ON A FUNCTIONAL-DIFFERENTIAL EQUATION ARISING FROM A TRAFFIC FLOW MODEL*

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Abstract. We provide a derivation in the context of a traffic flow model, and both analytical and numerical studies of the functional-differential equation

$$(z(s)+\alpha)^2 z'(s) = \beta(z(s+z(s)) - z(s)).$$

Here, α and β are positive parameters, and we are in particular investigating the existence and properties of non-constant "traveling-wave" type solutions.

Key words. Kinetic and macroscopic traffic flow models, traveling waves, functional-differential equations

AMS subject classifications. 34K99,35L65,82D99

1. Introduction.

The subject of interest of this article is the functional-differential equation

$$(z(s)+\alpha)^2 z'(s) = \beta(z(s+z(s)) - z(s)) \tag{1.1}$$

This equation arises in a search for traveling wave solutions of a macroscopic traffic flow model (first introduced in [14]) which takes the inherent non-locality of traffic seriously. We will refer to (1.1) as "jam" equation because it emerges from a search for braking waves. The model is related to the macroscopic traffic model first suggested by Aw and Rascle [1], and independently by Zhang [31]. Our paper is, in particular, a continuation of [15], where we presented mathematical and numerical analysis of the model. The emphasis in [15] was on a careful model refinement and on studies of its predictions under certain traffic scenarios, such as local speed limits, density perturbations or speed reductions. In particular, we investigated (numerically) whether traveling wave approximations would properly apply in suitable road intervals. The results in [15] gave an affirmative answer to this query, and they also showed that simplifications of the model in which the non-locality is removed by Taylor approximation to second order, while simpler and qualitatively satisfying, give significantly different results from a quantitative point of view.

In [15] no effort was made to analytically or numerically solve (1.1); rather, solutions of the full model of conservation type equations were checked to satisfy the traveling wave version (essentially (1.1)) in appropriate subdomains of the road. A study of (1.1) was deferred to the future, i.e., to this present work.

Functional-differential equations have been studied for centuries, but their theory suffers, to some extent, from a very limited toolbox. Existence and uniqueness questions can be much more subtle than for ordinary or partial differential equations. Analytic expansions of Cauchy-Kowalewskaya type are routinely applied, but they provide typically only locally defined solutions. The monograph [3] provides the framework and many examples for this approach. However, we state from the outset that our efforts to apply this methodology to (1.1) were not satisfactory.

Our work begins, in Section 2, with a brief review of the derivations given in the earlier references [14, 15]. This is also the place where crucial traffic parameters

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such as H (minimal safety distance), T (look-ahead time) and τ (individual reaction time) are introduced. We then focus on the functional-differential equations arising from a traveling wave ansatz in braking scenarios (similar considerations apply to acceleration cases, but will here not be discussed; the derivation and analysis of these scenarios requires model refinements and will be done elsewhere [25]). A preliminary discussion is provided in Section 8.

The crucial difficulty in our traveling wave equations is the non-locality arising from realistic driver behaviour. This non-locality can be removed via a Taylor expansion and crude truncation, as already discussed in [14, 15]. The truncated ("localized") equations admit beautiful traveling waves; however, as seen from the numerical experiments in [15], the truncation will in general cause a significant error, because the displacement $H + Tu(x,t)$, "removed" via Taylor expansion, is in general not small, and hence the truncation of the expansion will lead to errors. For that reason the inclusion of the full non-locality is not just a matter of theoretical interest; rather, it appears that it is essential for a detailed resolution of the velocity profiles in braking waves.

We review the derivation of the macroscopic model in Section 2, and the traveling wave ansatz and its localized version in Section 3.

In Section 4 we simplify the full (non-localized) traveling wave equation via affine transformations. The end result is equation (1.1) for $z = z(s)$ (where s is the variable $s = x + Vt$ and z is a rescaled speed variable)

$$(z(s) + \alpha)^2 z'(s) = \beta(z(s + z(s)) - z(s)).$$

There remain only two parameters α, β, combinations of the original model parameters. The main question of interest is for what values of $\alpha > 0, \beta > 0$ (1.1) will admit non-trivial (i.e., non-constant) solutions. This is a question not just of some practical interest for traffic flow studies. We hope to convince the reader that (1.1) is an object of pure mathematical interest in itself, and we follow a variety of avenues to answer the main question. While we have no complete solution, we will show evidence that non-trivial and stable traveling waves solving (1.1) will exist or fail to exist, depending on the values of the parameters. A remarkable practical aspect of our study is that it provides quite realistic upper and lower bounds for the traveling wave speeds.

In Section 5 we present two examples related to (1.1): First, a simpler linear functional-differential equation with a constant non-locality; for this example it is easy to construct explicit solutions, and they allow an interpretation in terms of elementary geometric operations. Secondly, we try a "Mott-Smith"- approximation [4] in the sense that we insert an ansatz $z(s) = A + B\tanh(-\sigma s)$ into the equation. It turns out that the equation is not satisfied (no surprise there), but if we force equality between the two sides by allowing $\beta = \beta(s)$ to be dependent on s, then the resulting functions β are asymptotically constant for large $|s|$; moreover, an exploration which we defer until Appendix 1 shows that there are many parameter choices (A, B, σ, α) for which $\beta(-\infty) = \beta(+\infty)$. The point of these examples is to provide evidence that there actually are equations closely related to (1.1) with non-trivial solutions of the desired class.

Section 6 returns to the real equation (1.1). In this section we describe a numerical approximation procedure to compute (approximate) solutions. The method is based on a dynamical systems idea and suggests rapid convergence for a select set of parameters. It is not to be expected that the traveling wave solutions we are looking for will exist for all (or even most) choices of parameters. In Appendix 2 we present a situation where the dynamical systems approach fails to converge. There may be no solution of (1.1) for the parameters and boundary values used there.

In Section 7 we describe an operator approach to the solvability question. This, the most abstract of our sections, is heavily motivated by geometric interpretation, and we introduce and discuss two operators T_1, T_2 acting on wave profiles such that a solution z of (1.1) of the desired type will satisfy $T_1 z = T_2 z$, a fixed point equation. We collect a number of intriguing properties of these operators, suggesting that a fixed point argument is a viable approach to the existence question, but we need some a priori assumptions to complete this argument. Whether these assumptions can be verified remains an avenue for future work.

While our problem arose in traffic flows models, it is treated here as an abstract mathematical challenge. Its practical relevance for traffic applications will depend on the degree to which the underlying model is accepted as realistic, and, if this is the case, whether high resolution of speed profiles can be exploited to improve traffic control. We discuss these matters in Section 8.

There is extensive mathematical literature on traffic flow models (see [15] and the references therein). Traffic models may roughly be classified as "microscopic" (keeping track of each car in a deterministic way, see for example [6, 7, 12, 13, 22, 24]), of stochastic type (there is much variation here; see [2, 28, 29]), or variations including both features, such as cellular automata [12, 26, 27]. The model which leads to Equation (1.1) is of macroscopic type, and earlier versions of macroscopic models and their analysis may be found in [1, 8, 9, 10, 11, 17, 18, 23, 30]. *Kinetic* models are a possible bridge between microscopic/ stochastic and macroscopic models, as discussed in [20, 5, 21].

An inherent problem in all traffic modelling is the complexity of drivers' reactions, which poses a serious obstacle to accuracy at all levels, and is a persistent source of criticism of all models. While validation of models from both a theoretical and practical point of view remains elusive for this "human" factor, traffic models often lead to intriguing mathematical challenges, like the functional-differential equation which is the object of our study.

2. The macroscopic model. Reference [15] introduced the following kinetic model for traffic on a single-lane (or homogenized over several lanes) highway:

$$\partial_t f + v\partial_x f + \partial_v (B(\rho, v - u^X) f) = 0 \tag{2.1}$$

where $f = f(x, v, t)$ is a kinetic car density such that $f\,dx\,dv$ will be the statistically expected car number in the space and speed domain $[x, x + dx] \times [v, v + dv]$, and ρ and u are the macroscopic density and speed, related to f via

$$\rho = \int_0^\infty f\,dv, \quad \rho u = \int_0^\infty v f\,dv.$$

The shorthand u^X in (2.1) stands for $u^X = u(x + H + Tv, t - \tau)$. Here, $H > 0$ is a constant safety distance (think of two car lengths) which drivers keep at low speeds, measured from car front to car front. T is a characteristic "look-ahead" time used to keep an appropriate distance to the lead car, and τ is the individual reaction time. The term $B(\ldots)$ in (2.1) denotes the braking or acceleration force applied by a reference driver at position x and moving with speed v at time t.

Equation (2.1) must be interpreted in a statistical sense (a comment which applies to all kinetic models, but is often ignored for models involving microscopic particles such as atoms or electrons), and although we have incorporated a non-locality, the model is likely to be overly simplistic from a practical point of view: we assume that the forces depend only on the local density (ρ) and on the relative speed of the reference driver with respect to the delay-observed average speed u^X at position $x + H + Tv$. While this makes sense, reality certainly requires random fluctuations in these forces, which are not included in (2.1). In addition, one could consider equations like (2.1) for many different lanes and include lane-changing terms on the right-hand sides. Some of this was done in [20] and subsequent papers [16]. Here, we aim at structural problems emerging from the non-locality, and we therefore keep things simple.

A reasonable ansatz for the braking or acceleration force is

$$B(\rho, v - u^X) = \begin{cases} -g_1(\rho)(v - u^X) & \text{if} \quad v - u^X > 0 \\ -g_2(\rho)(v - u^X) & \text{if} \quad v - u^X < 0 \end{cases} \tag{2.2}$$

and simple "reasonable" choices for g_1, g_2 are

$$g_1(\rho) = c_1\rho, \quad g_2(\rho) = c_2(\rho_{max} - \rho)$$

(the maximal density is $\rho_{max} = 1/H$, where H is the minimal safety distance between the fronts of two vehicles; in standing traffic we may have real bumper-to-bumper traffic, and then $\rho_{max} = 1/L$, where L is the average length of a car. One may guess that $H \approx 2L$). It must be stated here that (2.2) is overly simplistic, but as this is not a paper on the details of traffic modelling, we will not pursue the delicacies of driver behaviour. Some of these matters are discussed in [15], and in [22].

How does one go from a kinetic model to a macroscopic model? One method, used in [19], is to set up and study moment equations. Typically a closure procedure is needed to obtain a finite (closed) set of equations, and this closure will involve assumptions on the system at hand. There is a more direct (if rough) approach: in moderate to high traffic densities (which in reality are the *relevant* densities) one expects (based on observations) only small statistical fluctuations. Traffic is often described as "synchronized", meaning that all vehicles at time t and near position x will move at approximately the same speed. This motivates the ansatz $f(x, v, t) = \rho(x, t)\delta(v - u(x, t))$ in (2.1), and we have the following general result.

THEOREM 2.1. *Assume that $\rho = \rho(x, t)$ and $u = u(x, t)$ are of class C^1. Then the distribution $\rho(x, t)\delta(v - u(x, t))$ is a weak solution of (2.1) if and only if ρ and u satisfy the system of equations*

$$\rho_t + (\rho u)_x = 0 \tag{2.3}$$

$$u_t + uu_x - B(\rho, u - u^X) = 0. \tag{2.4}$$

REMARK 1. *The first equation is just the continuity equation, while the second equation is the equation for the speed. In equation (2.4) the meaning of the superscript $()^X$ has changed: Now, $u^X(x, t) = u(x + H + Tu(x, t), t - \tau)$. Notice how the dependent variable u here appears inside its own argument.*

Proof. The proof we present is a more transparent version of the proof given in [15]. The key idea is to exploit the different status of the variables x, t and v. For simplicity, assume that the (kinetic) density f vanishes rapidly as $x \to \pm\infty$, and as $v \to 0$ or $v \to \infty$ (this assumption certainly holds for the ansatz $\rho(x,t)\delta(v-u(x,t))$ while $u > 0$.) Then f is a weak solution of the kinetic model if for every test function $\phi(x,v,t)$, compactly supported in x,t and arbitrary but bounded in v, we have

$$\int\int\int \phi_t f + \phi_x v f + \phi_v B(\rho, v - u^X) f \; dx \; dv \; dt = 0. \tag{2.5}$$

We substitute in (2.5) a test function of the form $\phi(x,v,t) = \varphi(x,t)h(v)$. Set $\psi(x,t) := \phi(x,u(x,t),t) = \varphi(x,t)h(u(x,t))$, then

$$\begin{aligned} \psi_t &= \phi_t + \phi_v u_t = \varphi_t h + \varphi h' u_t \\ \psi_x &= \phi_x + \phi_v u_x = \varphi_x h + \varphi h' u_x. \end{aligned}$$

Rewrite $\phi_t = \psi_t - \partial_v \phi u_t$ and $\phi_x = \psi_x - \partial_v \phi u_x$ and substitute in (2.5), to find

$$\int\int\int [\psi_t f + \psi_x v f - \varphi h'(v) u_t f - \varphi h'(v) u_x v f + \varphi h'(v) B(\ldots) f] \; dv \; dx \; dt = 0,$$

and for $f = \rho\delta(v-u)$ this becomes

$$\int\int [\psi_t \rho + \psi_x \rho u] - \int\int \psi \frac{h'(u)}{h(u)} \rho [u_t + u u_x - B(\rho, u - u^X)] = 0,$$

provided we assume that h is bounded away from zero. We may then consider ψ as an arbitrary test function and observe that the last integral contains the extra degree of freedom $\frac{h'}{h}$. It follows that both integrals must vanish identically, and our result follows from this. □

REMARK 2. *It is a simple and natural idea to remove the nonlinearity in (2.4) by using a (formal) Taylor expansion. In a braking scenario, if we take* $B(\rho, u - u^X) = -g_1(\rho)(u - u^X)$ *and expand to first order* $u - u^X \approx -(H + Tu)u_x$*, we obtain the simpler equation*

$$u_t + u u_x - g_1(\rho)(H + Tu) u_x = 0. \tag{2.6}$$

In combination with the continuity equation, (2.6) is a generalization of the Aw-Rascle model [1], for which the speed transfer equation is usually written as

$$u_t + u u_x - \rho \frac{\partial p}{\partial \rho} u_x = 0.$$

Here,

$$p_\rho = \frac{\partial p}{\partial \rho}(\rho, u) = \frac{g_1(\rho)}{\rho}(H + Tu).$$

This generalizes the Aw-Rascle model inasmuch as p *depends on both* ρ *and* u*, while only a dependence on* ρ *was assumed in [1]. A similar equation applies to the acceleration scenario, and in combination these two equations form a traffic model of Hamilton-Jacobi type. One can easily consider models in which the second order terms in the Taylor expansion is retained, as already done in [14, 15]. The resulting models can be considered as Hamilton-Jacobi type models with diffusive corrections.*

3. On Traveling Waves. The models introduced above readily offer themselves to numerical analysis, in particular because the non-locality arises in a term with a time delay (i.e., the velocity profile needed in the calculation will have been computed in previous steps). Extra care, and modelling refinements, are needed in transition domains between braking and acceleration; as already mentioned, we plan to address these issues in future work [25].

Interesting quandaries arise in searching for traveling wave solutions, and this is the main theme addressed here. A traveling wave solution will be of the form $\rho = \rho(s)$, $u = u(s)$ where $s = x + Vt$ and V is the speed of the traveling wave. We will implicitly always assume $V > 0$, so with $s = x + Vt$ waves will be moving backwards in traffic. It is useful to note at this point that observations suggest realistic wave speeds $V \approx 20$ km/hr, or about 5.5 meters per second. We will pause intermittently in our progress to compare results with this benchmark.

With the traveling wave ansatz the continuity equation becomes $\frac{d}{ds}[\rho(u+V)] = 0$, which gives

$$\rho(s) = \frac{c_0 V}{u+V},$$

with an integration constant $c_0 > 0$ (in [14] we set $c_0 = \rho_{max}$, motivated by the observation that in standing traffic ($u = 0$) we expect $\rho = \rho_{max}$. However, other values of c_0 are perfectly consistent with the continuity equation).

Substituting this ρ into (2.4) and setting $g_1(\rho) = c_1\rho$, we find the equation for a traveling braking wave,

$$(u(s)+V)^2 u'(s) = c_0 c_1 V[u(s + (H - \tau V) + Tu(s)) - u(s)]. \tag{3.1}$$

This is a functional-differential equation. Observe how $u(s)$ shows up inside the argument of u itself. Equation (3.1) contains no fewer than 6 parameters $(H, T, \tau, c_0, c_1, V)$. We will shortly see that all but two parameters can be eliminated via affine transformations. Equation (3.1) suggests to restrict our discussion to $V < H/\tau$, so that for any $u(s) > 0$ we will have $Tu(s) + H - \tau V > 0$, a "causality" constraint.

3.1. Removing the nonlocality by Taylor expansion. This simple idea was already followed in [14, 15], but we include it here for completeness. It also provides further insight on the relationships between parameters. Here, it is implicitly assumed that u has sufficiently many derivatives.

A Taylor expansion to second order gives

$$u(s+(H-\tau V)+Tu(s))-u(s) = (H-\tau V+Tu(s))u'(s)+\frac{1}{2}(H-\tau V+Tu(s))^2 u''(s)+\dots$$

After neglecting terms of third and higher order and substituting into (3.1) one finds

$$u'' = 2\frac{(u+V)^2 - c_0c_1V(H-\tau V+Tu)}{c_0c_1V(H-\tau V+Tu)^2}u'. \tag{3.2}$$

This equation is easily studied in phase space u, u', because it follows from (3.2) that

$$\frac{du'}{du} = 2\frac{(u+V)^2 - c_0c_1V(H-\tau V+Tu)}{c_0c_1V(H-\tau V+Tu)^2}. \tag{3.3}$$

Every point $(u > 0, u' = 0)$ is a (trivial) solution of (3.2). If the condition

$$V < \frac{c_0 c_1 H}{1 + c_0 c_1 \tau}$$

is satisfied, the right-hand side of (3.3) is negative for sufficiently small (but non-negative) u, and in this case there are lots of traveling wave solutions for (3.2), two of which are depicted in Figure 1 (in phase space). The parameters taken for the (MAPLE) computations which produced this picture are

$$H = 10, T = 2, \tau = .25, V = 5, c_0 c_1 = 1.6$$

(if distances are measured in meters and times in seconds, then these quantities are within realistic ranges; except $c_0 c_1$, which is basically a guess).

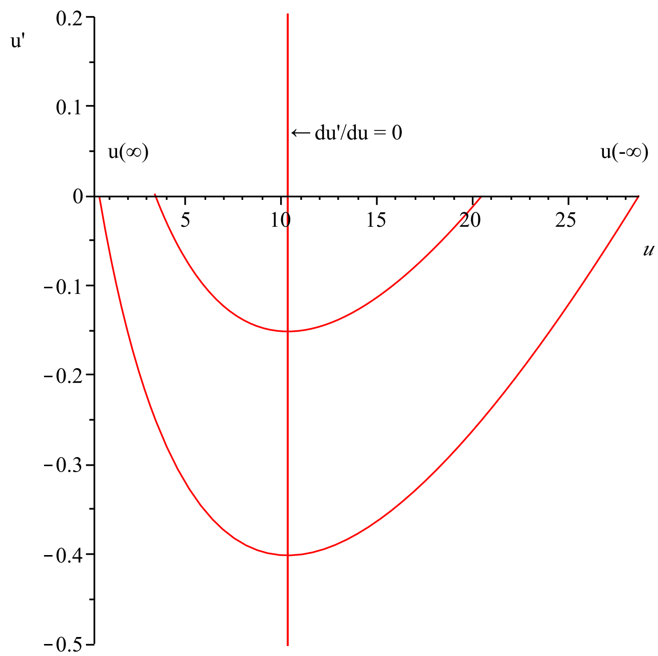

FIG. 3.1. *Localized braking waves*

Here, u_∞ is the (small) speed of traffic in front of the traveling braking wave, and $u_{-\infty}$ is the speed in the far back, not yet reached by the wave. If, at u_∞, $\frac{du'}{du} < 0$, then we have a braking wave connecting u_∞ to $u_{-\infty}$, and $u_{-\infty}$ is given via the condition that

$$\int_{u_\infty}^{u_{-\infty}} (\frac{du'}{du})\ du = 0.$$

Notice that the constraint on V

$$V < \frac{c_0 c_1 H}{1 + c_0 c_1 \tau} \tag{3.4}$$

is consistent with and in fact sharpens the earlier restriction that $V < \frac{H}{\tau}$.

A similar analysis applies to acceleration waves with the corresponding equation for acceleration scenarios. They fill the upper half of the phase portrait depicted in Figure 1. Examples for this (with $\tau = 0$) are given in [14].

4. The "Jam" equation. We return to the equation displaying the full non-locality,

$$(u(s) + V)^2 u'(s) = c_0 c_1 V [u(s + (H - \tau V) + T u(s)) - u(s)]. \tag{4.1}$$

A simple and natural idea is to try and reduce the number of parameters by affine transformations in both independent and dependent variables. It is straightforward to do that; elementary calculations show that if we set

$$\delta := \frac{1}{T}(H - \tau V), \quad z(s) := T(u(s) + \delta)$$

then $z = z(s)$ satisfies the simpler equation

$$(z(s) + \alpha)^2 z'(s) = \beta(z(s + z(s)) - z(s)). \tag{4.2}$$

with

$$\alpha = T(V - \delta) \text{ and } \beta = c_0 c_1 V T^2. \tag{4.3}$$

This is the "jam" equation already given in (1.1). It contains only the two parameters α, β and the (simpler) non-locality $z(s + z(s))$.

Our implicit assumption $u > 0$ translates into $z > T\delta = H - \tau V$. If we add the (somewhat arbitrary) assumption that $\alpha > 0$ then it follows that $\delta < V$, which means $H < V(T + \tau)$. While this final condition on V is less motivated than the earlier ones, it is instructive to write the sequence of constraints imposed so far:

$$\frac{H}{T + \tau} < V < \frac{c_0 c_1 H}{1 + c_0 c_1 \tau} < \frac{H}{\tau},$$

where the last estimate is obvious; note that the final upper bound depends only and H and τ. In reality, one has (approximately) $\tau \approx 1$ sec, $H \approx 8$ m, $T \approx 3$ sec. Inserting in the inequalities above one gets

$$\frac{H}{T + \tau} \approx 2 \text{ m/sec} < V < 8 \text{ m/sec} \approx \frac{H}{\tau},$$

which translates into Vs between 8 and 30 km/h — very much the observed range (5.5 m/sec).

We are looking for a special class of solutions of (4.2), namely, braking waves. A braking wave is a solution of (4.2) such that $\forall s \;\; z'(s) < 0$, and such that $z(-\infty) = a > b = z(\infty) > \delta > 0$. Here, a, b are (shifted and rescaled) speeds at $\pm\infty$. If $a = b$, the constant a is a trivial solution of (4.2) and it is immediate that every constant solves (4.2). We will implicitly assume that (4.2) is complemented with boundary conditions at infinity such that $a > b$. Two remarks are in order.

- Translation invariance: If $z = z(s)$ is a solution of the jam equation, then so is $S_{s_0}z(s) := z(s - s_0)$.
- Consistency: While $z(s)$ takes nonnegative values and is decreasing, we have $z(s+z(s)) - z(s) \leq 0$, consistent with $z'(s) \leq 0$. However, it is in general not true that $s \to z(s + z(s))$ will decrease if $z'(s) < 0$. See Section 7 for more details.

The hardest question we face is whether non-trivial braking waves as solutions of (4.2) actually exist. As the problem appears rather inaccessible to standard analytical tools, we now provide some related illuminating examples, and numerical evidence. We revisit the existence question in Section 7.

5. Related examples.

5.1. A linear example. Consider the much simpler *linear* example

$$z' = \beta(z(s + z_0) - z(s)),$$

where we have assumed a constant shift $z_0 > 0$ and a constant factor (1) multiplying z'. Clearly, every constant is a solution, but if we insert the ansatz $z(s) = Ce^{\alpha s}$, the equation reduces to $\alpha = \beta(e^{\alpha z_0} - 1)$. $\alpha = 0$ produces the already known constant solutions, but if $\beta z_0 < 1$ there is a unique positive α providing another (exponential) solution, and if $\beta z_0 > 1$ there is a unique negative α producing yet another solution. See Figure 2.

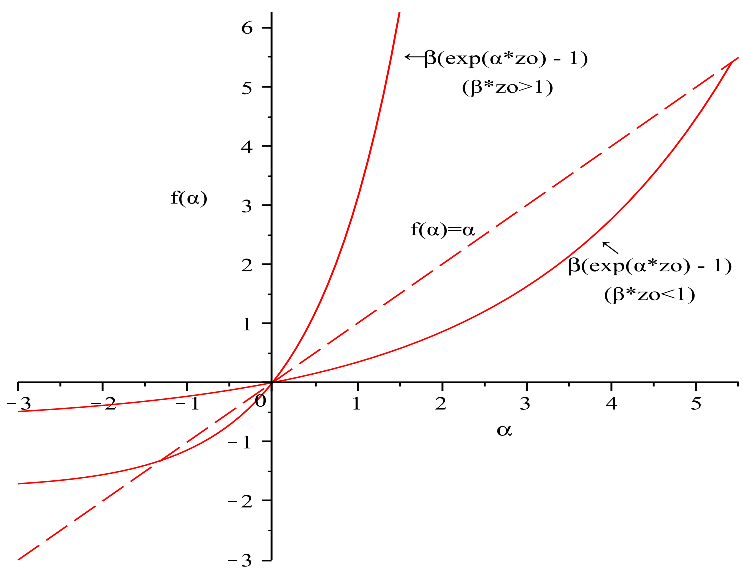

FIG. 5.1. *A linear example*

Furthermore, inserting the ansatz $z(s) = c_1 - c_2 s$ into the equation leads to

$$-c_2 = \beta(c_1 - c_2(s + z_0) - c_1 + c_2 s) = -\beta c_2 z_0.$$

This shows that there is another class of solutions if $\beta z_0 = 1$. Geometrically, this means that for this particular choice of β and z_0 the effects of moving the line $c_1 - c_2 s$ "down" (via the operation $z \to z + z'/\beta$) or "left" (via the operation $z(\cdot) \to z(\cdot + z_0)$) produce the same result.

5.2. The $\tanh-$ **(or "Mott-Smith") approximation.** One of the fundamental ideas of the "Mott-Smith" approximation (see, for example, [4] and the references therein) in fluid dynamics is to fit a hyperbolic tangent profile to a shock wave; the relevant dependent variables there are density, macroscopic flow speed, pressure and temperature. It is tempting to do the same here, where the traveling wave ansatz has already reduced the complexity to one dependent variable, $z(s)$.

Remarkably, if we consider the function $z(s) := A + B\tanh(-\sigma s)$, (it is a little sloppy to use the same same symbol z for this function, but we will do it anyway), and use the identities

$$\frac{d}{ds}\tanh(s) = 1 - \tanh^2(s)$$

$$\tanh(x+y) = \frac{\tanh(x) + \tanh(y)}{1 + \tanh(x)\tanh(y)}$$

we find that this z satisfies an equation

$$(z(s) + \alpha)^2 z'(s) = \beta(s)(z(s + z(s)) - z(s)), \tag{5.1}$$

where $\beta(s) =$

$$\frac{-\sigma(\alpha + A + B\tanh(-\sigma s))^2(1 + \tanh(-\sigma s)\tanh(-\sigma(A + B\tanh(-\sigma s))))}{\tanh(-\sigma(A + B\tanh(-\sigma s)))}. \tag{5.2}$$

Equation (5.1) is a rather trivial statement, because the $\beta = \beta(s)$ is not constant, as it was in (4.2). In fact, one could put any smooth function into the left- and right-hand sides of (4.2) and force an identity by computing the corresponding $\beta(s)$. The point of the above calculation is that the β given in (5.2) is asymptotically constant as $s \to \pm\infty$. It is possible to choose parameters such that $\beta(\infty) = \beta(-\infty)$, and such that $\beta(s)$ deviates from this constant only a little bit, and only very locally. In Figure 5.2 we show an example for such a $\beta = \beta(s)$. The parameters chosen for this example are $\sigma = 1$, $\alpha = 0.03296$, $a = 0.16$, and $b = 0.009$. Appendix 1 contains a brief graphical exploration of the context, and it will be shown there how parameters (as in the example) producing such functions $\beta(s)$ can be found.

6. Numerical experiments. It is not a straightforward task to approximate solutions of the jam equation numerically. There are three fundamental problems: First, the domain of interest is the entire real line. Any numerical procedure must include a truncation at large $|s|$, a potential source of errors. Second, the full (non-local) equation (4.2) includes the evaluation of the unknown z at $s+z(s)$; the latter will typically not fall onto a grid point, and we have to employ an interpolation process, another source of errors. And finally, how to start a numerical approximation?

The approach we used and present is based on the idea of considering the traveling wave as limit state of an "artificial" dynamical system (not to be confused with the traffic model (2.4), which together with the continuity equation is the "real" dynamical system). To this end, we introduce an *artificial* variable t (t can be thought of as an "artificial time") and allow $z = z(s,t)$. We require z to be a solution of

$$\partial_t z(s,t) + \frac{(z+\alpha)^2}{\beta}\partial_s z(s,t) + z(s,t) = z(s + z(s,t), t) \tag{6.1}$$

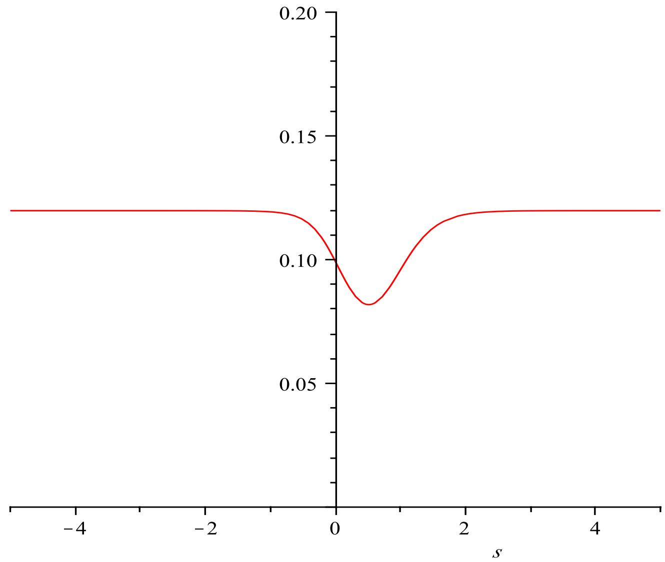

FIG. 5.2. *An example:* $\beta = \beta(s)$

Obviously, any steady solution of (6.1) will be a solution of (4.2). The former equation allows solution procedures based on explicit discretizations (in t). The discretizations in s involve interpolation to address the evaluation of the last term in (6.1). We complement (6.1) with boundary conditions $z(-\infty, t) = a, z(\infty, t) = b$.

As for the choice of an initial profile, we used the "Mott-Smith" approximants from the previous section, i.e., we set

$$z(s, 0) = A + B\tanh(-\sigma s),$$

where $A+B = a, A-B = b$. However, it is not clear from the outset how to choose the three parameters A, B and σ : Not all choices may be consistent with the existence of a traveling wave profile. We use some of the profiles obtained from the localized theory in Section 3.1 to estimate reasonable candidates for A, B and σ. Here as there we use parameter values $H = 10,\ T = 2,\ V = 5,\ \tau = .25, c_0c_1 = 1.6$ Our definition of δ then gives $\delta = \frac{H-\tau V}{T} = \frac{35}{8}$, and we recall that $z(s) = T(u(s) + \delta)$.

From (4.3), the resulting α and β are then 1.25 and 32. We can easily compute asymptotic values for z from asymptotic values for u and vice versa. For example, for the lower curve in Figure 3.1 we see that $u(-\infty) \approx 29,\ u(\infty) \approx 0.4,\ u'_{min} \approx -0.4$. From these estimates we compute $a \approx T(29+35/8) = 66.75, b \approx T(0.4+35/8) = 9.55$. This gives a good guess for possible boundary values for z, and we easily find $A = 38.15$ and $B = 28.6$.

The same curve allows an initial estimate for σ. If we take

$$z(s, 0) = A + B\tanh(-\sigma s)$$

and wish to match the steepest negative slope of this z to the steepest deceleration of u, we must set

$$-\sigma B = Tu'_{min},$$

A	29.71	a	34.535
B	4.825	b	24.885
$u(-\infty)$	12.8925	$u(\infty)$	8.067
σ	0.0051	"time" step	0.04

Table 6.1
Data for a "weak" jam

A	31.3055	a	44.2119
B	12.9064	b	18.3991
$u(-\infty)$	17.73095	$u(\infty)$	4.82455
σ	0.013061	"time" step	0.04

Table 6.2
A stronger initial profile

or, for the data under consideration, $\sigma = 0.02797$. This same procedure can be applied to all the possible triples $(u(\infty),\ u(-\infty),\ u'_{min})$ arising from the theory in Section 3.1. Or, equivalently, one can compute localized traveling waves directly for the dependent variable $z(s)$; this is how the initial data in our first numerical experiment (the weak jam: see Table 6.1) were constructed.

We show the results of three different numerical experiments: "weak", intermediate and strong braking profiles. The relevant data are given in Tables 6.1 to 6.3. In all runs we used $\alpha = 1.25,\ \beta = 32, \Delta t = 0.04, \Delta s = 8.5$. The σ given in each table is computed as described above; if the limit state is stable, then the value of σ can be varied without problems.

The braking wave corresponding to Table 6.1 is not depicted in Figure 3.1; it is a "weaker" wave in the sense that the speed difference is modest- the conversion gives $u(-\infty) \approx 12.9, u(+\infty) \approx 8.07$ (if we consider meters per second this translates into braking modestly, from 46 to 29 km per hour).

The step size (8.5) seems large but works well because of the small slope in our units. We computed inside a domain (-15,000, + 2,000) in order to avoid boundary errors to invade (eventually, this is unavoidable, as our sought after waves are not constant but only converge to constants for large $|s|$ - in the simulations, one has to use a cutoff). The time step was 0.04. We used an adaptive upwind scheme with excellent numerical stability. We are grateful for G. Russo for providing us with the scheme.

Figure 6.1 shows both the initial profile (the hyperbolic tangent) and the profile after 50,000 time steps (in red). We conducted a convergence test by inserting the final data file into both sides of the equation and subtraction. The result (not shown) was completely satisfactory. The experiment suggests that the solution of the jam equation exists and is stable for the data under consideration.

We next repeated this experiment for a stronger wave. The relevant data are given in Table 6.2.

For this stronger wave drivers have to brake from approximately 64 to 17.4 km per hour- a significant drop. In our simulation it appeared as if the scheme would converge, but letting the code run for a long time we made two observations: first, the profile seems to keep its shape but continues to wander (and hence does not really become stationary). See Figure 6.2, which shows the profile after $50,000$ time

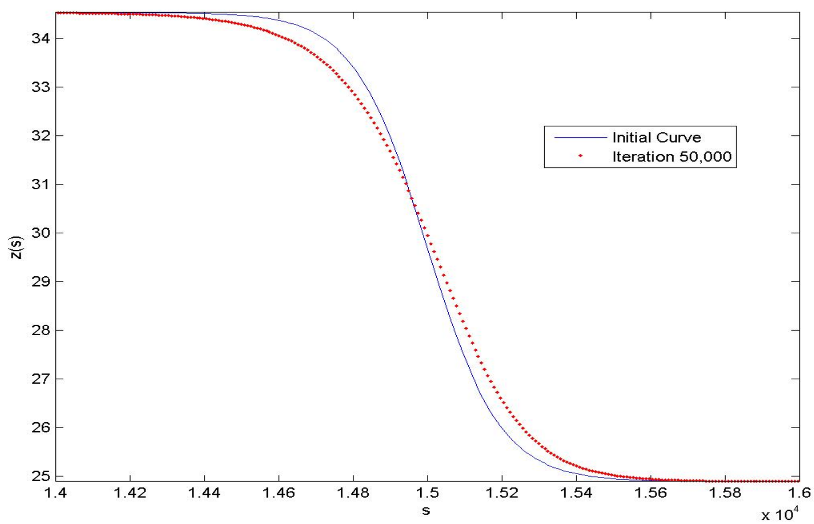

FIG. 6.1. *A numerical solution of the jam equation*

steps. And second, after insertion into the jam equation we found that the equation is not satisfied—the difference between the two sides of the jam equation keeps a disturbingly stubborn maximal value, which does not diminish with t. See Figure 6.3; the grid point 1,800 corresponds to $s = 1,800 \times 8.5 = 15,300$, so the error persists at the right (lower) end of the dotted wave in Figure 6.2. This may be a sign that the jam equation does not possess a solution of the desired kind for the chosen parameter values. We revisit the issue in section 7.

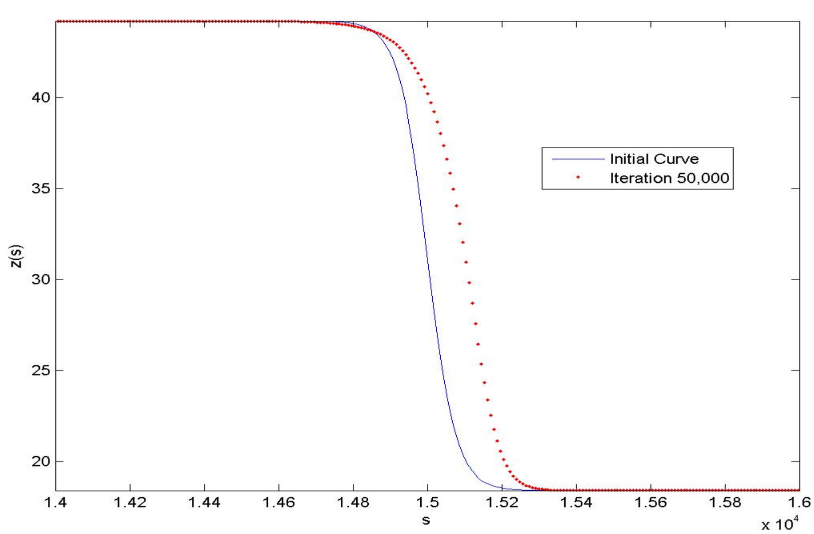

FIG. 6.2. *After* $50,000$ *steps*

A	38.125	a	66.75
B	28.625	b	9.5
$u(-\infty)$	29	$u(\infty)$	0.4
σ	0.013061	"time" step	0.04

TABLE 6.3
A very strong initial profile

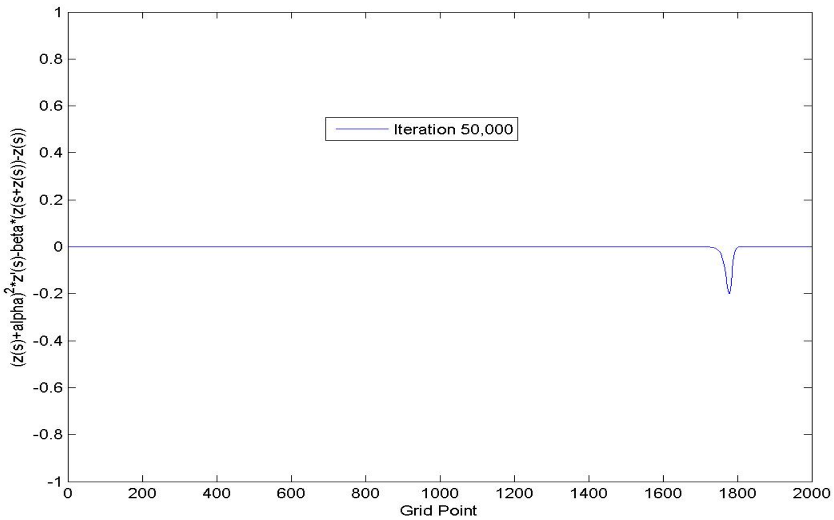

FIG. 6.3. *The residual error*

In a final experiment we used data for a very strong initial profile, as given in Table 6.3. The data are adapted from the strongest localized wave we depict in Section 3.1. These correspond to drivers braking from 104.4 to 0.4 km per hour (practically coming to a full stop from driving at the speed limit). The localized theory still produced a viable traveling wave.

For the jam equation, the numerics suggest that there is no convergence to a steady wave; instead, the profile wanders off towards ∞. See Figure 6.4. We also tested the profile by inserting it into the equation and found that a significant error persists. This is not surprising.

7. An operator approach.

7.1. The operators T_1 and T_2. We return to the equation (4.2) and present some steps towards its analytical solution. A function $z(s)$ is a solution of (4.2) exactly if it satisfies

$$T_1 z(s) = T_2 z(s) \tag{7.1}$$

where the operators T_1 and T_2 are defined by

$$T_1 : z(s) \to \frac{(z(s)+\alpha)^2}{\beta} z'(s) + z(s)$$

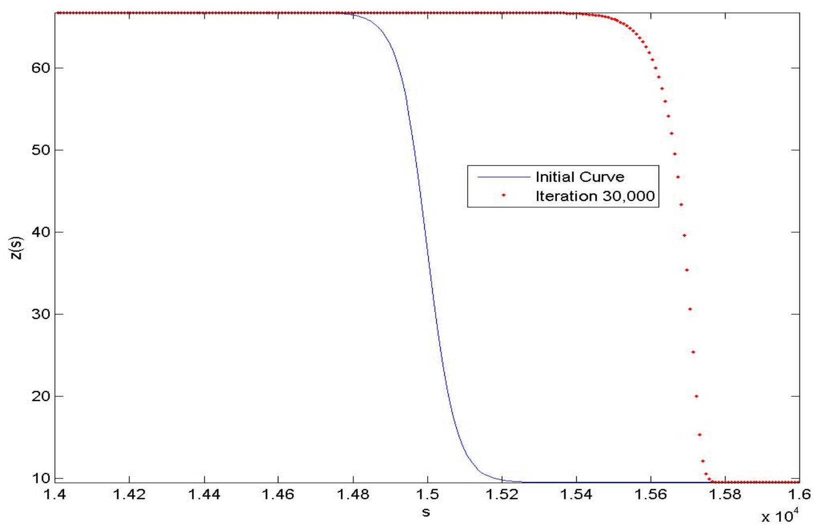

FIG. 6.4. *No convergence*

and

$$T_2 : z(s) \to z(s + z(s)).$$

Equation (7.1) is a fixed point equation, which we could rewrite as $z = T_1^{-1} T_2 z$ (if T_1 can be inverted) or as $z = T_2^{-1} T_1 z$ (if T_2 can be inverted). In this section we present what we can say about the operators T_1 and T_2. We begin by defining a class of functions W^1 which is invariant under T_2 and T_1^{-1}.

DEFINITION 7.1. *Let $0 < b < a$. By W^1 we denote all differentiable functions z on $\mathbf{R}$ such that for all s $0 < b < z(s) < a < \infty$, $\lim_{s \to -\infty} z(s) = a$, $\lim_{s \to \infty} z(s) = b$, and $-1 < z'(s) < 0$. W^1 depends on the choice of a, b, but as these are considered fixed we will suppress the dependence in the notation.*

LEMMA 7.2. *If $z \in W^1$ then $T_2 z \in W^1$.*

Proof. The limits and bounds on $T_2 z(s)$ are obvious. For the derivative compute

$$\frac{d}{ds} z(s + z(s)) = z'(s + z(s)) \cdot (1 + z'(s))$$

and observe that $1 + z'(s) \in (0, 1)$. □

The 45 degree rule. There is a geometric way of visualizing T_2. As depicted in Figure 7.1, observe that to find $h(s) := z(s + z(s))$ first identify the point $(s, z(s))$ on the graph of z. To find the point $s + z(s)$ on the real axis, follow the line through $(s, z(s))$ with slope -1 (hence the "45 degree" rule) to its intercept with the s-axis (alternatively, this is also the intercept of the circle through $(s, 0)$ with radius $z(s)$). Then find the value of z at this intercept- this is $h(s)$.

REMARK 3. *The 45 degree rule easily shows that if $z'(s) < -1$ on some interval, then the assertion of the Lemma will no longer hold: In general, $T_2 z$ will then not be decreasing. See Figure 7.2 for this situation.*

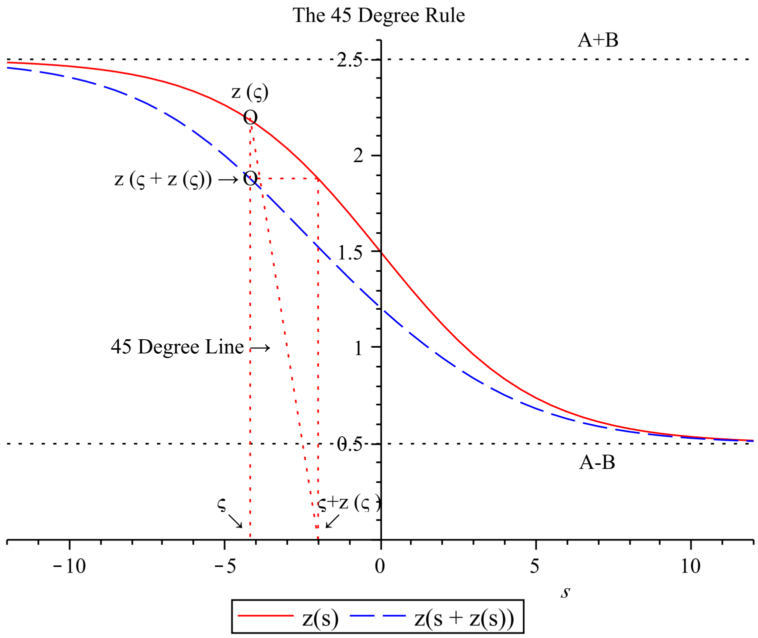

FIG. 7.1. *Geometric interpretation of T_2*

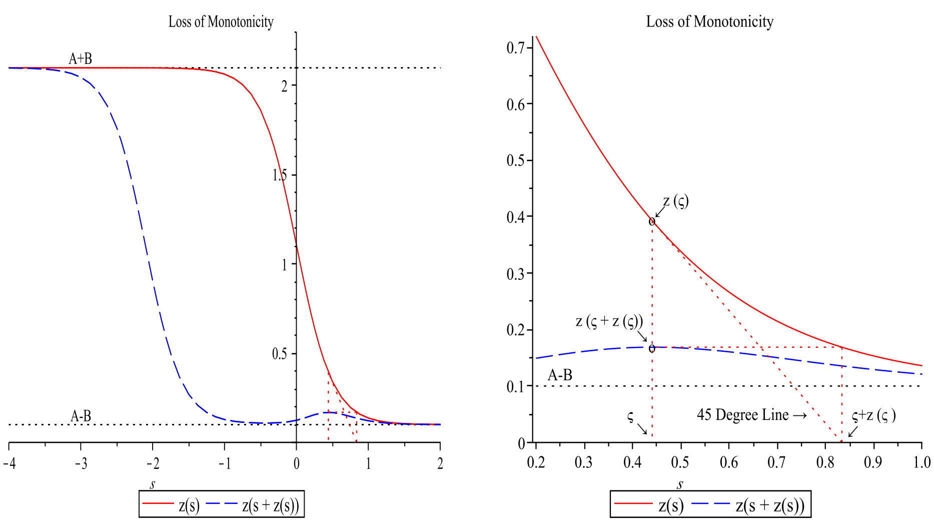

FIG. 7.2. *Loss of monotonicity. Detail at right*

T_2 also fails to preserve a different kind of monotonicity: Assume that z_1, z_2 are both decreasing functions in W_1 such that for all s

$$b < z_2(s) < z_1(s) < a.$$

It is then in general *not* true that for all s $T_2 z_2(s) < T_2 z_1(s)$. A graphical exploration using the 45 degree rule can be used to construct counterexamples. This observation shows that it would be difficult to solve (4.2) using monotonicity methods.

As T_2 moves functions in W_1 "to the left" and flattens them, T_1 moves such functions "down". It is certainly plausible that T_1 and T_2 will have exactly the same effect on some functions (which, if true, would assert the existence of solutions to our problem).

While T_1 moves z down it is not true that T_1 maps W_1 into itself. It is easy to construct examples of functions z in W_1 for which, say, $T_1 z$ violates the lower bound b. It turns out that a better way is to invert T_1.

7.2. Inverting T_1. The invertibility of T_1 on the class W_1 is given by the next theorem. Assume that h is continuous, strictly decreasing, and that $h(-\infty) = a > b = h(\infty) > 0$. We also recall that α and β are strictly positive constants.

THEOREM 7.3. *There is a unique function $z = z(s)$ such that*
- $(z+\alpha)^2 z' + \beta z = \beta h$
- $z(-\infty) = a, \quad z(\infty) = b$
- $\forall s \quad z'(s) < 0.$

Observe that $z = T_1^{-1} h$. We remark that for the simpler linear case where $z' + \beta z = \beta h$ a corresponding result holds, and it is easy to find an integral representation for z :

$$z(s) = \beta \int_0^\infty e^{-\beta x} h(s-x)\, dx.$$

The properties of z can be derived from this formula. The case at hand is more difficult.

Proof. Uniqueness: from the equation it is clear that every solution with the required properties will satisfy $z > h$ (this follows from $z = h - \frac{1}{\beta}(z+\alpha)^2 z'$). Write the equation as

$$\frac{d}{ds}\frac{1}{3}(z(s)+\alpha)^3 = -\beta(z(s) - h(s)).$$

Integration gives

$$\frac{1}{3}\left((b+\alpha)^3 - (a+\alpha)^3\right) = -\beta \int (z(s) - h(s))\, ds.$$

The left-hand-side depends only on a and b. Therefore, if we had two solutions $z_1.z_2$, it would follow that $\int (z_1(s) - z_2(s))\, ds = 0$. By continuity there has to be a s_0 where $z_1(s_0) = z_2(s_0)$, and this would imply $z_1 = z_2$ by the uniqueness theorem for ODEs.

Existence is a little less trivial, as we cannot start integration at $\pm\infty$. Instead, we use an approximation procedure. Choose a sequence $(s_n)_{n=1,2,\ldots}$ with $s_n \to -\infty$, and consider for each n solutions z_l, z_h of the ODE with $z_l(s_n) = h(s_n), \quad z_h(s_n) = a$. These solutions are well defined everywhere, but we only consider them for $s \geq s_n$. For $s < s_n$ we set $z_h(s) \equiv a, \quad z_l(s) = h(s)$. This means that the approximants do not satisfy the equation for $s \leq s_n$. The functions z_h and z_l defined in this way are continuous and differentiable for $s > s_n$, and they have the following properties.

a) for all s $h(s) \leq z_l(s) \leq z_h(s) \leq a,$ and for all $s > s_n$ $h(s) < z_l(s) < z_h(s) < a$.
b) $\lim_{s\to\infty} z_h(s) = b = \lim_{s\to\infty} z_l(s) = \lim_{s\to\infty} h(s)$. Similarly, all these limits as $s \to -\infty$ are a.

c) for $s > s_n$ $z_l'(s) < 0, z_h'(s) < 0$.

To prove these, note first that by uniqueness z_h and z_l cannot cross for $s > s_n$. Further, z_l cannot cross h, for if we had $z_l(t) = h(t)$ for some $t > s_n$ it would follow from the equation that $z_l'(t) = 0$ while $h'(t) < 0$, a contradiction. This shows a). Part c) is immediate from a) and from the equation. As for b), we only need to show the first equality. Clearly, $\lim_{s\to\infty} z_h(s)$ exists. If this limit is, say, $c > b$, we integrate as before to find

$$\frac{1}{3}\left((z_h(s)+\alpha)^3 - (a+\alpha)^3\right) = -\beta\int_{s_n}^{s}(z_h(\tau)-h(\tau))\,d\tau$$

and observe that the integral will diverge as $s \to \infty$, a contradiction.

To complete the proof we send $n \to \infty$, so $s_n \to -\infty$. We obtain two sequences of functions, $z_l^n(s)$ and $z_h^n(s)$, such that for $s > s_n$

$$h(s) \leq z_l^n(s) < z_l^{n+1}(s) < z_h^{n+1}(s) < z_h^n(s) < a.$$

It follows that both sequences must converge uniformly to limits z_l and z_h, both solutions of the equation with the same boundary conditions. By uniqueness, it follows that $z_l = z_h$. This completes the proof. □

By the previous results the operator T_2 maps W_1 into itself, and T_1^{-1} is defined on W_1. But it may well be that if $h \in W_1$ then $T_1^{-1}h \notin W_1$: the slope may become too steep. We next show that this will not happen if there are reasonable constraints on the parameters.

Lemma 7.4. *Assume that* $\beta(a-b) \leq (\alpha+b)^2$. *Then, if* $h \in W_1$ *and* $z = T_1^{-1}h$ *we have* $z \in W_1$.

Proof. All we have to do is to show that $|z'(s)| \leq 1$. But from the equation for z we have $|z'|(z+\alpha)^2 = \beta|z-h|$ (observe that for all s we have $z(s) > h(s)$.) Solving for $|z'|$ we obtain

$$|z'| = \frac{\beta(z-h)}{(z+\alpha)^2} \leq \frac{\beta(a-b)}{(b+\alpha)^2} \leq 1.$$

□

This result is straightforward and uses only the simplest estimates. Notice that the condition involves all four parameters. We can do better. Here is an alternative condition, not involving the parameter a, giving the same result. We are grateful to our colleague Rod Edwards for showing us this condition.

Lemma 7.5. *Suppose that* $\beta \leq 2(b+\alpha)$. *Then* $z \in W_1$.

Proof. We will assume that $z \notin W_1$ and show that then also $h \notin W_1$. To this end, assume that there is a s_1 such that $z'(s_1) < -1$. From the differential equation it is clear that z'' exists, and by differentiation we find

$$h' = z' + \frac{1}{\beta}[2(z')^2(z+\alpha) + z''(z+\alpha)^2].$$

We may further assume that s_1 is the location where z' assumes its minimum, and hence $z''(s_1) = 0$. Therefore

$$h'(s_1) = z'(s_1)\left[1 + \frac{2}{\beta} z'(s_1)(z(s_1) + \alpha)\right].$$

By using the assumption $z'(s_1) < -1$ twice in the above identity we obtain the estimate

$$h'(s_1) > \frac{2}{\beta}(z(s_1) + \alpha) - 1 > \frac{2}{\beta}(b + \alpha) - 1 \geq 0,$$

where the final estimate follows from the condition $\beta \leq 2(b + \alpha)$. Hence $h'(s_1) > 0$, contradicting $h \in W_1$. □

REMARK 4. *It is instructive to test whether the parameter values used in the simulations in Section 6 satisfy these conditions. A quick check shows that the data from Table 6.1 satisfy the conditions in both lemmas. The data from Table 6.2 violates the condition in Lemma 7.4 but satisifies the condition in Lemma 7.5, and the data from Table 6.3 violate both conditions.*

If we assume that the parameter constraints from either lemma are satisfied, then clearly $T = T_1^{-1}T_2$ maps W_1 into itself. Unfortunately, It does not seem to do so contractively, so we cannot use the Banach fixed point theorem to assert existence of a fixed point. The Schauder fixed point theorem is also not directly applicable; the problem is that while W_1 is a family of equicontinuous functions, it is not pre-compact because the domain is the whole real line, so the Arzela-Ascoli theorem is not applicable in direct form. Regrettably, while we know that an iteratively defined sequence $z_0 := h \in W_1, \;\; z_{n+1} = Tz_n$ will stay in W_1, we cannot assert that there will be a convergent subsequence. This difficulty is related to the translation invariance of (4.2).

It is useful to visualize how the sequence $\{z_n\}$ could fail to converge: There could be a) progressive "flattening", where the slopes of the z_n would converge to zero on compact subsets, and the z_n themselves may (or may not) approach a constant in $[b, a]$ on compact sets, while the boundary conditions are violated in the limit. Or, b) the wave could wander away to plus or minus infinity, leaving again a constant limit a or b but violating one of the boundary conditions in the limit. One needs an additional compactness constraint to prevent this behaviour. A condition of the following type suffices.

Assumption. Suppose that there is a nonempty subset $W_2 \subset W_1$ with the following two characteristics.

a) $T : W_2 \to W_2$ (invariance under T)
b) There are function h_l, h_u, both in W_1, such that for all $z \in W_2$ and all $s \in \mathbf{R}$

$$h_l(s) \leq z(s) \leq h_u(s).$$

The existence of such a W_2 will certainly depend on the choice of parameters. If there is such a set W_2 then one can easily prove that W_2 is compact with respect to the topology of uniform convergence. Therefore, the Schauder fixed point theorem applies on W_2 to the operator T, and we have

THEOREM 7.6. *If a W_2 with the properties a) and b) exists, then T has a fixed point in W_2.*

This result is not satisfactory, as we have no good criteria for the existence of such a W_2. Of course every fixed point itself is a possible element of such a set. Natural candidates for h_l and h_u are suitably scaled and shifted hyperbolic tangents (see Section 5), i.e., $h_l(s) = A+B\tanh(-\sigma_1(s+s_1))$ and $h_u(s) = A+B\tanh(-\sigma_2(s-s_1))$, but it is not clear under which conditions the set $\{z \in W_1; h_l \le z \le h_u\}$ is invariant under $T_2T_1^{-1}$. This is a challenge for the future; the numerical experiments from Section 6 provide some guidance. For example, it seems reasonable to expect that a W_2 as described will exists for the first example discussed there.

8. Concluding Remarks and an Applied Perspective. We have shown how the "jam" equation (1.1) arises from a kinetic traffic model in a formal high-density limit (where traffic is locally synchronized) and via a travelling wave ansatz. Removing the non-locality via Taylor approximation provides easily solvable ordinary differential equations with convincing travelling wave profiles. Further, we investigated functional-differential equations similar to the jam equation from a geometric point of view, and we explored a hyperbolic-tangent approximation to the expected wave profiles. Some numerical experiments were presented in Section 6. These experiments suggest that non-trivial solutions of (4.2) will exist for reasonable choices of the four parameters (α, β, a, b) but not for all choices (for example, as seen in Figure 6.4, our numerical procedure may fail to converge and produce instead a profile wandering off to ∞. The functional-analytic discussion in Section 7 is consistent with these observations: we were able to identify non-trivial solutions of (4.2) as fixed points of suitable operators T_1, T_2 and managed to find function sets invariant under $T_1^{-1}T_2$. This required constraints on the parameter set which are consistent with the results in Section 6. For the final Theorem we needed an additional invariance assumption to overcome the lack of compactness in our function sets. It remains an interesting (and presumably hard) open problem to prove that this assumption really holds for suitable parameters.

The applicability of our work is manifold: first, we obtained reasonable bounds on the dimensions of possible wave speeds for traveling waves. Second, there is the relevance to identify accurate speed profiles in braking waves, which may prove useful for traffic guidance systems in congested domains (for example, optimal speed limits to guarantee safety and maximize flux at the same time). Our models may also give useful information on boundary conditions $(u(infty), u(\infty))$ for which traveling waves exist.

A similar theory can be developed for acceleration waves. The localized version of the corresponding "unjam" equation was already described in [14].

For practical purposes transition regimes between braking and acceleration will require additional modelling ingredients, because otherwise unrealistic behaviour may result. This is work in progress. For example, the model introduced in this paper can naturally be coupled with the more common models where drivers are at liberty to brake or accelerate according to a fundamental diagram.

9. Appendix. Here it is explained which parameter choices (α, a, b) will produce Mott-Smith approximants for which $\beta(-\infty) = \beta(\infty)$. Recall $a = A + B, b = A - B$. From the formula (5.2) for $\beta(s)$ we have

$$\beta(-\infty) = -\frac{\sigma(a+\alpha)^2(1+\tanh(-\sigma a))}{\tanh(-\sigma a)}$$

and

$$\beta(\infty) = -\frac{\sigma(b+\alpha)^2(1-\tanh(-\sigma b))}{\tanh(-\sigma b)}$$

so both will be equal if (after multiplying both by σ)

$$(9.1) \qquad (a\sigma+\alpha\sigma)^2\frac{1-\tanh(\sigma a)}{\tanh(\sigma a)} = (b\sigma+\alpha\sigma)^2\frac{1+\tanh(\sigma b)}{\tanh(\sigma b)}.$$

So our question whether we can have a $\beta(s)$ which approximates the same constant as $s \to \pm\infty$ will be answered in the affirmative if we find solutions of (9.1) in acceptable ranges.

Proposition 9.1. *Let $\sigma > 0$. Then (9.1) possesses solutions $a > b > 0$ if $\alpha > 0$ is sufficiently small (relative to σ).*

Proof. First note that σ scales all the variables (α, a, b) in (9.1). We can therefore just set $\sigma = 1$. Then (9.1) simplifies to

$$(9.2) \qquad (a+\alpha)^2\frac{1-\tanh a}{\tanh a} = (b+\alpha)^2\frac{1+\tanh b}{\tanh b}.$$

For $\alpha = 0$ the right-hand side is $b^2(1+\tanh b)/\tanh b$, and by L'Hôpital's rule $\lim_{b\to 0} b^2(1+\tanh b)/\tanh b = 0$. Therefore, for all $\epsilon > 0$ there is an $\alpha(\epsilon)$ such that for all $\alpha < \alpha(\epsilon)$

$$\inf_{b>0}(\alpha+b)^2(1+\tanh b)/\tanh b \leq \epsilon,$$

although

$$\lim_{b\to 0}(\alpha+b)^2(1+\tanh b)/\tanh b = \infty.$$

The two sides of equation (9.2) are depicted in Figure 9.1, where x is written for a and b respectively. For small enough α there is an interval (dependent on α) $I := [a_1, a_2]$ such that the left-hand side of (9.2) is increasing as a function of a on I, and

$$\min_{a\in I}(a+\alpha)^2(1-\tanh a)/\tanh a \geq \min_{b>0}(b+\alpha)^2(1+\tanh b)/\tanh b.$$

For each $a \in I$ there are then 2 values for b (one for $a = a_1$) such that (9.2) holds. See Figure 9.1.
□

The corresponding $\beta(s)$ for $a = 0.6,\ b = 0.01,\ \sigma = 1$ and $\alpha = 0.05$ is plotted in Figure 9.2

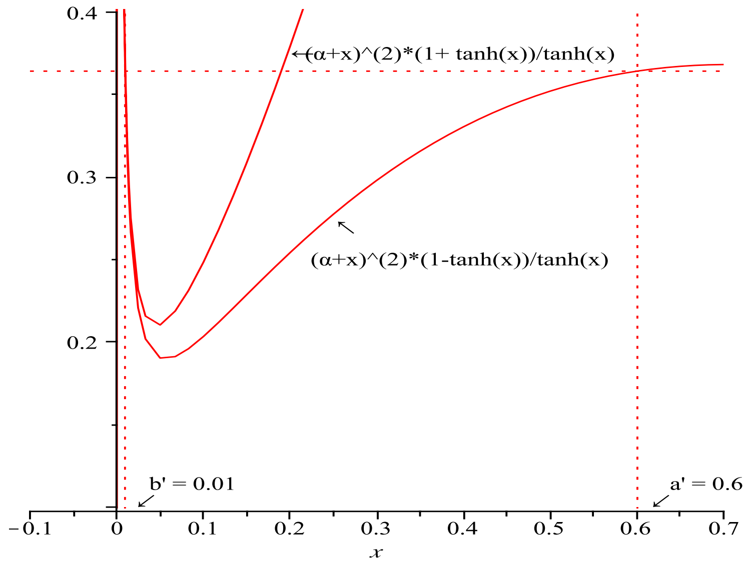

FIG. 9.1. *Geometric background for Equation 9.2*

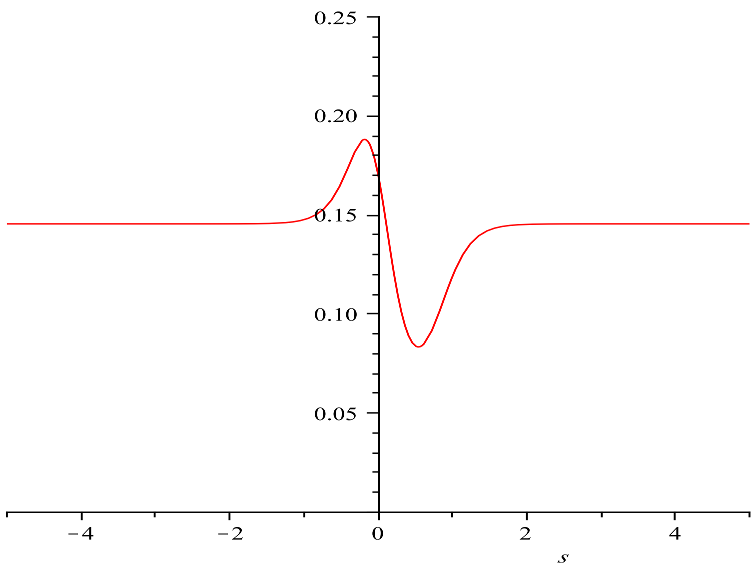

FIG. 9.2. *A possible $\beta(s)$ from Equation 9.2*

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REFERENCES

[1] A. Aw and M. Rascle, *Resurrection of "second order" models of traffic flow.*, SIAM J. Appl. Math., 60 (2000), pp. 916–938.
[2] E. Ben-Naim, P. L. Krapinsky, S. Redner, *Kinetics of clustering in traffic flows*, Physical Rev. E 50(2), (1994) pp. 822–829.
[3] S. S. Cheng, W. Li, *Analytic Solutions of Functional Equations*, World Scientific (2008)
[4] C. Cercignani, A. Frezzotti, P. Grosfils, *The structure of an infinitely strong shock wave*, Phys. Fluids 11, (1999), 2757–2765.
[5] J. Dolbeault and R. Illner, *Entropy methods for kinetic models of traffic flow*, Commun. Math. Sci, 1 (2003), pp. 401–423.
[6] I. Gasser, T. Seidel, G. Sirito, and B. Werner, *Bifurcation Analysis of a Class of Car Following Traffic Models II: Variable Reaction Times and Agressive Drivers*, Bulletin of the Institute of Mathematics, Academica Sinica (New Series), 2 (2007), pp. 587–607.
[7] I. Gasser, G. Sirito, and B. Werner, *Bifurcation analysis of a class of 'car following' traffic models.*, Physica D, 197 (2004), pp. 222–241.
[8] J. Greenberg, *Extensions and amplifications of a traffic model of Aw and Rascle.*, SIAM J. Appl. Math., 62 (2001), pp. 729–745.
[9] ———, *Congestion redux.*, SIAM J. Appl. Math., 64 (2004), pp. 1175–1185.
[10] ———, *Traffic congestion – an instability in a hyperbolic system*, Bulletin of the Institute of Mathematics, Academica Sinica (New Series), 2 (2007), pp. 123–138.
[11] J. Greenberg, A. Klar, and M. Rascle, *Congestion on multilane highways.*, SIAM J. Appl. Math., 63 (2003), pp. 818–833.
[12] D. Helbing, *Traffic dynamics. New physical concepts of modelling. (Verkehrsdynamik. Neue physikalische Modellierungskonzepte.)*, Berlin: Springer. xii, 308 p. DM 128.00; öS 934.40; sFr 113.00 , 1997.
[13] D. Helbing, A. Hennecke, V. Shvetsov, and M. Treiber, *Micro- and macro-simulation of freeway traffic.*, Math. Comput. Modelling, 35 (2002), pp. 517–547.
[14] M. Herty, R. Illner,*On stop-and-go waves in dense traffic.*, Kinetic and Related Models 1(3)(2008), pp. 437-452.
[15] M. Herty, R. Illner, *Analytical and numerical investigations of refined macroscopic traffic flow models*, Kinetic and Related Models 2(3) (2010), pp. 311-334.
[16] M. Herty, R. Illner, A. Klar, and V. Panferov, *Qualitative properties of solutions to systems of Fokker-Planck equations for multilane traffic flow.*, Transp. Theory Stat. Phys., 35 (2006), pp. 31–54.
[17] M. Herty and A. Klar, *Modelling, simulation and optimization of traffic flow networks*, SIAM J. Sci. Comp., 25 (2003), pp. 1066-1087.
[18] M. Herty and M. Rascle, *Coupling conditions for a class of second-order models for traffic flow.* , SIAM J. Math. Anal., 38 (2006), pp. 595-616.
[19] R. Illner, C. Kirchner, and R. Pinnau, *A derivation of the Aw–Rascle traffic models from fokker-planck type kinetic models*, Quarterly Appl. Math., 67(1) (2009), pp. 39–45
[20] R. Illner, A. Klar, and T. Materne, *Vlasov-Fokker-Planck models for multilane traffic flow.*, Commun. Math. Sci., 1 (2003), pp. 1–12.
[21] A. Klar, R .Wegener, *A hierarchy of models for multilane vehicular traffic. I. Modeling.*, SIAM J. Appl. Math., Vol. 3 (1999), pp. 983–1001
[22] B. Kerner, *The Physics of traffic*, Springer, Berlin, 2004.
[23] J. P. Lebacque, *Les modèles macroscopiques de trafic*, Annales des Ponts 67, 3rd trim, (1993), pp 28-45.
[24] P. I. Richards, *Shock waves on the highway*, Oper. Res., 4 (1956), pp. 42–51.
[25] A. Richardson, *M.Sc. Thesis*, University of Victoria (2011), in preparation.
[26] L. Santen, A. Schadschneider, M. Schreckenberg, *Towards a realistic microscopic description of highway traffic*, J. Phys A, Vol. 33, (2000), pp. 477–485
[27] S. Marinosson, R. Chrobok, A. Pottmeier, J. Wahle, M. Schreckenberg, *Simulation framework for the autobahn traffic in North Rhine-Westphalia*, Cellular automata 315–324, Lecture Notes in Comput. Sci., 2493, Springer, Berlin (2002)
[28] T. Alperovich, A. Sopasakis, *Stochastic description of traffic flow*, J. Stat. Phys., Vol. 133 (2008), pp. 1083-1105
[29] A. Sopasakis, M.A. Katsoulakis, *Stochastic modeling and simulation of traffic flow: asymmetric single exclusion process with Arrhenius look-ahead dynamics*, SIAM J. Appl. Math., Vol. 66 (2006), pp. 921-944.
[30] M. Treiber and D. Helbing, *Macroscopic simulation of widely scattered synchronized traffic states.*, J. Phys. A, Math. Gen., (1999).

[31] H. M. Zhang, *A non–equilibrium traffic model devoid of gas–like behavior*, Tans. Res. B, Vol. 36 (2002), pp. 275–290

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