2000).
12. Guenther, M., Klar, A., Materne, T., and Wegener, R., *preprint* (2002).
13. Illner, R., Klar, A., and Materne, T., *to appear in Communications in Mathematical Sciences* (2002).