On Stop-and-Go Waves in Dense Traffic

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Abstract

From a Vlasov-type kinetic equation with nonlocal braking and acceleration forces, taken as a traffic model for higher densities, we derive macroscopic equations generalizing the second order model of conservation laws suggested by Aw and Rascle [1] and Zhang [2]. The nonlocality remains present in these equations, but more conventional, local equations are derived by using suitable Taylor expansion. A second order model of this type is discussed in some detail and is shown to possess traveling wave solutions that resemble stop-and-go waves in dense traffic. A phase space analysis suggests that inside the class of such traveling waves there are steady solutions of type $\rho\delta(v-u)$ that are stable inside the class of traveling wave solutions.

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1 Introduction

Every driver has had this experience: you are driving on the freeway, following a seemingly endless line of cars, all moving along at the same speed and similar distances from their lead car, when suddenly you see brake lights flash up ahead. Within seconds the driver in front of you hits the brakes, you have no choice but to follow suit, and so on backwards down the line. As car distances shrink in the process, the slowdown seems to reinforce itself, until you crawl along at ridiculously slow speed. Seconds, sometimes minutes pass, until suddenly, up ahead, car speeds pick up. Before long your lead car is also picking up speed. Eventually, everybody drives along much as before. You wonder what caused the jam. Curiously, there is no evidence of an accident, construction site or lane reduction.

What you have experienced is an instability inherent to the dynamical process of traffic flow. It is a commonplace phenomenon, widely observed, frustrating in its utter uselessness, and expensive; gasoline is wasted every time a car has to reduce and then regain speed in the process.

Empirical observations on such stop-and-go waves can be found, for example, in [3, 4, 5, 6]. In [5] B. Kerner reports 1997 observations on traffic equilibria and moving jams from the German autobahn. His observations show a bivalued fundamental diagram (defined as the measured functional relationship between density and flux, or density and average speed) for equilibrated traffic in moderate densities, and many density-speed "equilibria" for larger densities. These equilibria would persist for some time and then give way to moving jams as described above. The observations led Kerner [5, 7] and others [8] to formulate conjectures on the existence and stability of equilibria in traffic at high densities.

There have been attempts to model stop-and go waves via microscopic and macroscopic traffic models: we mention the papers [9, 10, 11, 12] and [13, 14] and the references given there.

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Here, our first objective is to obtain a new model from a kinetic approach. We take as our starting point a model of Fokker-Planck type suggested in [15][16], originally developed to compute a fundamental diagram for multilane traffic flow; the model produced a multi-valued fundamental diagram that qualitatively reproduced Kerner's observations in moderately dense traffic. The fundamental diagram computed there extended into denser traffic and predicted it to be one-valued for such densities. However, this latter feature is in contrast to Kerner's observations.

We mention in passing that the entropy analysis in [17] for the Fokker-Planck models shows that inside the class of spatially homogeneous solutions high density points on the fundamental diagram are stable, but this result does not exclude instability because of the restriction to spatial homogeneity.

The discrepancy is addressed in this current work. We first adjust the Fokker-Planck model from [15] to high-density situations; the key simplifying assumptions are that diffusion (random fluctuations in the acceleration or braking force) is small, that lane-changing is rare and can be ignored, and that the speed-dependent look-ahead distance functions are identical for braking and acceleration scenarios. As we omit diffusion, the resulting kinetic equation loses Fokker-Planck character and is really a Vlasov equation with a nonlocal force term. We next search for special distributional solutions of this Vlasov equation equation of type $f(x, v, t) = \rho(x, t)\delta(v - u(x, t))$. It turns out that solutions of this type must satisfy a set of macroscopic conservation equations with an unconventional and interesting nonlocality in the momentum equation.

The next step is to consider an approximation to (an example of) this system of macroscopic equations by removing the nonlocality via a Taylor expansion; truncation at the linear level produces momentum equations of Hamilton-Jacobi type which generalize the conservation type model due to Aw–Rascle [1, 2]. We choose to truncate at the quadratic level, producing a nonlinear momentum equation of Hamilton-Jacobi type with diffusive corrections.

Finally, we investigate whether these latter equations possess traveling wave solutions. The simple ansatz $(\rho(x, t), u(x, t)) = (\rho(x + vt), u(x + vt))$ with $v > 0$ (meaning that the wave travels backwards relative to a rest state along the road) leads to ordinary differential equations (ODEs) which are readily solvable in phase space $\{(u, u') \in \Re_+ \times \Re\}$. It turns out that under reasonable conditions on the model parameters, there are traveling "brake" and "acceleration" waves connecting high speed to low speed states. Every v in a parameter dependent range produces a phase diagram in which unstable and stable domains (relative to the class of traveling waves under consideration) are easily identified.

Our example model is simplistic but gives qualitatively reasonable brake and acceleration behaviour. The results promise that avoidance of moving jams on freeways by speed restrictions via traffic guidance systems is a possibility.

2 Modeling

Throughout this paper x, v, t will denote position (on the road), speed $\in [0, \infty)$, and time.
$f = f(x, v, t)$, $\rho = \rho(x, t), u = u(x, t)$ denote the kinetic density, the macroscopic density and the macroscopic (average) speed of cars on a highway (freeway) lane. As lane changing in high density is difficult, it is rare (sometimes, of course, you have to change lanes) and we ignore it. Our analysis is therefore confined to one lane or to a lane-homogenized scenario. By definition, we have the relationships

$$\rho = \int f \, dv, \quad \rho u = \int v f \, dv.$$

As the reaction of drivers is always driven by what they see ahead of themselves, we introduce the shorthand

$$u^X = u(x + H + Tv, t),$$

and similarly $\rho^X = \rho(x + H + Tv, t)$. Here, H is considered a minimal safety distance, and T is a characteristic reaction time which multiplies the driver's speed. Note how the (independent) speed variable enters into the macroscopic variables through this definition.

Our Vlasov model for high density traffic with a braking/acceleration force B (and without diffusion) is

$$\partial_t f + v\partial_x f + \partial_v \left(B(\rho, v - u^X)f\right) = 0 \tag{1}$$

where we are making the implicit assumption that B depends only on the density (at (x,t)) and at the relative speed (with respect to u^X) of the reference vehicle at (x,v). It is easy to think of and discuss more general scenarios (see, for example, [15][16]); for example, B could also depend on u^X, or on ρ^X rather than ρ. Such cases can be considered and are subject to the same analysis as follows; we confine our attention to the above case for simplicity.

Let $\phi \in C_0^\infty((-\infty,\infty)\times(0,\infty)\times(0,\infty))$ be a test function, $\phi = \phi(x,v,t)$.

f is called a weak solution of (1) if for all such ϕ

$$T_f(\phi) := \int\int\int \phi_t f + \phi_x v f + \phi_v(B(\rho, v - u^X)f)dx\ dv\ dt = 0. \tag{2}$$

Proposition. The distribution valued function $\rho(x,t)\delta(v - u(x,t))$ is a weak solution of (1) in the sense of (2) if and only if almost everywhere

$$\rho_t + (\rho u)_x = 0 \tag{3a}$$

$$\rho\left(u_t + u_x u - B(\rho, u - u^X)\right) = 0 \tag{3b}$$

Proof. Let $\partial_i\phi(\xi_1,\xi_2,\xi_3) = \frac{\partial}{\partial \xi_i}\phi(\xi_1,\xi_2,\xi_3)$. First, assume that ρ, u satisfy (3). Then, for $f(x,v,t) = \rho(x,t)\delta(v - u(x,t))$ we obtain from (2),

$$T_f(\phi) = \int\int \partial_3\phi(x,u(t,x),t)\rho(x,t) + \partial_1\phi(x,u(x,t),t)u(x,t)\rho(x,t) + \tag{4a}$$

$$\partial_2\phi(x,u(t,x),t)\rho(x,t)B(\rho(x,t),u(x,t) - u(x + H + Tu(x,t),t))dx\ dv\ dt \tag{4b}$$

We exploit that the speed variable is an independent variable in the kinetic model. To this end, set $\psi(x,t) := \phi(x,u(x,t),t)$. Then, clearly $\psi_t() = \partial_3\phi() + \partial_2\phi()u_t$, and (4) reads

$$\begin{aligned} T_f(\phi) = \int\int \partial_t\psi(t,x)\rho(x,t) - \partial_2\phi(x,u(x,t),t)u_t(x,t)\rho(x,t) + \\ +\partial_x\psi(t,x)\rho(x,t)u(x,t) - \partial_2\phi(x,u(x,t),t)u_x(x,t)u(x,t)\rho(x,t) + \\ +\partial_2\phi(x,u(x,t),t)\rho(x,t)B(\rho(x,t),u(x,t) - u(x + H + Tu(x,t),t))dx\ dt = 0 \end{aligned}$$

The result is immediate from this integral identity.

Conversely, let $f(x,v,t) = \rho(x,t)\delta(v - u(x,t))$ and satisfy (2). Then, $T_f(\phi) = 0$ and using the same trick as before we obtain (3) by taking variations in ϕ and ψ, respectively. □

In the following, we will further analyze the equations (3) and discuss specific properties.

2.1 The choice of $B(\rho, u - u^X)$

1. Let us mention and briefly discuss the simplest example $B(\rho, u - u^X) = -g(\rho)(u - u^X)$ for some function $g(\rho)$.

   Expanding into a Taylor series at u and substituting into (9), we find the Burgers equation at zeroth order (and our model is the model for 1D pressureless gas dynamics). If we retain the linear term, the momentum equation becomes

$$u_t + u_x u + g(\rho)(H + Tu)u_x = 0, \tag{5}$$

which is the momentum equation associated with the Aw–Rascle model (see [1]) with

$$\rho \frac{\partial p}{\partial \rho} = -g(\rho)(H + Tu). \tag{6}$$

In [18] this model was obtained from a Fokker-Planck model by moment approximations.

We point out a modeling weakness in the force term $g(\rho)(u - u^X)$. The function g is usually assumed to be of the type $c\rho^\gamma$ with some $\gamma > 0$, making it monotone increasing in ρ. The rationale is that one has to be braking harder in denser traffic; unfortunately, the simplicity of the ansatz implies that one also accelerates harder in denser traffic, an unreasonable assumption. A moment's thought shows that this problem cannot be avoided as long as one tries to write the force term as a tensor product of $g(\rho)$ and a function of $u - u^X$. The next example addresses this point.

2. For the second, our most important example, we make the more reasonable ansatz

$$B(\rho, v - u) = \begin{cases} -g_1(\rho)(v - u) & \text{if } \quad v - u > 0 \\ -g_2(\rho)(v - u) & \text{if } \quad v - u < 0 \end{cases} \tag{7}$$

and $g_2(\rho)$ should be a decreasing function of ρ. Although more complicated choices are possible, see below, we proceed with the simple setting

$$g_1(\rho) = c_1 \rho, \quad g_2(\rho) = c_2(\rho_{max} - \rho).$$

As seen in the forthcoming sections this choice is already sufficient to observe realistic traffic phenomena.

3. In [[15]] similar dependencies were assumed, but (in the absence of lane changing) the braking and acceleration forces there were given as

$$B(\rho, v - u) = \begin{cases} -c_1(\rho)(v - u)^2 & \text{if } \quad v - u > 0 \\ c_2(\rho_{max} - \rho)(v - u)^2 & \text{if } \quad v - u < 0 \end{cases} \tag{8}$$

This ansatz makes the constants c_1 and c_2 dimensionless, an attractive feature. However, we will not pursue this example further in this work.

For the time being our attention will be focussed on the general case, and shortly thereafter on choice (7).

2.2 Removing the nonlocality u^X

Next, we formally derive nonlinear partial differential equations from (9) in which the nonlocality u^X is removed. We obtain approximate equations by Taylor expansion.

Recall the general nonlocal conservation law (3):

$$\rho_t + (\rho u)_x = 0 \tag{9a}$$

$$\rho \left(u_t + u_x u - B(\rho, u - u^X) \right) = 0 \tag{9b}$$

Let $w := u - u^X$. Assuming that at $w = 0$ the force B has one-sided first and second order partial derivatives (this is clearly true for the mentioned examples), we distinguish the cases $u > u^X$ and $u < u^X$, and expand for $u > u^X$

$$B(\rho, u - u^X) = B(\rho, 0+) + B_u(\rho, 0+)(u - u^X) + \frac{1}{2} B_{uu}(\rho, 0+)(u - u^X)^2 + \dots \tag{10}$$

and for $u < u^X$

$$B(\rho, u - u^X) = B(\rho, 0-) + B_u(\rho, 0-)(u - u^X) + \frac{1}{2} B_{uu}(\rho, 0-)(u - u^X)^2 + \dots \tag{11}$$

If $B(\rho, w)$ is as in (7), the braking scenario $u > u^X$ gives $B(\rho, u - u^X) = -g_1(\rho)(u - u^X)$, and this is of course the exact Taylor expansion of B in w. The corresponding expansion holds for the acceleration case. It is now natural to also expand

$$(u - u^X) = -u_x(H + Tu) - \frac{1}{2} u_{xx}(H + Tu)^2 + \dots,$$

where all functions are evaluated at x, t. This assumes, of course, that u has the smoothness required for such an expansion, a nontrivial assumption.

In the case of (7), we have to make another small leap of faith to proceed with the discussion: We replace the condition

$$u - u^X > 0$$

(braking) by

$$u_x < 0$$

and, similarly, the condition $u - u^X < 0$ (acceleration) by $u_x > 0$. Mathematically it is clear that this cannot be rigorous, but it is accurate to first order with respect to a Taylor expansion. It also produces simpler, tractable equations, in which the nonlocalities disappear.

Summarizing, in the case of (7) we obtain up to second order from (9), the following system of equations:

$$\rho_t + (\rho u)_x = 0 \tag{12a}$$

$$u_t + uu_x - g_i(\rho)[u_x(H + Tu) + \frac{1}{2} u_{xx}(H + Tu)^2] = 0. \tag{12b}$$

where $i = 1$ while $u_x < 0$ and $i = 2$ while $u_x > 0$. This dependence makes (12) an equation of Hamilton–Jacobi type with a diffusion term.

Remark. If we consider only first–order expansions we have to remove the diffusion term and we recognize the structure of the Aw–Rascle model, albeit with density dependent factors $g_i(\rho)$ that change from braking to acceleration scenarios. The same procedure using (5) yields the Aw–Rascle model with a 'traffic pressure' law given by (6).

There are several rough approximations in the transition from (3) to (12): Very simple assumptions on B; truncated Taylor expansions; and replacement of $u - u^X > 0$ (or < 0) by $u_x < 0$ (or > 0) (it would seem more accurate to include the second order correction in our condition, as we have kept this condition in our Taylor approximation; however, the present condition is easier to implement).

The critical reader may object that this neglect of rigour could severely compromise the relevance of our model. This is possible. But, we have to keep in mind that even our starting model is only a rough model which incorporates qualitatively reasonable forces. As we are not talking about basic physics, but about forces enacted by humans reacting to complex scenarios, it is questionable whether much more accuracy is possible. Complexity yes, accuracy probably only with difficulty.

In particular, one needs to keep in mind that we need to include the possibility $u - u^X = 0$; in this case, no force is applied, as none is necessary. For (9), things can happen when $u_x = 0$, so at least at such locations the reduced model clearly differs from the original one. We should simply keep in mind that spots where drivers can change from braking to acceleration, or vice versa, need special attention. In reality these are the onset or offset points for stop-and-go waves. We discuss them later.

3 On traveling wave solutions for (12)

The equation 12 possesses traveling wave solutions that suggest a reasonable explanation of the stop-and-go waves (moving jams) described by Kerner [5].

Recall the choice of B from (7)

$$g_1(\rho) = c_1\rho, \quad g_2(\rho) = c_2(\rho_{max} - \rho).$$

Using the ansatz $s = x + vt, \rho(x,t) = \rho(s), \quad u(x,t) = u(s)$ in (12) leads to the simplified continuity equation

$$\frac{d}{ds}(\rho(u+v)) = 0, \tag{13}$$

or $\rho(s) = \frac{C}{u(s)+v}$. We simplify further by assuming that $v > 0$ (this means that we are looking for a wave traveling backwards along the road) and that $\rho = \rho_{max}$ if $u = 0$: Hence $C = \rho_{max}v$, and we have

$$\rho(s) = \frac{\rho_{max}v}{u(s)+v}. \tag{14}$$

Remark. It is obvious that every pair of constants $(\rho, u) \in [0, \rho_{max}] \times [0, \infty)$ is a steady solution of (7). Most of these trivial equilibria are unrealistic. The condition that $u = 0$ for maximal density is a reasonable restriction, and it entails (via (14)) a functional speed-density relationship for each positive v.

To proceed we abbreviate $\frac{d}{ds} ='$. The equation for u reduces to two different ordinary differential equations (ODEs), depending on whether we are in the braking ($u' < 0$) or acceleration ($u' > 0$) scenario. In the first case, we obtain from (12)

$$vu' + uu'(1 - c_1\rho T) - c_1\rho H u' = \frac{1}{2}c_1\rho(H + Tu)^2 u''. \tag{15}$$

Remark. If the Taylor expansion of $u - u^X$ were truncated after the linear term there would be no diffusion term. In that case, our system would be of Hamilton-Jacobi type without diffusion, and the right-hand side of (15) would vanish. It is easily seen that the only traveling wave solutions in this case are constants.

We insert (14) into (15) and simplify. The result is

$$u'' = F_b(u)u', \tag{16}$$

where

$$F_b(u) := 2\frac{(u+v)^2 - c_1\rho_{max}v(H+Tu)}{c_1\rho_{max}v(H+Tu)^2}.$$

We attach the index b to emphasize that F_b addresses the braking scenario. Analogous calculations for the acceleration case result in

$$u'' = F_a(u)u', \tag{17}$$

with

$$F_a(u) := 2\frac{(u+v)^2 - c_2\rho_{max}u(H+Tu)}{c_2\rho_{max}u(H+Tu)^2}.$$

The equations (16,17) can (in principle) be explicitly integrated: For (16), suppose that $H_b(u)$ is an antiderivative of F_b. Then (16) can be rewritten as $u' = H_b(u) + C$, or $\frac{du}{H_b(u)+C} = ds$, and after one more integration u will be given in implicit form. However, this procedure (standard in ODE theory) does not easily provide the structural information we seek. It is better to conduct a phase plane analysis in the phase plane with coordinates u and $z := u'$. Then, all points on $z \equiv 0$ are steady states. Now, consider first the braking case. Suppose that the minimal value of u is u_0, assumed at $s = +\infty$, and u decreases for all s. Writing $u'' = z'$, equation (16) is

$$\frac{dz}{du} = F_b(u), z < 0,$$

and equation (17) is

$$\frac{dz}{du} = F_a(u), z > 0$$

or using the Heaviside function $H(\cdot)$,

$$\frac{dz}{du} = F_a(u)H(z) + F_b(u)H(-z). \tag{18}$$

In phase space, a "deceleration" or "braking" traveling wave will be a parametric curve $(u(s), z(s))$ such that the following properties hold:

1. $u(\infty) = u_0, z(\infty) = 0$
2. $u'(s) = z(s) < 0$ for all s, there is a limit $\lim_{s\to-\infty} u(s) = u_{-\infty} < \infty$, and $\lim_{s\to-\infty} z(s) = 0$.

Similarly, an "acceleration" traveling wave will be a curve solving (17) such that

1. $u(-\infty) = u_0, z(-\infty) = 0$
2. $u'(s) = w(s) > 0$ for all s, there is a limit $\lim_{s\to\infty} u(s) = u_\infty < \infty$, and $\lim_{s\to\infty} z(s) = 0$.

Remark. A moving jam or stop-and-go wave should be thought of as a concatenation of an acceleration with a braking wave. It appears that it would take forever to reach the state (u_0, ρ_0) in braking, that one could stay forever in that state, and if one chooses to accelerate out of it, that would take forever, too. However, these are just artifacts of the modeling process: our waves are heteroclicinc orbits connecting steady solutions, and it is the nature of ODE modeling that the (stable or unstable) dynamics near such dynamics display exponential decay or growth. We discuss this further in the next section.

The following conditions on the functions F_a and F_b are sufficient to guarantee the existence of traveling wave solutions. We repeat our assumptions for completeness. Assume (as everywhere in this section and as in (7)) that $g_1(\rho) = c_1\rho,\ \text{ and }\ g_2(\rho) = c_2(\rho_{max} - \rho)$.

Theorem 1. *If $0 < v < \rho_{max}c_1H$ then for each v in that range and for each sufficiently small $u_0 = \inf u(s)$ (with the range for u_0 depending on v) there is a traveling deceleration wave solution of (12). If $c_2\rho_{max}T > 1$ and u_0 is sufficiently small, then there is an acceleration wave beginning with u_0.*

Proof. We cover the acceleration and deceleration cases separately. First, we repeat the definition of F_a and F_b :

$$F_a(u) := 2\frac{(u+v)^2 - c_2\rho_{max}u(H+Tu)}{c_2\rho_{max}u(H+Tu)^2}.$$

$$F_b(u) := 2\frac{(u+v)^2 - c_1\rho_{max}v(H+Tu)}{c_1\rho_{max}v(H+Tu)^2}.$$

In the acceleration case, the assumed condition implies that there is an $\alpha > 0$ such that $F_a(\alpha) = 0$ and $F_a(u) > 0$ for $0 < u < \alpha, F_a(u) < 0$ for $u > \alpha$. Note that α depends on v; if $v = 0$ we get $\alpha = 0$, and no acceleration wave emerges from any positive u_0.

One easily sees that for every $u_0 < \alpha$ there is a finite $u_\infty > \alpha$ such that

$$F_a(u_0) > 0, F_a(u_\infty) < 0$$

and $\int_{u_0}^{u_\infty} F_a(u)\ du = 0$. But this means that z, defined by integrating $\frac{dz}{du} = F_a(u)$ with the initial condition $z(u_0) = 0$, is a traveling wave connection between the states u_0 and u_∞.

The deceleration case is dealt with similarly. Studying $F_b(u)$, we observe that

$$F_b(0) = 2\frac{v^2 - c_1\rho_{max}Hv}{c_1\rho_{max}vH^2},$$

which is negative by the first condition. Clearly, $F_b(u)$ becomes positive for large enough u, the numerator is monotone increasing, and there is a unique $\beta > 0$ such that $F_b(\beta) = 0$. One checks with little effort that for each $u_0 < \beta$ there is a $u_{-\infty} > 0$ such that $\int_{u_0}^{u_{-\infty}} F_b(u)\ du = 0$. This shows the traveling wave connection in the braking regime. If $u_0 < \min\{\alpha, \beta\}$ then the two regimes connect and produce a full range traveling wave solution. This completes the proof. □

Remarks.

We remark that u and $z = u'$ are continuous at the "switch" from braking to acceleration, but u'' is in general discontinuous. This is clear, as we change models at the switch. It is also consistent with common sense.

Observe that if $\beta > \alpha$ then the range of u between α and β is stable in the class of traveling wave solutions with group speed v. This may be significant for the avoidance of stop-and-go waves by traffic guidance systems. The parameters α and β are given by

$$\alpha = \frac{2v - c_2\rho_{max}H + \sqrt{-4vc_2\rho_{max}H + c_2^2\rho_{max}^2H^2 + 4v^2c_2\rho_{max}T}}{2c_2\rho_{max}T - 2},$$

$$\beta = v(\frac{1}{2}c_1\rho_{max}T - 1) + \frac{1}{2}\sqrt{v^2\left(T^2\rho_{max}^2c_1^2 - 4c_1\rho_{max}\right) + 4c_1v\rho_{max}H}.$$

We provide a phase diagram showing the qualitative behaviour of (18) for the combined parameter values of $H = 1, T = 2$, $\rho_{max} = 1$ and $c_1 = 1.6$ and $c_2 = 1$. For the the maximal stability interval obtained by solving $\max_v (\beta - \alpha)$ and is given for the group velocity $v_{max} = 0.335$. Figure 1 shows the phase–diagram for $v = \frac{1}{2}$ and $v = v_{max}$.

4 Conclusions and Outlook

Starting with a Vlasov model (with nonlocal force terms) for traffic flow, we took these steps.

1. we derived macroscopic equations for special solutions of the type $f = \rho\delta(u - v)$, i.e., equilibrated traffic
2. we considered different braking and acceleration forces, acting nonlocally;
3. using Taylor expansions we removed the nonlocality, and then looked for traveling wave solutions to the arising PDEs;
4. for a suitable range of parameters such traveling waves exist.

   The associated phase diagram provides information on the stability of steady solutions
5. Suitable concatenations of braking and acceleration waves can be interpreted as stop– and go–waves.

Much work remains to be done and is on the agenda of future research.

1. First, the present stability observations apply only relative to a class of traveling waves with wave speed v. It is tempting, to search for a stability domain that will apply relative to all (or at least a large set of) wave speeds. This means that we have to analyse the behavior of the intervals $[\alpha(v), \beta(v)]$ as v varies.

2. The restriction that $\rho = \frac{\rho_{max} v}{u+v}$ arose from the requirement that standing traffic will correspond to maximal density. Formally, one does not have to work with this requirement (although it seems very reasonable); what other normalization are possible and physically meaningful?

3. Near steady states ($u = u_0, z = 0$) the model 12 should not be taken too seriously; in reality there is some noise on top of the dynamics, and this noise will of course be the trigger for leaving an unstable equilibrium (stable equilibria are all right, as even with noise the system will remain at or near the equilibrium).

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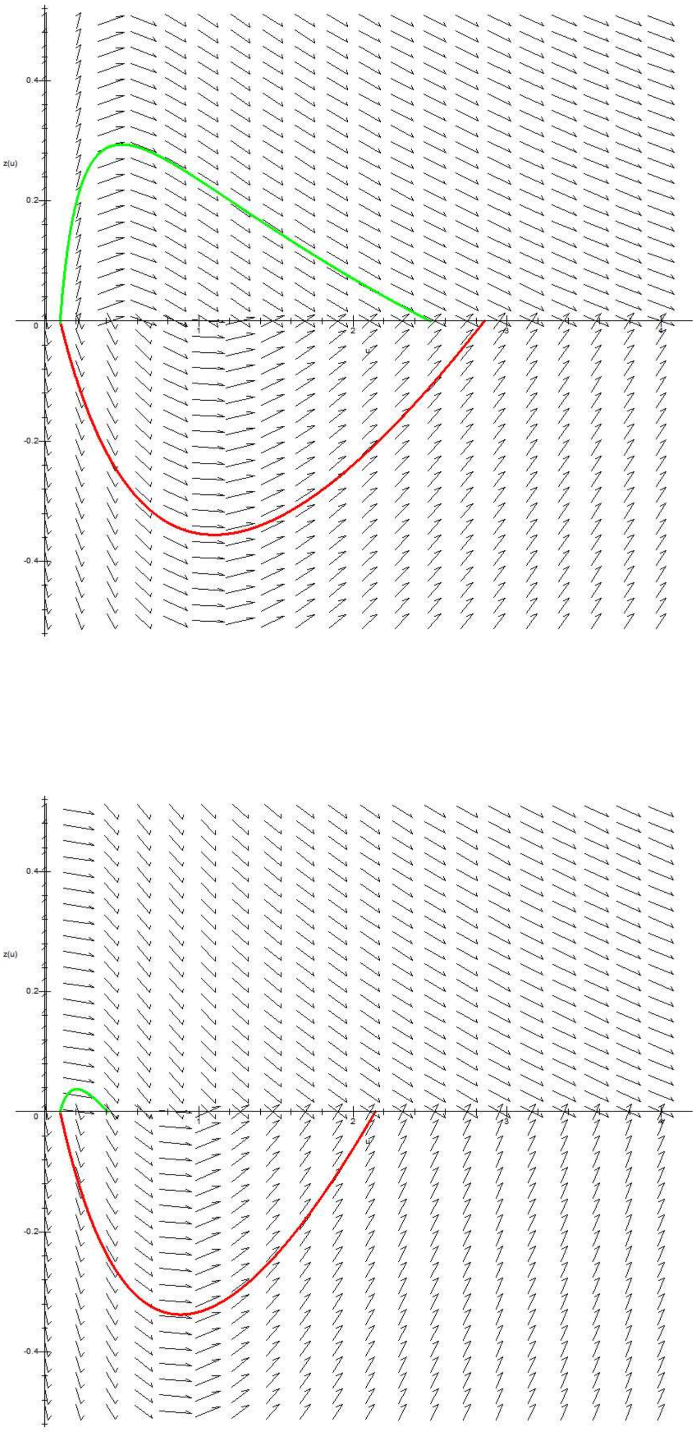

Figure 1: Vector field of the ODE (18) with $v = \frac{1}{2}$ (top) and $v = v_{max}$, respectively. A stop-and–go wave connecting three steady states by first a deceleration (red) wave and then acceleration (green) wave are depicted. The combined wave has the property described in the introduction.