COMMUTATIVITY PRESERVING LINEAR MAPS AND LIE AUTOMORPHISMS OF TRIANGULAR MATRIX ALGEBRAS

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1. Introduction and Statement of Main Results

A linear map φ on an algebra $\mathbb{A}$ is said to be *commutativity preserving* if $\varphi(a)$ commutes with $\varphi(b)$ for every pair of commuting elements a, b in $\mathbb{A}$. It is said to be *commutativity preserving in both directions* when the condition $ab = ba$ holds if and only if $\varphi(a)\varphi(b) = \varphi(b)\varphi(a)$. It is called a *Lie homomorphism* if $\varphi([a, b]) = [\varphi(a), \varphi(b)]$, where $[x, y]$ denotes $xy - yx$. The expression $[x, y]$ is referred to as the *Lie bracket.*

Commutativity preserving linear maps on spaces of matrices or operators have been considered by several authors, see [CJR], [CL], [O], [R] and [W]. In this article, we consider the algebra $\mathcal{T}_n = \mathcal{T}_n(\mathbb{F})$ of upper triangular n by n matrices over an arbitrary field $\mathbb{F}$. We characterize linear maps on $\mathcal{T}_n$ that preserve commutativity in both directions. As a consequence we characterize the Lie automorphisms of $\mathcal{T}_n$.

We observe that every algebra automorphism of any algebra is evidently a Lie automorphism. It is well-known that the algebra automorphisms of $\mathcal{T}_n$ are all *inner,* i.e if $\varphi : \mathcal{T}_n \to \mathcal{T}_n$ is an algebra automorphism, then there exists an invertible element X of $\mathcal{T}_n$ such that $\varphi(A) = X^{-1}AX$ for all $A \in \mathcal{T}_n$, (cf. [D, p. 234]). (This result will not be used here. Indeed, it follows as a corollary of our results.) To describe other Lie automorphisms, we make use of a particular permutation matrix J given by

$$J = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & & & & \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix}, \tag{1}$$

i.e $J = [\delta_{i,n+1-i}]$, where δ is the Kronecker delta symbol. If T^t denotes the transpose of T, then it is straightforward to verify that the map

$$\psi_{0}(T) = -JT^tJ \tag{2}$$

is a Lie automorphism of $\mathcal{T}_n$. The content of Theorem 5 is that every Lie automorphism is either an algebra automorphism or a composition of ψ_0 and an algebra automorphism. We thank the referee for pointing out that Lie automorphisms of $\mathcal{T}_n(R)$ have been characterized in [Đo] by D. Đoković, where R is a commutative ring in which the only idempotents are 0 and 1 .

Evidently every Lie automorphism φ of an algebra $\mathbb{A}$ preserves commutativity in both directions, as does the map $c\varphi + g$ for a non-zero scalar c and a linear map g mapping $\mathbb{A}$ into its centre. In view of Theorem 5, the assertion of Theorem 4 is that the above describes all linear maps on $\mathcal{T}_n$ that preserve commutativity in both directions. As in several other algebras, the linear maps that preserve zero Lie brackets in both directions differ only slightly from those that preserve all Lie brackets.

We note that it is well-known, and quite easy to prove, that the centre of $\mathcal{T}_n(\mathbb{F})$ is $\mathbb{F}I$. Thus linear maps from $\mathcal{T}_n$ into the centre are given by $T \mapsto f(T)I$, where $f : \mathcal{T}_n \to \mathbb{F}$ is a linear functional. We shall use the term "linear functional on $\mathcal{T}_n$" to denote both the linear map f of $\mathcal{T}_n$ into $\mathbb{F}$ as well as the corresponding map of $\mathcal{T}_n$

into $\mathbb{F}I$. Such a linear functional f will be called a *generalized trace* if $f(AB-BA)=0$ for every A and B in $\mathcal{T}_n$. We shall determine the commutators and generalized traces presently. First we state the result about the centre as a formal Lemma.

Lemma 1. *The centre of* $\mathcal{T}_n(\mathbb{F})$ *is* $\mathbb{F}I$.

Lemma 2. *The linear span of all commutators* $[A,B]$; $A,B\in\mathcal{T}_n$ *is the space of strictly upper triangular* n *by* n *matrices.*

Proof. It is easy to see that $[A,B]$ has zero diagonal. On the other hand, if $\{E_{ij} : 1\le i\le j\le n\}$ denote the usual matrix units, then $E_{ij}=[E_{ii},E_{ij}]$ for $i<j$ and so every strictly upper triangular matrix is a sum of commutators.

□

Lemma 3. *Let* $\tau : \mathcal{T}_n \to \mathbb{F}$ *be a generalized trace, i.e. a linear functional satisfying* $\tau(AB-BA)=0$ *for all* $A,B\in\mathcal{T}_n$. *Then there exist scalars* $b_1,b_2,\ldots,b_n$ *such that* $\tau([a_{ij}])=\sum_{i=1}^n b_i a_{ii}$, *i.e.* $\tau(A)=\mathrm{tr}\,(AB)$, *where* $B=\mathrm{diag}\,(b_1,b_2,\ldots,b_n)$ *is a diagonal matrix and* "tr" *denotes the usual trace.*

Proof. This is immediate from Lemma 2.

□

Now we state our main results:

Theorem 4. *Let* $\varphi : \mathcal{T}_n\to\mathcal{T}_n$ $(n\ge 3)$ *be a linear map. The following conditions are equivalent.*

- (a) φ *preserves commutativity in both directions.*
- (b) *There exists a non-zero scalar* $c\in\mathbb{F}$, *a linear functional* f *on* $\mathcal{T}_n$ *and an invertible matrix* $S\in\mathcal{T}_n$ *such that either*

  (i) $\varphi(T)=cS^{-1}TS+f(T)I$ *for every* $T\in\mathcal{T}_n$,
  or
  (ii) $\varphi(T)=cS^{-1}JT^tJS+f(T)I$ *for every* $T\in\mathcal{T}_n$, *where* J *is the matrix defined by equation* (1).

The above result is false for $n=2$ as will be shown in Proposition 8.

Theorem 5. *Let* $\varphi : \mathcal{T}_n\to\mathcal{T}_n$ *be a linear map. Then* φ *is a Lie automorphism of* $\mathcal{T}_n$ *if and only if* φ *takes one of the following forms:*

$$\varphi(T)=S^{-1}TS+\tau(T)I,$$

or

$$\varphi(T)=-S^{-1}JT^tJS+\tau(T)I,$$

where $S \in \mathcal{T}_n$ *is invertible,* τ *is a generalized trace on* $\mathcal{T}_n$ *such that* $\tau(I) \neq -1$, *and* J *is the permutation matrix defined by equation* (1).

Remarks. 1. By Lemma 3, we get that $\tau(A) = \operatorname{tr}(AD)$ where D is a diagonal matrix with $\operatorname{tr}(D) \neq -1$.

2. For $n \geq 3$, Theorem 5 follows readily from Theorem 4 and some easy calculations. For $n = 2$, a proof is given in the next section.

As a Corollary to Theorem 5, we obtain the following well-known result:

Corollary 6. *Every algebra automorphism of* $\mathcal{T}_n$ *is inner.*

We also get the following companion result:

Corollary 7. *A map* $\varphi : \mathcal{T}_n \to \mathcal{T}_n$ *is an algebra anti-isomorphism if and only if there exists an invertible matrix* S *such that*

$$\varphi(T) = S^{-1} J T^t J S,$$

where J *is the matrix defined by equation* (1).

Let us now fix some terminology which we shall require in the sequel. By $\mathcal{D}_n$ we denote the set of diagonal $n \times n$ matrices over a field $\mathbb{F}$, and for an $n \times n$ matrix A, we write $\sigma(A)$ for the set of eigenvalues of A. Given two vectors u and v in $\mathbb{F}^n$, we shall denote by $u \otimes v$ the matrix uv^t, which we may associate with the operator $(u \otimes v)(z) = v^t z\, u$ for each $z \in \mathbb{F}^n$. If $\{e_k\}_{k=1}^n$ denotes the standard basis for $\mathbb{F}^n$ (i.e. $e_1 = (1, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0)$, etc.), then the standard matrix units for $\mathcal{T}_n$ are denoted by $E_{ij} = e_i \otimes e_j$.

Given a subset $\mathbb{B}$ of an algebra $\mathbb{A}$, we denote the *commutant* of $\mathbb{B}$ by

$$\mathbb{B}' = \{A \in \mathbb{A} : AB = BA \quad \text{for all} \quad B \in \mathbb{B}\}.$$

The *double commutant* of $\mathbb{B}$ is nothing more than $\mathbb{B}'' = (\mathbb{B}')'$.

2. The Exceptional Case $n = 2$

Theorem 4 is false in the case $n = 2$. This follows from our next result.

Proposition 8. *Let* $\varphi : \mathcal{T}_2 \to \mathcal{T}_2$ *be a linear map. Then*

(a) φ *is commutativity preserving if and only if* $\varphi(I) \in \mathbb{F} I$, *or the range of* φ *is a commutative subspace of* $\mathcal{T}_2$.

(b) φ *preserves commutativity in both directions if and only if* $\varphi(I) \in \mathbb{F} I$ *and the range of* φ *is non-commutative.*

Proof. (a) If $\operatorname{ran}(\varphi)$, the range of φ, is commutative, then φ obviously preserves commutativity. Next, we assume that $\varphi(I) \in \mathbb{F} I$. If $A, B \in \mathcal{T}_2$ are commuting matrices, it is easy to verify that $\{I, A, B\}$ are linearly dependent. Hence $\{I, \varphi(A), \varphi(B)\}$ are also linearly dependent and $\varphi(A)$ commutes with $\varphi(B)$.

Conversely, if φ preserves commutativity and $\operatorname{ran}(\varphi)$ contains two non-commuting matrices C and D then $\{I, C, D\}$ are linearly independent and hence span all of $\mathcal{T}_2$. Therefore $\{\operatorname{ran}(\varphi)\}'$, the commutant of $\operatorname{ran}(\varphi)$, if $\mathbb{F}I$. Since $\varphi(I)$ commutes with $\operatorname{ran}(\varphi)$, we get $\varphi(I) \in \mathbb{F}I$.

(b) If φ preserves commutativity in both directions, then it follows that $\operatorname{ran}(\varphi)$ must be non-commutative since $\mathcal{T}_2$ is. Now part (a) implies that $\varphi(I) \in \mathbb{F}I$. To prove the converse, assume that $\varphi(I) \in \mathbb{F}I$ and $\operatorname{ran}(\varphi)$ is non-commutative. Upon adding an appropriate linear functional to φ if necessary, we obtain a linear mapping φ_1 of $\mathcal{T}_2$ into itself such that $\varphi_1(I) = I$ and $\operatorname{ran}(\varphi_1)$ is non-commutative. As in the proof of part (a), there exist $C, D \in \operatorname{ran}(\varphi_1)$ such that $\{I, C, D\}$ span $\mathcal{T}_2$, i.e. φ_1 is surjective and hence bijective. From (a), we see that both φ and φ^{-1} preserve commutativity. Thus φ preserves commutativity in both directions.

□

Example. Let $\varphi(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}) = \begin{bmatrix} (a+c)/2 & b \\ 0 & a+b \end{bmatrix}$. It follows from Proposition 8 that φ preserves commutativity in both directions. If $N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then $\varphi(N)$ is a non-zero idempotent. However, the forms described in Theorem 4 would lead to $\varphi(N)$ being a sum of a nilpotent and a scalar. This map is thus a counterexample to the assertion of Theorem 4 for $n = 2$.

Next, for the sake of completeness we shall prove Theorem 5 for $n = 2$. First we observe that when $n = 2$, the two forms of Theorem 5 coincide. Indeed, if $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$, then $-JA^tJ = \begin{bmatrix} -c & -b \\ 0 & -a \end{bmatrix} = S^{-1}AS - \operatorname{tr}(A)I$, where $S = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

Proof of Theorem 5 for $n = 2$. Assume that φ is a Lie automorphism of $\mathcal{T}_2$. Since $\operatorname{span}\{E_{12}\}$ is the set of all commutators, it follows that $\varphi(E_{12}) = rE_{12}$ for a nonzero scalar r. Also $\varphi(I) \in \mathbb{F}I$ by Proposition 8. It follows that there exists a linear map g from the diagonal algebra $\mathcal{D}_2$ into itself and a scalar s such that

$$\varphi(A) = g(\operatorname{diag} A) + \begin{bmatrix} 0 & rb + s(a-c) \\ 0 & 0 \end{bmatrix},$$

when $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ and $\operatorname{diag} A$ refers to the diagonal of A. Let $C = \begin{bmatrix} r & -s \\ 0 & 1 \end{bmatrix}$ and define ψ on $\mathcal{T}_2$ by $\psi(T) = \varphi(C^{-1}TC)$. Then ψ is a Lie automorphism of $\mathcal{T}_2$ and it follows from direct computation that $\psi(A) = g(\operatorname{diag} A) + \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$, i.e. $\psi(E_{12}) = E_{12}$ and $\psi|\mathcal{D}_2 = g$. Thus if $D = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}$, we have $[g(D), E_{12}] = [\psi(D), \psi(E_{12})] =$

$\psi([D, E_{12}]) = \psi((d_1 - d_2)E_{12}) = (d_1 - d_2)E_{12} = [D, E_{12}]$. Thus $[g(D) - D, E_{12}] = 0$, and since $g(D)$ is diagonal, we have that $g(D) - D$ is just a scalar. It follows that $\psi(A) = A + h(A)$ for some linear functional h, and hence $\varphi(T) = CTC^{-1} + \tau(T)$ for a linear functional τ. It is easy to see that τ must be a generalized trace. This proves that "only if" assertion of the Theorem. The converse is obvious.

□

3. The Proof of Theorem 4

The "if" part of Theorem 4 is obvious. We shall prove the converse via several lemmas and propositions. Throughout this section, we shall assume that φ is a linear map on $\mathcal{T}_n$ that preserves commutativity in both directions.

Lemma 9. (a) *The null space of* φ *is included in* $\mathbb{F}I$.
(b) *If* τ *is a linear functional, then* $\varphi + \tau$ *preserves commutativity in both directions and is bijective if and only if* $(\varphi + \tau)(I) \neq 0$.

Proof. Assertion (a) follows from the fact that $\mathbb{F}I$ is the centre of $\mathcal{T}_n$. The second assertion is trivial.

□

In view of the above Lemma, we may assume without loss of generality that φ is *bijective* and that $\varphi(I) = I$. This will be assumed throughout the remainder of this section.

Lemma 10. *If* $A \in \mathcal{T}_n$, *and* $\{A\}'' = \operatorname{span}\{I, A\}$, *then* $A = \lambda I + \beta B$, *where* $\lambda, \beta \in \mathbb{F}$ *and either* $B^2 = 0$ *or* $B^2 = B \neq 0$.

Proof. Now $A^2 \in \{A\}'' = \operatorname{span}\{I, A\}$, so $A^2 = \alpha I + \gamma A$ for some $\alpha, \gamma \in \mathbb{F}$. As such, A satisfies a polynomial of degree at most 2, and hence has at most two eigenvalues.

If $\sigma(A) = \{\lambda\}$, then $\sigma(A - \lambda I) = \{0\}$, and $(A - \lambda I)$ also satisfies a polynomial of degree 2, so $(A - \lambda I)^2 = 0$. Set $B = A - \lambda I$ to get $A = \lambda I + B, B^2 = 0$.

If $\sigma(A) = \{\lambda_1, \lambda_2\}$ with $\lambda_1 \neq \lambda_2$, then $B = (\lambda_2 - \lambda_1)^{-1}(A - \lambda_1 I)$ satisfies $\sigma(B) = \{0, 1\}$. Again, $\{I, B, B^2\} \subseteq \{A\}'' = \operatorname{span}\{I, A\}$, and so B satisfies a polynomial of degree 2. But then $B(B - I) = 0$, i.e. $B^2 = B$.

Thus $A = \lambda_1 I + (\lambda_2 - \lambda_1)B, \quad B^2 = B \neq 0$.

□

In what follows, commutants and double commutants are always relative to the algebra $\mathcal{T}_n$ and not the full algebra $\mathbb{M}_n$ of $n \times n$ matrices over $\mathbb{F}$. It may be useful to keep in mind the fact that for any subset $\mathcal{S}$ of $\mathcal{T}_n$, $\{\varphi(\mathcal{S})\}' = \varphi(\{\mathcal{S}\}')$.

Lemma 11. *Let* R *be a rank one matrix in* $\mathcal{T}_n$. *Then* $\{R\}'' = \operatorname{span}\{I, R\}$.

Proof. Let $R = x \otimes y$. As $R \in \mathcal{T}_n$, there exists an integer $k \in \{1, 2, \ldots, n\}$ such that $x \in \operatorname{span}\{e_1, e_2, \ldots e_k\}$ and $y \in \operatorname{span}\{e_k, e_{k+1}, \ldots, e_n\}$. We consider three cases:

(a) $1 < k < n$. Let $C = e_1 \otimes v$, where $v \in \{x\}^\perp$. Then $RC = CR = 0$. If $A \in \{R\}''$, then $AC = CA$, i.e., $Ae_1 \otimes v = e_1 \otimes A^t v$, and hence $A^t v = a_{11} v$ for every $v \in \{x\}^\perp$. Similarly, if $u \in \{y\}^\perp$ and $D = u \otimes e_n$, then $RD = DR$ and hence $AD = DA$, leading to $Au \otimes e_n = u \otimes A^t e_n$. But we already have that $A^t e_n = a_{11} e_n$ and so $Au = a_{11} u$ for every $u \in \{y\}^\perp$. Let $B = A - a_{11} I$. Then $Bu = 0 = B^t v$ for every $u \in \{y\}^\perp$ and every $v \in \{x\}^\perp$. It follows that $B = \beta x \otimes y$ for a scalar β , and so $A \in \text{span}\,\{I, R\}$.

(b) $k = 1$. In this case, $R = e_1 \otimes y$. If $u \in \{y\}^\perp \cap \text{span}\,\{e_1, e_2\}$, $j \geq 2$, and $C = u \otimes e_j$, then $RC = CR = 0$. If $A \in \{R\}''$, then $AC = CA$, i.e., $Au \otimes e_j = u \otimes A^t e_j$. Thus $Au = a_{22} u$ and $A^t e_j = a_{22} e_j$ for $j \geq 2$. Let $B = A - a_{22} I$. Then $B^t e_j = 0$ for $j \geq 2$. Next, we take $v \in \{y\}^\perp$, $D = v \otimes e_n$. We then have $RD = DR = 0$, and hence $Bv \otimes e_n = v \otimes B^t e_n$. We conclude that $Bv = 0$ for every $v \in \{y\}^\perp$. Thus $B = \beta e_1 \otimes y$ for some scalar β . Hence $A \in \text{span}\,\{I, R\}$.

(c) $k = n$. The proof is similar to (b).

In each case, we have $\{R\}'' \subseteq \text{span}\,\{I, R\}$. The reverse inclusion is obvious.

□

For each $1 \leq i \leq j \leq n$, we set $F_{ij} := \varphi(E_{ij})$, and also write F_m for F_{mm}. In view of the fact that $\{\varphi(A)\}' = \varphi(\{A\}')$ for every $A \in \mathcal{T}_n$, it is an immediate consequence of Lemmas 10 and 11 that each F_{ij} is of the form $F_{ij} = \lambda_{ij} I + \beta_{ij} B_{ij}$, where $\lambda_{ij}, \beta_{ij} \in \mathbb{F}$ and either $B_{ij}^2 = 0$, or $B_{ij}^2 = B_{ij} \neq 0$. In fact, more is true.

Lemma 12. *There exists* $1 \leq m \leq n$ *such that* $F_m = \lambda_{mm} I + \beta_{mm} B_{mm}$, *with* $B_{mm}^2 = B_{mm} \neq 0$.

Proof. Suppose otherwise. Then for each $1 \leq m \leq n$, write $N_m = \beta_{mm} B_{mm}$ to get $F_m = \lambda_{mm} I + N_m$, $N_m^2 = 0$.

For some $1 \leq i \leq j \leq n$, $\varphi(E_{ij}) = \lambda_{ij} I + \beta_{ij} B_{ij}$ with $\beta_{ij} \neq 0$ and $B_{ij}^2 = B_{ij} \neq 0$. For otherwise, $\text{diag}\,(\varphi(T)) \in \mathbb{F} I$ for all $T \in \mathcal{T}_n$, contradicting the surjectivity of φ. For this fixed i and j, consider $F_i = \lambda_{ii} I + N_i$, $F_j = \lambda_{jj} I + N_j$ and $F_{ij} = \lambda_{ij} I + \beta_{ij} B_{ij}$. Let $\{I, E_{ii}, E_{ij}, E_{jj}, U_5, U_6, \ldots, U_{n(n+1)/2}\}$ be a basis for $\mathcal{T}_n$ and let $\tau : \mathcal{T}_n \to \mathbb{F} I$ be the linear functional satisfying:

$$
\begin{aligned}
\tau(I) &= 0 \\
\tau(E_{ii}) &= -\lambda_{ii} I, \qquad \tau(E_{jj}) = -\lambda_{jj} I \\
\tau(E_{ij}) &= -\lambda_{ij} I \\
\tau(U_n) &= 0, \qquad 5 \leq m \leq n(n+1)/2.
\end{aligned}
$$

Set $\rho := \varphi + \tau$. Since $\rho(I) = (\varphi + \tau)(I) = I + 0 \neq 0$, Lemma 9 shows that ρ satisfies all of the conditions of our original problem, as well as satisfying $\rho(E_{ii}) = N_i$, $\rho(E_{ij}) = \beta_{ij} B_{ij}$, $\rho(E_{jj}) = N_j$ and $\rho(I) = I$. As such we may relabel so as to assume that $\varphi = \rho$. We shall obtain a contradiction through the following three steps:

(i) We have $N_i^2 = N_j^2 = 0$ and $[E_{ii}, E_{jj}] = 0$ forces $[N_i, N_j] = 0$. Moreover, $[(E_{ii} + E_{jj}), E_{ij}] = 0$ implies $[B_{ij}, N_i + N_j] = 0$.

(ii) CLAIM: $B_{ij}N_i + N_iB_{ij} = N_i$ and $B_{ij}N_j + N_jB_{ij} = N_j$.

PROOF OF CLAIM: $\varphi(E_{ii} + E_{ij}) = N_i + \beta_{ij}B_{ij}$.

By Lemma 10 and Lemma 11, $(N_i + \beta_{ij}B_{ij})$ satisfies a quadratic polynomial. Since $\sigma(N_i + \beta_{ij}B_{ij}) = \{0, \beta_{ij}\}$, that polynomial must be $p(t) = t\,(t - \beta_{ij})$. We therefore have $(N_i + \beta_{ij}B_{ij})^2 = \beta_{ij}(N_i + \beta_{ij}B_{ij})$, from which follows $N_iB_{ij} + B_{ij}N_i = N_i$.

The second equality is proved in a similar fashion, ending the proof of the claim.

(iii) With $N = N_i + N_j$, it follows from (ii) that $NB_{ij} + B_{ij}N = N$. On the other hand, $NB_{ij} = B_{ij}N$ since $E_{ii} + E_{jj}$ commutes with E_{ij} . It follows that $NB_{ij} = NB_{ij}^2 + B_{ij}NB_{ij} = 2NB_{ij}$. Therefore $NB_{ij} = 0$, and so $B_{ij}N = 0$. Thus $\varphi(E_{ii} + E_{jj}) = N = 0$. This contradicts the injectivity of φ , proving the Lemma.

□

Proposition 13. *For* $1 \le i \le n$, $\quad F_i = \varphi(E_{ii}) = \lambda_i I + \alpha_i Q_i$, *where* $0 \ne \alpha_i \in \mathbb{F}$, $\lambda_i \in \mathbb{F}$, *and* $I \ne Q_i^2 = Q_i \ne 0$.

Proof. By Lemma 12, there exists at least one j, $1 \le j \le n$ for which is is true. Suppose now that for some $1 \le i \ne j \le n$, $F_i = \varphi(E_{ii}) = \lambda_i I + N_i$ where $N_i^2 = 0$. (This is the only other alternative, by Lemmas 10 and 11.)

We can extend $\{I, E_{ii}, E_{jj}\}$ to a basis for $\mathcal{T}_n$, say $\{I, E_{ii}, E_{jj}, U_4, U_5, \ldots, U_{n(n+1)/2}\}$. Define a linear functional $\tau : \mathcal{T}_n \to \mathbb{F}I$ via $\tau(E_{ii}) = -\lambda_i I$, $\tau(E_{jj}) = -\lambda_j I$, $\tau(I) = 0 = \tau(U_m)$, $4 \le m \le n(n+1)/2$. Using the arguments of Lemma 9, we see that $\rho := \varphi + \tau$ satisfies the conditions of our original problem, as well as satisfying $\rho(E_{jj}) = \alpha_j Q_j$, $\rho(E_{ii}) = N_i$, $\rho(I) = I$. We relabel so that $\varphi = \rho$.

Next, $[E_{ii}, E_{jj}] = 0$ implies $[Q_j, N_i] = 0$. Since $R := E_{ii} + E_{jj}$ is an idempotent, it follows that $\{R\}'' = \text{span}\{I, R\}$. Hence $S := \varphi(E_{ii} + E_{jj}) = N_i + \alpha_j Q_j$ satisfies $\{S\}'' = \text{span}\{I, S\}$. It follows that S satisfies a quadratic equation. But $\sigma(N_i + \alpha_j Q_j) = \{0, \alpha_j\}$, and hence the equation must be $z(z - \alpha_j) = 0$. As such,

$$(N_i + \alpha_j Q_j)^2 = \alpha_j(N_i + \alpha_j Q_j).$$

We may expand this equation to obtain:

$$N_i^2 + \alpha_j N_i Q_j + \alpha_j Q_j N_i + \alpha_j^2 Q_j^2 = \alpha_j N_i + \alpha_j^2 Q_j,$$

or

$$2Q_j N_i = N_i.$$

Thus $2Q_j^2 N_i = 2Q_j N_i = Q_j N_i$, implying that $Q_j N_i = 0$, and hence $N_i = 2Q_j N_i = 0$, contradicting the injectivity of φ.

We conclude that $\varphi(E_{jj}) = \lambda_j I + \alpha_j Q_j$, $\alpha_j \ne 0$, $Q_j^2 = Q_j \ne 0$ for all $1 \le j \le n$.

□

Remark. It should be noted that the choice of Q_i above is not unique. If $F_i = \lambda_i I + \alpha_i Q_i$ with $Q_i^2 = Q_i \ne 0$, then $Q_i' = I - Q_i$ is again a nonzero idempotent, and $F_i =$

$(\lambda_i+\alpha_i)I+(-\alpha_i)Q_i'$. As we shall now see, this is the only latitude we have in determining $\alpha_i,\ 1\le i\le n$.

Proposition 14. *For* $1\le i\le n,\ F_i=\varphi(E_{ii})=\lambda_i I+\alpha Q_i,$ *where* $0\ne\alpha\in\mathbb{F},\ \lambda_i\in\mathbb{F}$ *and* $Q_i^2=Q_i\ne 0.$ *In other words, the* α_i*'s from Proposition 13 may be chosen to be identical.*

Proof. It clearly suffices to show that $\alpha_i=\alpha_1,\ 2\le i\le n$. To that end, fix $2\le j\le n$. Let $\tau:\mathcal{T}_n\to\mathbb{F}I$ be a linear functional satisfying $\tau(I)=0,\tau(E_{11})=-\lambda_1 I,\ \tau(E_{jj})=-\lambda_j I$ and $\tau(U_m)=0$, $4\le m\le n(n+1)/2$, where $\{I,E_{11},\ E_{jj},\ U_4,\dots,U_{n(n+1)/2}\}$ is a basis for $\mathcal{T}_n$.

If $\rho:=\varphi+\tau$, then by Lemma 9, ρ preserves commutativity in both directions, while $\rho(E_{11})=\alpha_1 Q_1$ and $\rho(E_{jj})=\alpha_j Q_j$ by Proposition 13. Thus $\rho(E_{11}+E_{jj})=\alpha_1 Q_1+\alpha_j Q_j$. Again, since $(E_{11}+E_{jj})$ satisfies a quadratic equation, we see as before that so must $(\alpha_1 Q_1+\alpha_j Q_j)$. Note also that $[E_{11},E_{jj}]=0$, forcing $[Q_1,Q_j]=0$. By Lemma 10, $\alpha_1 Q_1+\alpha_j Q_j=\lambda I+\beta B$, where $B^2=0$ or $B^2=B\ne 0$.

Since Q_1 and Q_j are commuting idempotents, they can be simultaneously diagonalized. Therefore $\sigma(\alpha_1 Q_1+\alpha_j Q_j)$ includes:

(a) α_1 if $\operatorname{ran} Q_1\not\subseteq\operatorname{ran}(Q_j)$;
(b) α_j if $\operatorname{ran} Q_j\not\subseteq\operatorname{ran}(Q_1)$;
(c) $\alpha_1+\alpha_j$ if $\operatorname{ran} Q_1\cap\operatorname{ran} Q_j\ne\{0\}$;
(d) 0 if $\operatorname{ran} Q_1+\operatorname{ran} Q_j\ne\mathbb{F}^n$.

Furthermore, since $(\alpha_1 Q_1+\alpha_j Q_j)=\lambda I+\beta B$ satisfies a quadratic equation, $\sigma(\alpha_1 Q_1+\alpha_j Q_j)$ consists of either one or two points. As such, we must have one of the following possibilities (keeping in mind that $0\ne Q_i\ne I$ for all i):

(i) $\alpha_1=\alpha_j$, and either $\operatorname{ran} Q_1\cap\operatorname{ran} Q_j=\{0\}$ or $\operatorname{ran} Q_1+\operatorname{ran} Q_j=\mathbb{F}^n$.
(ii) $\alpha_1=-\alpha_j$. Replace Q_j by $Q_j':=I-Q_j$. As in the Remark preceding this Proposition, we see that the corresponding $\alpha_j'=-\alpha_j=\alpha_1$.
(iii) $\operatorname{ran} Q_1=\operatorname{ran} Q_j$. This implies that $Q_1=Q_j$ since the two idempotents commute. This contradicts injectivity.
(iv) $\operatorname{ran} Q_1\cap\operatorname{ran} Q_j=0$, and $\operatorname{ran} Q_1+\operatorname{ran} Q_j=\mathbb{F}^n$, i.e. $\mathbb{F}^n=\operatorname{ran} Q_1\oplus\operatorname{ran} Q_j$. Then $Q_1+Q_j=I$, contradicting injectivity again.

□

Proposition 15. *For* $1\le i\le n,$ *let* $F_i=\varphi(E_{ii})=\lambda_i I+\alpha Q_i$ *with* $0\ne\alpha\in\mathbb{F},\ \lambda_i\in\mathbb{F}$ *and* $Q_i^2=Q_i\ne 0,$ *as derived from the previous proposition. Then for* $1\le i\ne j\le n,$ *either*

(a) $Q_iQ_j=Q_jQ_i=0;$ *or*
(b) $(I-Q_i)(I-Q_j)=(I-Q_j)(I-Q_i)=0.$

Proof. As in the proof of Proposition 14, we have that $Q_i\,Q_j=Q_j\,Q_i$, and $\sigma(Q_i+Q_j)$ has at most two points, implying that either $\operatorname{ran} Q_i\cap\operatorname{ran} Q_j=\{0\}$ or $\operatorname{ran} Q_i+\operatorname{ran} Q_j=$

$\mathbb{F}^n$. The former case is equivalent to assertion (a), while the latter is equivalent to (b).

$\square$

Proposition 16. *For* $1 \le i \le n$, *let* $F_i = \varphi(E_{ii}) = \lambda_i I + \alpha Q_i$ *with* $\alpha \ne 0$, $\lambda_i \in \mathbb{F}$, *and* $Q_i^2 = Q_i \ne 0$ *as derived from Proposition 14. Then either*

(a) $Q_i Q_j = 0 = Q_j Q_i$ *for all* $1 \le i \ne j \le n$, *or*
(b) $(I - Q_i)(I - Q_j) = (I - Q_j)(I - Q_i)$ *for all* $1 \le i \ne j \le n$.

Remark. The difference between this Proposition and the previous is the "for all" quantifier.

Proof. Suppose

(i) $Q_i Q_j = 0 = Q_j Q_i$

(ii) $(I - Q_i)(I - Q_k) = 0 = (I - Q_k)(I - Q_i)$

and (iii) $Q_j Q_k = 0 = Q_k Q_j$.

Then
$$\begin{aligned} 0 = Q_j 0 &= Q_j (I - Q_i)(I - Q_k) \\ &= (Q_j - 0)(I - Q_k) \\ &= Q_j - 0 \\ &= Q_j, \quad \text{a contradiction.} \end{aligned}$$

Alternatively, suppose (i) $Q_i Q_j = 0 = Q_j Q_i$

(ii) $(I - Q_i)(I - Q_k) = 0 = (I - Q_k)(I - Q_i)$

and (iii) $(I - Q_j)(I - Q_k) = 0 = (I - Q_k)(I - Q_j)$

Then
$$\begin{aligned} 0 = Q_i 0 &= Q_i (I - Q_j)(I - Q_k) \\ &= (Q_i - 0)(I - Q_k) \\ &= (I - Q_k), \quad \text{implying that} \quad I = Q_k, \quad \text{a contradiction.} \end{aligned}$$

Thus it is impossible to simultaneously have $Q_i Q_j = 0$ and $(I - Q_i)(I - Q_k) = 0$ for any $1 \le k \ne i \le n$. Combining this with Proposition 15, the statement follows.

$\square$

Remark. Suppose that with the above notation, we have

$$(I - Q_i)(I - Q_j) = 0 = (I - Q_j)(I - Q_i), \quad 1 \leq i \neq j \leq n.$$

Then we may set $\delta_i = \lambda_i + \alpha$, $\gamma = -\alpha$ and $V_i = (I - Q_i)$ to get $\varphi(E_{ii}) = \delta_i I + \gamma V_i$ and $V_i V_j = 0 = V_j V_i$, $1 \leq i < j \leq n$. As such, there is no loss of generality in assuming a priori that $Q_i Q_j = 0 = Q_j Q_i$.

Lemma 17. *Let* $\{Q_i\}_{i=1}^n$ *be* n *mutually disjoint non-zero commuting idempotents in* $\mathcal{T}_n$. *Then there exists an invertible operator* $R \in \mathcal{T}_n$ *and a permutation* $\pi : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ *such that*

$$R^{-1} Q_i R = E_{\pi(i)\pi(i)}, \quad 1 \leq i \leq n.$$

Proof. We shall argue by induction on n, the dimension of the underlying vector space. If $n = 1$, there is nothing to prove. Suppose therefore that the result holds when the dimension of the space is $(n-1)$. The fact that the Q_i 's are non-zero mutually disjoint idempotents forces $\sigma(Q_i) = \{0, 1\}$, $1 \leq i \leq n$. Since there are n such idempotents acting on an n-dimensional space, each idempotent must have precisely one "1" on the diagonal with the remaining diagonal entries being "0".

Choose $\pi(1) \in \{1, 2, \ldots, n\}$ such that $\langle Q_{\pi(1)} e_1, e_1 \rangle = 1$, $\langle Q_{\pi(1)} e_k, e_k \rangle = 0$, $2 \leq k \leq n$. Since $Q_{\pi(1)}$ is an idempotent in $\mathcal{T}_n$, we also have $\langle Q_{\pi(1)} e_j, e_k \rangle = 0$, $2 \leq j \leq n$, $j \leq k \leq n$. Thus

$$Q_{\pi(1)} = \begin{bmatrix} 1 & y_{12} & y_{13} & \ldots & y_{1n} \\ & 0 & 0 & \ldots & 0 \\ & & 0 & & \vdots \\ & & & \ddots & \\ & & & & 0 \end{bmatrix}. \quad \text{Set} \quad R_1 = \begin{bmatrix} 1 & -y_{12} & -y_{13} & \ldots & & -y_{1n} \\ & 1 & 0 & \ldots & & 0 \\ & & 1 & & & \vdots \\ & & & \ddots & 1 & 0 \\ & & & & & 1 \end{bmatrix}.$$

Obviously $R_1 \in \mathcal{T}_n$ and is invertible. Then $R_1^{-1} Q_{\pi(1)} R_1 = E_{11}$. Moreover, if we set $G_k = R_1^{-1} Q_k R_1$, $1 \leq k \leq n$, then $\{G_k\}_{k=1}^n$ forms a set of commuting non-zero idempotents (mutually disjoint) with $G_{\pi(1)} = E_{11}$. Since G_j commutes with $G_{\pi(1)}$ for all $1 \leq j \leq n$, and since $\langle G_j e_1, e_1 \rangle = 0$ from above, we see that for $1 \leq j \neq \pi(1) \leq n$,

$$G_j = \begin{bmatrix} 0 & 0 \\ 0 & G'_j \end{bmatrix},$$

where G'_j is the compression of G_j to the subspace $\text{span}\{e_2, e_3, \ldots, e_n\}$.

By our induction hypothesis, there exists R_2, a triangular operator acting on $\text{span}\{e_2, e_3, \ldots, e_n\}$ such that $\{R_2^{-1} G'_j R_2\}_{1 \leq j \neq \pi(1) \leq n}$ is a permutation of the diagonal matrix units of the corner algebra (i.e. the compression of $\mathcal{T}_n$ to $\text{span}\{e_2, e_3, \ldots, e_n\}$, which we may identify with $\mathcal{T}_{n-1}$).

Next, we set $R = \begin{bmatrix} 1 & 0 \\ 0 & R_2 \end{bmatrix} R_1$, so that $R \in \mathcal{T}_n$, R is invertible, and it is readily verified that $\{R^{-1}Q_jR\}_{j=1}^n$ is a permutation of $\{E_{jj}\}_{j=1}^n$.

□

Observation 18. At this point, it may be worth reviewing our situation. We currently have (without loss of generality)

$$\varphi(E_{ii}) = \lambda_i I + \alpha Q_i,$$

where $0 \neq \alpha \in \mathbb{F}$, $\lambda_i \in \mathbb{F}$ and $Q_i^2 = Q_i \neq 0$, $1 \leq i \leq n$. Moreover, by the Remark following Proposition 16, we may also assume that $Q_iQ_j = Q_jQ_i = 0$, $1 \leq i \neq j \leq n$.

Set $\rho_1 = R^{-1}\varphi R$, where R is the invertible matrix from Lemma 17. Clearly ρ_1 is still bijective and preserves commutativity in both directions, and $\rho_1(E_{ii}) = \lambda_i I + \alpha E_{\pi(i)\pi(i)}$, $1 \leq i \leq n$ for some permutation π of $\{1, 2, \ldots, n\}$. Next, let $\tau_1 : \mathcal{T}_n \to \mathbb{F}I$ be the linear functional determined by

$$\begin{aligned} \tau_1(E_{ii}) &= -\lambda_i I & & 1 \leq i \leq (n-1) \\ \tau_1(I) &= 0 = \tau_1(E_{ij}) & & 1 \leq i < j \leq n. \end{aligned}$$

Set $\rho_2 = \rho_1 + \tau_1$. Then $\rho_2(I) = I$ and so by Lemma 9, ρ_2 is also bijective and commutativity preserving in both directions. Moreover

$$\begin{aligned} \rho_2(E_{ii}) &= \alpha E_{\pi(i)\pi(i)} & & 1 \leq i \leq (n-1) \\ \rho_2(E_{ij}) &= \rho_1(E_{ij}) & & 1 \leq i < j \leq n \\ \rho_2(I) &= I. \end{aligned}$$

Next, let τ_2 be the linear functional determined by

$$\begin{aligned} \tau_2(E_{ij}) &= 0 & & \text{if} \quad (i,j) \neq (n,n) \\ \tau_2(E_{nn}) &= (\alpha - 1)I. \end{aligned}$$

Define $\rho_3 := \rho_2 + \tau_2$ to obtain $\rho_3(E_{ii}) = \alpha E_{\pi(i)\pi(i)}$, $1 \leq i \leq n$. Finally, set $\rho = (\alpha^{-1})\rho_3$. Again, ρ is bijective and commutativity preserving, and now $\rho(E_{ii}) = E_{\pi(i)\pi(i)}$, $1 \leq i \leq n$.

Note that $\rho = (\alpha^{-1})(R^{-1}\varphi R) + \alpha^{-1}(\tau_1 + \tau_2)$. Our next step is to show that either (a) $\pi(k) = k$, $1 \leq k \leq n$ or (b) $\pi(k) = (n+1) - k$, $1 \leq k \leq n$.

Lemma 19. *Suppose* $1 \le i < j \le n$, *and* $(i,j) \ne (1,n)$. *Then* $\dim\{E_{ij}\}' < \dim\{E_{1n}\}' = n(n+1)/2 - 1$.

Proof. It is readily verified that $\{E_{1n}\}' = \{[a_{ij}] \in \mathcal{T}_n : a_{11} = a_{nn}\}$ and therefore $\dim\{E_{1n}\}' = n(n+1)/2 - 1$.

Also, $\{E_{ij}\}' = \{[a_{ij}] \in \mathcal{T}_n : a_{ii} = a_{jj},\ a_{ki} = 0 \text{ if } k \ne i, a_{jm} = 0 \text{ if } m \ne j\}$. The dimension of the latter space is $n(n+1)/2 - (n+i-j) < n(n+1)/2 - 1$ unless $i = 1,\ j = n$.

□

Proposition 20. *Let* π *be the permutation of* $\{1,2,\dots,n\}$ *such that* $\rho(E_{ii}) = E_{\pi(i)\pi(i)}$, $1 \le i \le n$, *where* ρ *is the bijective map which is commutativity preserving in both directions, as obtained in Observation 18. Then either*

(i) $\pi(j) = j, \quad 1 \le j \le n$, *or*
(ii) $\pi(j) = (n+1) - j, \quad 1 \le j \le n.$

Proof. Step One. We have $\rho(E_{ii}) = E_{\pi(i)\pi(i)}$, $1 \le i \le n$. Let $H_{ij} = \rho(E_{ij})$, $1 \le i < j \le n$. We claim that there exist scalers γ_{ij} and μ_{ij} such that $H_{ij} = \gamma_{ij}(E^0_{\pi(i)\pi(j)}) + \mu_{ij} I$ where $E^0_{\pi(i)\pi(j)} = E_{\pi(i)\pi(j)}$ or $E^0_{\pi(i)\pi(j)} = E_{\pi(j)\pi(i)}$ (depending upon whether $\pi(i) < \pi(j)$ or $\pi(i) > \pi(j)$ respectively).

To see this, note that for $k \notin \{i,j\}$, $[E_{ij}, E_{kk}] = 0$, and hence $[H_{ij}, E_{\pi(k)\pi(k)}] = 0$. Thus

$$
\begin{aligned}
(H_{ij} e_{\pi(k)}) \otimes e_{\pi(k)} &= H_{ij}(e_{\pi(k)} \otimes e_{\pi(k)}) \\
&= (e_{\pi(k)} \otimes e_{\pi(k)}) H_{ij} \\
&= e_{\pi(k)} \otimes (H^t_{ij} e_{\pi(k)}),
\end{aligned}
$$

and so $H_{ij} e_{\pi(k)} \in \mathbb{F} e_{\pi(k)}$, $H^t_{ij} e_{\pi(k)} \in \mathbb{F} e_{\pi(k)}$. We deduce that $H_{ij} = \gamma_{ij} E^0_{\pi(i)\pi(j)} + C_{ij}$, where C_{ij} is a diagonal matrix. Note that $\gamma_{ij} \ne 0$, since C_{ij} lies in the span of $\{E_{kk}\}_{k=1}^n$, but H_{ij} does not, and ρ is bijective.

Next, fix ℓ, m so that $j \ne \ell \le m \ne i$. Then $[E_{\ell m}, E_{ij}] = 0$, forcing $[H_{\ell m}, H_{ij}] = 0$. Upon expanding, we get

$$\gamma_{\ell m}[E^0_{\pi(\ell)\pi(m)}, C_{ij}] + \gamma_{ij}[E^0_{\pi(i)\pi(j)}, C_{\ell m}] = 0.$$

If we temporarily denote C_{ij} by $\operatorname{diag}(x_k)$ and $C_{\ell m}$ by $\operatorname{diag}(y_k)$, the equation implies that

$$\gamma_{\ell m}(x_{\pi(m)} - x_{\pi(\ell)}) E^0_{\pi(\ell)\pi(m)} = 0$$

$$\text{and} \qquad \gamma_{ij}(y_{\pi(j)} - y_{\pi(i)}) E^0_{\pi(i)\pi(j)} = 0,$$

and hence that $x_{\pi(m)} - x_{\pi(\ell)} = 0$ for all $j \neq \ell \leq m \neq i$. But this implies that $C_{ij} = x_{\pi(i)}I$, which we relabel as $\mu_{ij}I$. We therefore have $H_{ij} = \gamma_{ij}E^0_{\pi(i)\pi(j)} + \mu_{ij}I$.

Step Two. Let $\tau : \mathcal{T}_n \to \mathbb{F}I$ be the linear functional determined by $\tau(E_{ii}) = 0,\ 1 \leq i \leq n$ and $\tau(E_{ij}) = -\mu_{ij}I,\ 1 \leq i < j \leq 1$. Then $\rho_0 = \rho + \tau$ is bijective and preserves commutativity in both directions, while $\rho_0(E_{ii}) = E_{\pi(i)\pi(i)},\ 1 \leq i \leq n$ and $\rho_0(E_{ij}) = \gamma_{ij}E^0_{\pi(i)\pi(j)},\ 1 \leq i < j \leq n$.

Step Three. We now claim that $\{\pi(1), \pi(n)\} = \{1, n\}$. Indeed, consider $H_{1n} = \gamma_{1n}E^0_{\pi(1)\pi(n)}$. Then $\dim\{E_{1n}\}' = n(n+1)/2-1$, so $\dim\{H_{1n}\}' = n(n+1)/2-1$, forcing $E^0_{\pi(1)\pi(n)} = E_{1n}$, by Lemma 19. Thus either $\pi(1) = 1,\ \pi(n) = n$, or $\pi(1) = n,\ \pi(n) = 1$.

Step Four. Suppose $\pi(1) = n,\ \pi(n) = 1$. Let $J = \begin{bmatrix} 0 & \dots & 0 & 1 \\ \vdots & \ddots & & \\ & \ddots & & 0 \\ 0 & \ddots & & \vdots \\ 1 & 0 & \dots & 0 \end{bmatrix}$ be the operator defined in Section 1. Let $\rho_0'(A) = J(\rho_0(A))^tJ$ for all $A \in \mathcal{T}_n$. Again, ρ_0' is bijective, preserves commutativity in both directions, and $\rho_0'(E_{11}) = E_{11},\ \rho_0'(E_{nn}) = E_{nn}$. Therefore, by replacing ρ_0 by ρ_0' if necessary, we may assume that $\pi(1) = 1,\ \pi(n) = n$. (This replacement gives rise to the second possibility of Theorem 4.)

Step Five. We have $E_{\pi(1)\pi(1)} = E_{11},\ E_{\pi(n)\pi(n)} = E_{nn}$. Thus $\rho_0(E_{kn}) = E^0_{\pi(k)n} = E_{\pi(k)n},\ 1 \leq k \leq n$. This implies that the compression of ρ_0 to $\operatorname{span}\{e_1, \dots, e_{n-1}\}$ is again a bijective, commutativity preserving (in both directions) map. The associated permutation, call it π_c, is simply a restriction of π to $\{1, 2, \dots, n-1\}$. The above arguments show that $\pi(n-1) = \pi_c(n-1) \in \{1, n-1\}$. But $\pi_c(1) = \pi(1) = 1$ and π_c is injective, so $\pi(n-1) = \pi_c(n-1) = (n-1)$. Continuing in this manner we see that $\pi(k) = k,\ 1 \leq k \leq n$.

□

Proposition 21. *Let ρ_0 be a linear, bijective map preserving commutativity in both directions and satisfying $\rho_0(E_{ii}) = E_{ii},\ 1 \leq i \leq n,\ \rho_0(E_{ij}) = \gamma_{ij}E_{ij},\ 1 \leq i < j \leq n$, where γ_{ij} are non-zero scalars. Then there exists an invertible matrix D such that*

$$\rho_0(A) = D^{-1}AD \quad \textit{for all} \quad A \in \mathcal{T}_n.$$

Proof. We see immediately that ρ_0 is a Hadamard multiplier – i.e. $\rho_0([a_{ij}]) = [\gamma_{ij}a_{ij}]$, where we set $\gamma_{ii} = 1,\ 1 \leq i \leq n$. Set $d_1 = 1$, and for $2 \leq r \leq n$, set $d_r = d_{r-1}\gamma_{r-1\,r}$. Then

$$D\rho_0(A)D^{-1} = [d_i\gamma_{ij}d_j^{-1}a_{ij}] = [k_{ij}a_{ij}],$$

where $k_{ij} = d_i \gamma_{ij} d_j^{-1}$. Note that $k_{ii} = 1,\ 1 \le i \le n$ and that $k_{i\,i+1} = 1,\ 1 \le i \le n-1$. Furthermore, the map $A \mapsto D\rho_0(A)D^{-1}$ preserves commutative in both directions.

We claim that $k_{ij} = 1$ for all $1 \le i \le j \le n$. For suppose otherwise. Fix $1 \le \ell \le (n-2)$ maximal with respect to the condition that there exists $m > \ell + 1$ such that $k_{\ell m} \ne 1$. Let $A = e_{\ell\,\ell+1} + e_{\ell m}$ and $B = e_{\ell+1\,n} - e_{mn}$. Then $[A, B] = (e_{\ell n} - 0 + 0 - e_{\ell n}) = 0$, and so A and B commute. On the other hand,

$$[D\rho_0(A)D^{-1}, D\rho_0(B)D^{-1}] = e_{\ell n} - k_{\ell m} e_{\ell n} \ne 0,$$

a contradiction. Thus $k_{ij} = 1$ for all $1 \le i \le j \le n$, and hence $D\rho_0(A)D^{-1} = A$ for all $A \in \mathcal{T}_n$. In other words, $\rho_0(A) = D^{-1}AD$ for all $A \in \mathcal{T}_n$, completing the proof.

□

Conclusion

The ingredients required to complete the proof of Theorem 4 are now all present. We have an invertible matrix $D \in \mathcal{T}_n$ (in fact D is diagonal) such that

$$\rho_0(A) = D^{-1}AD \quad \text{for all} \quad A \in \mathcal{T}_n.$$

Now from Step Two of Proposition 20, there exists a linear functional τ such that $\rho_0 = \rho + \tau$, where from Observation 18

$$\rho = \alpha^{-1}(R^{-1}\varphi R) + \alpha^{-1}(\tau_1 + \tau_2),$$

$0 \ne \alpha \in \mathbb{F}$, $R \in \mathcal{T}_n$ is invertible and τ_1, τ_2 are linear functionals. Putting these together, we obtain:

$$D^{-1}AD = \alpha^{-1}(R^{-1}\varphi R) + \alpha^{-1}(\tau_1 + \tau_2) + \tau,$$
$$\text{or} \qquad \varphi = \alpha(RD^{-1})A(DR^{-1}) + (\alpha\tau - \tau_1 - \tau_2).$$

Letting $c = \alpha$, $S = DR^{-1}$ and $f = \alpha\tau - \tau_1 - \tau_2$, we recover the first statement of Theorem 4. The second possibility of Theorem 4 arises if we must use ρ_0' instead of ρ_0 in Step Four of Proposition 20.

□

Remark. In a related article [MS], we describe the Lie automorphisms of nest algebras acting on a Hilbert space.

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References

[CL] Chan, G.H. and Lim, M.H., *Linear transformations on symmetric matrices that preserve commutativity*, Linear Algebra Appl. **47** (1982), 11-22.

[CJR] Choi, M.D., Jafarian, A.A. and Radjavi, H., *Linear maps preserving commutativity*, Linear Algebra Appl. **87** (1987), 227-241.

[D] Davidson, K.R., *Nest algebras*, Pitman Research Notes in Mathematics, no. 191, Longman Scientific and Technical, London and New York, 1988.

[Đo] Đoković, D., *Automorphisms of the Lie algebra of upper triangular matrices over a connected commutative ring*, J. Algebra **170** (1994), 101-110.

[MS] Marcoux, L.W. and Sourour, A.R., *Lie isomorphisms of nest algebras*, preprint.

[O] Omladič, M., *On operators preserving commutativity*, J. Functional Analysis **66** (1986), 105-122.

[R] Radjavi, H., *Commutativity-preserving operators on symmetric matrices*, Linear Algebra Appl. **61** (1984), 219-224.

[W] Watkins, W., *Linear maps that preserve commuting pairs of matrices*, Linear Algebra Appl. **14** (1976), 29-35.