CONJUGATION-INVARIANT SUBSPACES AND LIE IDEALS IN NON-SELFADJOINT OPERATOR ALGEBRAS

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ABSTRACT. We show that a weakly closed subspace $\mathcal{S}$ of a nest algebra $\mathcal{A}$ is closed under conjugation by invertible elements in $\mathcal{A}$, i.e. that $a^{-1}\mathcal{S}a = \mathcal{S}$ if and only if $\mathcal{S}$ is a Lie ideal. A similar result holds for not-necessarily closed subspaces of algebras of infinite multiplicity. Furthermore, we give an explicit characterisation of weakly closed Lie ideals in a nest algebra.

1. INTRODUCTION.

1.1. Let $\mathcal{A}$ be an associative algebra with identity. A *Lie ideal* in $\mathcal{A}$ is a linear manifold $\mathcal{L}$ satisfying $[\mathcal{A}, \mathcal{L}] \subseteq \mathcal{L}$, where $[a, b] = ab - ba$ for all $a, b \in \mathcal{A}$. A linear manifold $\mathcal{S}$ in $\mathcal{A}$ is said to be *conjugation-invariant*, or *invariant under conjugation by invertible elements*, if $a^{-1}\mathcal{S}a \subseteq \mathcal{S}$ for every invertible element $a \in \mathcal{A}$. The connection between these two notions has been investigated by several authors. One of the earliest such investigations is by Herstein [11, 12]. In [24], it was show that the two notions are equivalent for norm-closed subspaces of $\mathcal{B}(\mathcal{H})$, the algebra of all bounded operators on a Hilbert space $\mathcal{H}$. The equivalence was established in [8] for not-necessarily closed linear manifolds in $\mathcal{B}(\mathcal{H})$. Similar results for von Neumann algebras and certain C^*-algebras can be found in [17] and [15].

1.2. In this paper we shall examine the relationship between these two notions for three main classes of non-selfadjoint operator algebras. These are: *nest algebras*, *triangular uniformly hyperfinite (TUHF) algebras*, and *algebras of infinite multiplicity* (which can also include some selfadjoint examples).

Our results show that if a linear manifold $\mathcal{L}$ in a nest algebra (resp. TUHF algebra) is weakly (resp. norm) closed, then $\mathcal{L}$ is a Lie ideal if and only if it is conjugation-invariant. If $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is a weakly closed algebra of infinite multiplicity, then the same conclusion holds with no topological assumptions about the linear manifold.

1.3. We end this introduction by giving the easy proof of the implication that in a Banach algebra, closed subspaces invariant under conjugation are Lie ideals. This result is in a manuscript [24] written by Topping in 1970 which was never published due to his untimely death.

1.4. **Theorem.** (Topping). *Let $\mathcal{A}$ be a Banach algebra. If $\mathcal{S}$ is a closed, conjugation-invariant subspace of $\mathcal{A}$, then $\mathcal{S}$ is a Lie ideal in $\mathcal{A}$.*

Proof. Let $a \in \mathcal{A}$ and $x \in \mathcal{S}$. Then $f(t) = e^{ita}xe^{-ita} \in \mathcal{S}$ for every $t \in \mathbb{R}$. Upon taking derivatives at $t = 0$, we get that $ax - xa \in \mathcal{S}$. Thus $\mathcal{S}$ is a Lie ideal. □

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2. Nest algebras.

2.1. The following definitions can all be found in [2], and we refer the reader to this source for general information on nest algebras.

A *nest* $\mathcal{N}$ on a Hilbert space $\mathcal{H}$ is a chain of closed subspaces of $\mathcal{H}$ which is closed under the operations of taking arbitrary intersections and closed linear spans, and which includes $\{0\}$ and $\mathcal{H}$ itself. To each subspace $N \in \mathcal{N}$ we can associate the orthogonal projection $P(N)$ of $\mathcal{H}$ onto N. We can and do identify $\mathcal{N}$ with the chain of projections $\{P(N) : N \in \mathcal{N}\}$ obtained from $\mathcal{N}$, and speak of elements of the nest as subspaces or projections interchangeably.

Corresponding to each nest $\mathcal{N}$ is the algebra $\mathcal{T}(\mathcal{N})$ of all operators T on $\mathcal{H}$ such that $TN \subseteq N$ for all $N \in \mathcal{N}$. This is always closed in the weak operator topology. The *diagonal* $\mathcal{D}(\mathcal{N})$ of a nest algebra $\mathcal{T}(\mathcal{N})$ is the von Neumann subalgebra $\mathcal{T}(\mathcal{N}) \cap \mathcal{T}(\mathcal{N})^*$. The *core* $\mathcal{C}(\mathcal{N})$ of the nest algebra $\mathcal{T}(\mathcal{N})$ is the von Neumann algebra generated by the projections $\{P(N) : N \in \mathcal{N}\}$, or equivalently, $\mathcal{C}(\mathcal{N})$ is the commutant of $\mathcal{D}(\mathcal{N})$. The core $\mathcal{C}(\mathcal{N})$ is always a subalgebra of the diagonal $\mathcal{D}(\mathcal{N})$.

Given two elements $E, F \in \mathcal{N}$ with $E < F$, $F - E$ is called an *interval* of the nest. The nonzero minimal intervals are called *atoms*. A nest is said to be *atomic* if the atoms of the nest span $\mathcal{H}$ and said to be *continuous* if it has no atoms.

2.2. Let $\mathbb{A}$ denote the set of atoms of a nest $\mathcal{N}$. The *atomic-diagonal* of $\mathcal{T}(\mathcal{N})$ is the weakly closed algebra $\mathcal{D}_{\text{atom}}(\mathcal{N})$ generated by $\{P\mathcal{B}(\mathcal{H})P : P \in \mathbb{A}\}$. There is a canonical expectation Δ from $\mathcal{T}(\mathcal{N})$ onto $\mathcal{D}_{\text{atom}}(\mathcal{N})$ defined by

$$\Delta(T) = \sum_{P \in \mathbb{A}} PTP.$$

A subset $\mathcal{M}$ of $\mathcal{T}(\mathcal{N})$ will be called *atomic-diagonal disjoint* if $\Delta(T) = 0$ for every $T \in \mathcal{M}$.

2.3. **Associative ideals.** We shall make use of the following characterisation of weakly closed associative ideals in $\mathcal{T}(\mathcal{N})$ from [6].

Theorem. (Erdos-Power). *Let $\mathcal{I}$ be a weakly closed associative ideal in $\mathcal{T}(\mathcal{N})$. Then there exists a left order-continuous order-homomorphism $E \mapsto \widetilde{E}$ from $\mathcal{N}$ into $\mathcal{N}$ such that $\widetilde{E} \leq E$ for each $E \in \mathcal{N}$ and*

$$\mathcal{I} = \{T \in \mathcal{B}(\mathcal{H}) : (I - \widetilde{E})TE = 0 \textit{ for all } E \in \mathcal{N}\}.$$

Conversely, for every left order-continuous order-homomorphism $E \mapsto \widetilde{E}$ with $\widetilde{E} \leq E$ for all $E \in \mathcal{N}$, the above set $\mathcal{I}$ defines a weakly closed associative ideal.

We shall also require the following Lemma from [13].

2.4. **Lemma.** *Let $\mathcal{I}$ be a weakly closed associative ideal in $\mathcal{T}(\mathcal{N})$ and let $E \mapsto \widetilde{E}$ be the corresponding order-homomorphism of the nest $\mathcal{N}$. Then*

$$\mathcal{I} = \{T \in \mathcal{T}(\mathcal{N}) : (E - \widetilde{E})T(E - \widetilde{E}) = 0 \textit{ for all } E \in \mathcal{N}\}.$$

In the following, $\operatorname{span} \mathcal{M}$ denotes the linear span of a set $\mathcal{M}$, and $\mathrm{wk} - \mathrm{cl}(\mathcal{M})$ denotes the closure of $\mathcal{M}$ in the weak operator topology.

2.5. **Proposition.** *Let $\mathcal{T}(\mathcal{N})$ be a nest algebra.*

(a) *Let $\mathcal{I}$ be an atomic-diagonal disjoint weakly closed associative ideal in $\mathcal{T}(\mathcal{N})$. Then*

$$\mathcal{I} = \text{wk} - \text{cl span}\{PT(I-P) : T \in \mathcal{I}, P \in \mathcal{N}\}.$$

(b) *Let $\mathcal{I}$ be a weakly closed associative ideal in $\mathcal{T}(\mathcal{N})$, and let*

$$\mathcal{I}^+ = \text{wk} - \text{cl span}\{P\mathcal{I}(I-P) : P \in \mathcal{N}\}.$$

Then $\mathcal{I}^+$ is an atomic-diagonal disjoint weakly closed associative ideal. Furthermore, $\mathcal{I}^+ = \{T \in \mathcal{I} : \Delta(T) = 0\}$.

Proof. (a) Let $\mathcal{I}^+ = \text{wk} - \text{cl span}\{P\mathcal{I}(I-P) : P \in \mathcal{N}\}$. It is easy to see that $\mathcal{I}^+$ is a weakly closed associative ideal, and $\mathcal{I}^+ \subseteq \mathcal{I}$. Let $E \mapsto \widetilde{E}$ and $E \mapsto \widehat{E}$ be the order-homomorphisms of the nest associated with $\mathcal{I}$ and $\mathcal{I}^+$ respectively. Thus $\widehat{E} \leq \widetilde{E}$. We will show that $\widehat{E} = \widetilde{E}$ and so $\mathcal{I}^+ = \mathcal{I}$.

To prove this, assume to the contrary that $\widehat{E_1} < \widetilde{E_1}$ for some $E_1 \in \mathcal{N}$. For any $X \in \mathcal{B}(\mathcal{H})$, the operator $S := (\widetilde{E_1} - \widehat{E_1})X(I - E_1)$ is evidently in $\mathcal{I}$. Now observe that if E_1 has no immediate predecessors, then by the left order-continuity of the function $E \mapsto \widehat{E}$, there exists an element $F \in \mathcal{N}$ with $F < E_1$ and $\widetilde{F} > \widehat{E_1}$. Consider a rank one operator $T \in \mathcal{T}(\mathcal{N})$ satisfying $T = (\widetilde{F} - \widehat{E_1})T(E_1 - F)$, i.e., $T = \xi \otimes \eta^*$ with $\eta \in (E_1 - F)\mathcal{H}$ and $\xi \in (\widetilde{F} - \widehat{E_1})\mathcal{H}$. If E_1 has an immediate predecessor, take $F = (E_1)_-$ and T a rank one operator in $\mathcal{T}(\mathcal{N})$ satisfying $T = (\widetilde{E_1} - \widehat{E_1})T(E_1 - F)$. In either case, we have

$$T = (G - \widehat{E_1})T(E_1 - F)$$

where $G = \widetilde{E_1}$ or $\widetilde{F}$ and $\widehat{E_1} < G \leq \widetilde{E_1}$. We now show that $T \in \mathcal{I}$.

We consider three cases. First consider the case $E \geq E_1$. Then $(I - \widetilde{E})TE = 0$, since $(I - \widetilde{E})(G - \widehat{E_1}) = 0$. Next, consider the case $E \leq F$. It follows that $(I - \widetilde{E})TE = 0$ due to the fact that $(E_1 - F)E = E - E = 0$. In the third case, $F < E < E_1$. Hence $G = \widetilde{F}$ and so $(I - \widetilde{E})TE = (E - \widetilde{E})(\widetilde{F} - \widehat{E_1})T(E_1 - F)E$, and since $(I - \widetilde{E})(\widetilde{F} - \widehat{E_1}) = 0$, we again get that $(I - \widetilde{E})TE = 0$, proving that $T \in \mathcal{I}$.

Next, we prove that $T \in \mathcal{I}^+$. This follows from the equation $T = FT(I - F)$, which we verify presently. In the case when $G = \widetilde{F}$, i.e. $T = (\widetilde{F} - \widehat{E_1})T(E_1 - F)$, the result follows immediately since $F > \widetilde{F} - \widehat{E_1}$ and $E_1 - F < I - F$. In the remaining case, $F = (E_1)_-$ and also $\widetilde{E_1} \leq (E_1)_-$ since $\mathcal{I}$ is atomic-diagonal disjoint. Hence $\widetilde{E_1} \leq F$ and so $F(\widetilde{E_1} - \widehat{E_1}) = \widetilde{E_1} - \widehat{E_1}$, which again leads to $T = FT(I - F)$. This proves that $T \in \mathcal{I}^+$.

However,

$$\begin{aligned}(I - \widehat{E_1})TE_1 &= (I - \widehat{E_1})(G - \widehat{E_1})T(E_1 - F)E_1 \\ &= (G - \widehat{E_1})T(E_1 - F) \\ &= T \neq 0.\end{aligned}$$

This contradicts Theorem 2.3.

(b) This follows easily, given (a). □

3. Lie ideals in nest algebras.

3.1. Let $\mathcal{L}$ be a weakly closed Lie ideal in $\mathcal{T}(\mathcal{N})$. As in [13], we define

$$\mathcal{I} = \mathcal{I}_{\mathcal{L}} := \text{wk} - \text{cl span}\{PT(I-P) : T \in \mathcal{L},\ P \in \mathcal{N}\}.$$

3.2. **Proposition.** *Let $\mathcal{L}$ be a weakly closed Lie ideal in $\mathcal{T}(\mathcal{N})$ and let $\mathcal{I}$ be defined as above. Then $\mathcal{I}$ is a weakly closed associative ideal in $\mathcal{T}(\mathcal{N})$ that is atomic-diagonal disjoint. Furthermore, $\mathcal{I} \subseteq \mathcal{L}$.*

Proof. The fact that $\mathcal{I}$ is atomic-diagonal disjoint is obvious. The other assertions are proven in [13].

3.3. We shall now show that there is a larger ideal $\mathcal{J}$, constructed from $\mathcal{I}$ by adding some atoms, and that

$$\mathcal{J}^\circ \subseteq \mathcal{L} \subseteq \mathcal{J} + \mathcal{C}_\mathcal{J},$$

where $\mathcal{J}^\circ$ denotes the set of trace-zero operators in $\mathcal{J}$ (to be defined later), and $\mathcal{C}_\mathcal{J}$ is a certain subalgebra of the core $\mathcal{C}(\mathcal{N})$. This is a refinement of the results of [13] where it was shown that there exists a subalgebra $\mathcal{D}_\mathcal{I}$ of the diagonal $\mathcal{D}(\mathcal{N})$ such that $\mathcal{I} \subseteq \mathcal{L} \subseteq \mathcal{I} + \mathcal{D}_\mathcal{I}$. Indeed, the inclusions we obtain in this article are "tight enough" to conclude that *every* linear manifold $\mathcal{M}$ satisfying $\mathcal{J}^\circ \subseteq \mathcal{M} \subseteq \mathcal{J} + \mathcal{C}_\mathcal{J}$ is automatically a Lie ideal. Examples in [13] show that there are subspaces between $\mathcal{I}$ and $\mathcal{I} + \mathcal{D}_\mathcal{I}$ that are not Lie ideals.

3.4. We now describe certain enlargements of a weakly closed associative ideal by certain atoms. Let $\mathcal{I}$ be a weakly closed atomic-diagonal disjoint ideal in $\mathcal{T}(\mathcal{N})$. Let $E \mapsto \widetilde{E}$ be the corresponding order-homomorphism of the nest. In this case $\widetilde{E} \le E_-$. We associate with $\mathcal{I}$ a certain subset $\mathbb{A}_\mathcal{I}$ of the atoms of $\mathcal{N}$ which are said to be *atoms adjacent to* $\mathcal{I}$. These are defined by:

$$\mathbb{A}_\mathcal{I} = \{E - E_- : \widetilde{E} = E_-\}.$$

For any arbitrary subset $\mathbb{A}_1$ of $\mathbb{A}_\mathcal{I}$, we define the *saturation* of $\mathcal{I}$ by $\mathbb{A}_1$ to be the ideal

$$\mathcal{I} \vee \mathbb{A}_1 := \mathrm{wk}-\mathrm{cl}\{\mathcal{I} + \sum\{P\mathcal{B}(\mathcal{H})P : P \in \mathbb{A}_1\}\}.$$

It is straightforward to verify that $\mathcal{I} \vee \mathbb{A}_1$ is also a weakly closed associative ideal in $\mathcal{T}(\mathcal{N})$ and that the corresponding order-homomorphism of the nest is $E \mapsto \overline{\overline{E}}$, where

$$\overline{\overline{E}} = \begin{cases} E & \text{if } \widetilde{E} = E_- \text{ and } E - E_- \in \mathbb{A}_1 \\ \widetilde{E} & \text{otherwise} \end{cases}.$$

3.5. For every ideal $\mathcal{J}$, we associate a Lie ideal $\mathcal{J}^\circ$, called the *zero-trace part* of $\mathcal{J}$, defined by

$$\mathcal{J}^\circ = \{A \in \mathcal{J} : \operatorname{tr} PAP = 0 \text{ for every atom } P \text{ in the nest with } \dim P < \infty\}.$$

In particular, for the ideals $\mathcal{I} \vee \mathbb{A}_1$ defined above, we have

$$\begin{aligned}(\mathcal{I} \vee \mathbb{A}_1)^\circ \quad = \quad & \mathrm{wk}-\mathrm{cl}\{\mathcal{I} + \sum\{P\mathcal{B}(\mathcal{H})P : P \in \mathbb{A}_1,\ \dim P = \infty\} \\ & + \sum\nolimits_n\{\sum P(\mathfrak{sl}_n)P : P \in \mathbb{A}_1,\ \dim P = n < \infty\}\}\end{aligned}$$

where $\mathfrak{sl}_n$ denotes the Lie ideal of zero-trace matrices in $\mathbb{M}_n$. Next, we borrow a notation from associative ring theory. If $\mathcal{A}$ is an algebra and $\mathcal{L}$ a Lie ideal in $\mathcal{A}$, we define

$$[\mathcal{A} : \mathcal{L}] := \{a \in \mathcal{A} : [a, x] \in \mathcal{L} \text{ for all } x \in \mathcal{A}\}.$$

In analogy with ring theory terminology, this may be called the "Lie residual quotient" of $\mathcal{A}$ by $\mathcal{L}$. It may be viewed as the "lifting" of the "centre" of $\mathcal{A}/\mathcal{L}$ to $\mathcal{A}$.

3.6. **Lemma.**

(i) *If $\mathcal{L}$ is a Lie ideal in an associative algebra $\mathcal{A}$, then $[\mathcal{A} : \mathcal{L}]$ is a Lie ideal that includes $\mathcal{L}$. Furthermore, every linear manifold $\mathcal{M}$ satisfying $\mathcal{L} \subseteq \mathcal{M} \subseteq [\mathcal{A} : \mathcal{L}]$ is a Lie ideal.*

(ii) *If, in addition, $\mathcal{A}$ is a weakly closed operator algebra and $\mathcal{L}$ is weakly closed in $\mathcal{A}$, then $[\mathcal{A} : \mathcal{L}]$ is weakly closed.*

Proof. Obvious. □

We next define a certain subalgebra of the core associated with a weakly closed ideal. For a weakly closed ideal $\mathcal{I}$ in $\mathcal{T}(\mathcal{N})$ with corresponding order-homomorphism $E \mapsto \widetilde{E}$ of $\mathcal{N}$, define

$$\mathcal{C}_{\mathcal{I}} := \{D \in \mathcal{C}(\mathcal{N}) : \text{ For every } E \in \mathcal{N}, \text{ there exists a scalar } \lambda_E \text{ such that } (E - \widetilde{E})D(E - \widetilde{E}) = \lambda_E(E - \widetilde{E})\}$$

The proof of the following Proposition is straightforward, and is omitted.

3.7. **Proposition.** *Let $\mathcal{I}$ be a weakly closed ideal in a nest algebra $\mathcal{T}(\mathcal{N})$. Then $\mathcal{C}_{\mathcal{I}}$ is a von Neumann algebra.*

3.8. **Proposition.** *Let $\mathcal{I}$ be a weakly closed ideal in a nest algebra $\mathcal{T}(\mathcal{N})$ and let $\mathcal{I}^\circ$ be the zero-trace part of $\mathcal{I}$. Then*

(a) $[\mathcal{T}(\mathcal{N}) : \mathcal{I}^\circ] = [\mathcal{T}(\mathcal{N}) : \mathcal{I}] = \mathcal{I} + \mathcal{C}_{\mathcal{I}}$.

(b) *If, furthermore, $\mathcal{I}$ is atomic-diagonal disjoint, $\mathbb{A}_1 \subseteq \mathbb{A}_{\mathcal{I}}$, and $\mathcal{J} = \mathcal{I} \vee \mathbb{A}_1$ is the saturation of $\mathcal{I}$ by $\mathbb{A}_1$, then $\mathcal{C}_{\mathcal{I}} = \mathcal{C}_{\mathcal{J}}$.*

Proof. (a) The first assertion, namely that $[\mathcal{T}(\mathcal{N}) : \mathcal{I}^\circ] = [\mathcal{T}(\mathcal{N}) : \mathcal{I}]$, is obvious. Next, let $A = B + D$ with $B \in \mathcal{I}$ and $D \in \mathcal{C}_{\mathcal{I}}$. If $T \in \mathcal{T}(\mathcal{N})$, then $[T, A] = [T, B] + [T, D]$. The first commutator $[T, B]$ is evidently in $\mathcal{I}$. As to the second commutator, we have that for $\widetilde{E} < E$,

$$(E - \widetilde{E})(TD - DT)(E - \widetilde{E}) = (E - \widetilde{E})(T\lambda_E - \lambda_E T)(E - \widetilde{E}) = 0.$$

It follows from Lemma 2.4 that $TD - DT \in \mathcal{I}$. Thus $[T, A] \in \mathcal{I}$. This proves that $\mathcal{I} + \mathcal{C}_{\mathcal{I}} \subseteq [\mathcal{T}(\mathcal{N}) : \mathcal{I}]$.

To prove the reverse inclusion, assume that $A \in [\mathcal{T}(\mathcal{N}) : \mathcal{I}]$. Let $\pi_{\mathcal{C}}$ be an expectation of $\mathcal{T}(\mathcal{N})$ onto the core $\mathcal{C}(\mathcal{N})$. (Such expectations always exist; see ([2]; Theorem 8.5). Let $B = A - \pi_{\mathcal{C}}(A)$. For $\widetilde{E} < E$, we have

$$\begin{aligned} 0 &= (E - \widetilde{E})[A, T](E - \widetilde{E}) \\ &= [(E - \widetilde{E})A(E - \widetilde{E}), (E - \widetilde{E})T(E - \widetilde{E})]. \end{aligned}$$

Thus $(E - \widetilde{E})A(E - \widetilde{E})$ is in the commutant of the nest algebra $(E - \widetilde{E})\mathcal{T}(\mathcal{N})(E - \widetilde{E})$ on the Hilbert space $(E - \widetilde{E})(\mathcal{H})$. As the commutant of a nest algebra is trivial ([2]; Cor. 19.5), we get that $(E - \widetilde{E})A(E - \widetilde{E}) = \lambda_E(E - \widetilde{E})$. for a scalar λ_E. Hence

$$\begin{aligned} (E - \widetilde{E})B(E - \widetilde{E}) &= (E - \widetilde{E})A(E - \widetilde{E}) - (E - \widetilde{E})\pi_{\mathcal{C}}(A)(E - \widetilde{E}) \\ &= \lambda_E(E - \widetilde{E}) - \pi_{\mathcal{C}}((E - \widetilde{E})A(E - \widetilde{E})) \\ &= \lambda_E(E - \widetilde{E}) - \pi_{\mathcal{C}}(\lambda_E(E - \widetilde{E})) \\ &= 0. \end{aligned}$$

Therefore $B \in \mathcal{I}$.

The above calculation also shows that $(E-\widetilde{E})\pi_{\mathcal{C}}(A)(E-\widetilde{E}) = \lambda_E(E-\widetilde{E})$. Thus $\pi_{\mathcal{C}}(A) \in \mathcal{C}_{\mathcal{I}}$ and so $A = B + \pi_{\mathcal{C}}(A) \in \mathcal{I} + \mathcal{C}_{\mathcal{I}}$. This ends the proof of part (a).

(b) Assume that $E \mapsto \overline{\overline{E}}$ is the order-homomorphism of $\mathcal{N}$ associated with $\mathcal{J}$. Thus $\overline{\overline{E}} = E$ if $E - \widetilde{E} \in \mathbb{A}_1$, and $\overline{\overline{E}} = \widetilde{E}$ otherwise. The inclusion $\mathcal{C}_{\mathcal{I}} \subseteq \mathcal{C}_{\mathcal{J}}$ is quite obvious. To prove the reverse inclusion, assume that $D \in \mathcal{C}_{\mathcal{J}}$ and consider $(E-\widetilde{E})D(E-\widetilde{E})$ for $\widetilde{E} < E$. If $E - \widetilde{E}$ is an atom, then $(E-\widetilde{E})D(E-\widetilde{E}) = \lambda_E(E-\widetilde{E})$ due to the fact that $D \in \mathcal{C}(\mathcal{N})$ and so its compression to any atom must be a scalar. If $E - \widetilde{E}$ is not an atom, then $\widetilde{E} = \overline{\overline{E}}$, and so

$$(E-\widetilde{E})D(E-\widetilde{E}) = (E-\overline{\overline{E}})D(E-\overline{\overline{E}}) = \lambda_E(E-\widetilde{E})$$

due to the fact that $D \in \mathcal{C}_{\mathcal{J}}$. In either case, we find that $D \in \mathcal{C}_{\mathcal{I}}$ $\square$

We are now ready to state the main result of this section. Recall that for an algebra $\mathcal{A}$, we denote the group of invertible elements of $\mathcal{A}$ by $\mathcal{A}^{-1}$.

3.9. **Theorem.** *Let $\mathcal{L}$ be a weakly closed subspace of a nest algebra $\mathcal{T}(\mathcal{N})$. The following conditions are equivalent:*

(a) *$\mathcal{L}$ is conjugation-invariant, i.e., $A^{-1}\mathcal{L}A \subseteq \mathcal{L}$ for every $A \in \mathcal{T}(\mathcal{N})^{-1}$.*
(b) *$\mathcal{L}$ is a Lie ideal in $\mathcal{T}(\mathcal{N})$.*
(c) *There exist an atomic-diagonal disjoint ideal $\mathcal{I}$ of $\mathcal{T}(\mathcal{N})$ and a subset $\mathbb{A}_1$ of the atoms $\mathbb{A}_{\mathcal{I}}$ adjacent to $\mathcal{I}$ such that*
$$(\mathcal{I} \vee \mathbb{A}_1)^\circ \subseteq \mathcal{L} \subseteq [\mathcal{T}(\mathcal{N}) : \mathcal{I} \vee \mathbb{A}_1] = (\mathcal{I} \vee \mathbb{A}_1) + \mathcal{C}_{\mathcal{I}}.$$
(d) *There exists an associative ideal $\mathcal{J}$ of $\mathcal{T}(\mathcal{N})$ such that*
$$\mathcal{J}^\circ \subseteq \mathcal{L} \subseteq [\mathcal{T}(\mathcal{N}) : \mathcal{J}] = \mathcal{J} + \mathcal{C}_{\mathcal{J}}.$$

Proof. That (a) implies (b): follows from Theorem 1.4.

To prove that (b) implies (c), assume that $\mathcal{L}$ is a Lie ideal and that $\mathcal{I} = \mathcal{I}_{\mathcal{L}}$ is the corresponding associative and atomic-diagonal disjoint ideal defined in paragraph 3.1. Let $\mathbb{A}_1$ be the subset of the atoms $\mathbb{A}_{\mathcal{I}}$ adjacent to $\mathcal{I}$ defined by

$$\mathbb{A}_1 = \{P \in \mathbb{A}_{\mathcal{I}} : P\mathcal{L}P \not\subseteq \mathbb{C}P\}.$$

That $\mathcal{I} \subseteq \mathcal{L}$ follows from Proposition 3.2. Next, we show that $(\operatorname{span}\mathbb{A}_1)^\circ \subseteq \mathcal{L}$. Assume that $P \in \mathbb{A}_1$. It is readily verified that $P\mathcal{L}P$ is a weakly closed Lie ideal in $P\mathcal{B}(\mathcal{H})P$. Since $P\mathcal{L}P \not\subseteq \mathbb{C}P$, it follows from [8] that $P\mathcal{L}P = P\mathcal{B}(\mathcal{H})P$ or $P\mathcal{L}P = P(\mathfrak{sl}_n)P$ - the latter alternative may be present when $\dim P < \infty$. This easily leads to $(\mathcal{I} \vee \mathbb{A}_1)^\circ \subseteq \mathcal{L}$.

Next let $L \in \mathcal{L}$ and let $X \mapsto \overline{\overline{X}}$ be the Erdos-Power function associated with the ideal $\mathcal{I} \vee \mathbb{A}_1$. For an arbitrary $T \in \mathcal{T}(\mathcal{N})$ and $E \in \mathcal{N}$, consider

$$S = (E-\overline{\overline{E}})(LT-TL)(E-\overline{\overline{E}}).$$

If $\overline{\overline{E}} = E$, then $S = 0$. Otherwise $\overline{\overline{E}} = \widetilde{E}$ and $S = (E-\widetilde{E})(LT-TL)(E-\widetilde{E})$. Let $\mathcal{H}_1 = (E-\widetilde{E})\mathcal{H}$, $\mathcal{N}_1 = (E-\widetilde{E})\mathcal{N}$, $L_1 = (E-\widetilde{E})L(E-\widetilde{E})$, $\mathcal{L}_1 = (E-\widetilde{E})\mathcal{L}(E-\widetilde{E})$ restricted to $\mathcal{H}_1$, and $T_1 = (E-\widetilde{E})T(E-\widetilde{E})$. We have a nest algebra $\mathcal{T}(\mathcal{N}_1)$ in $\mathcal{B}(\mathcal{H}_1)$ and a Lie ideal $\mathcal{L}_1$ of $\mathcal{T}(\mathcal{N}_1)$ with the corresponding "off diagonal" associative ideal $\mathcal{I}_{\mathcal{L}_1}$ equal to 0, thus $P\mathcal{L}_1(I-P) = (I-P)\mathcal{L}_1P = 0$ for every $P \in \mathcal{N}_1$. Hence $\mathcal{L}_1$ is included in the diagonal $\mathcal{D}(\mathcal{N}_1)$. Furthermore, $S = L_1T_1 - T_1L_1$. We shall show that $S = 0$. Towards this it suffices to show that L_1 is a scalar multiple of the identity in $\mathcal{B}(\mathcal{H}_1)$.

We consider two cases according to whether $\widetilde{E} = E_-$ or not. If $\widetilde{E} = E_-$, then $\mathcal{N}_1$ is trivial, $\mathcal{T}(\mathcal{N}_1) = \mathcal{B}(\mathcal{H}_1)$ and $\mathcal{L}_1$ is a weakly closed Lie ideal in $\mathcal{B}(\mathcal{H}_1)$. Also, the atom $E - E_- \notin \mathbb{A}_1$, and so, by the very definition of $\mathbb{A}_1$, we have $\mathcal{L}_1$ included in the space of all scalar multiples of the identity on $\mathcal{H}_1$ as required. In the case $\widetilde{E} < E_-$, the result follows from ([13]; Thm. 12). For the sake of completeness, we include a proof.

Let $P \in \mathcal{N}_1$, $A = PL_1P$, $B = (I-P)L_1(I-P)$ and let $X \in P\mathcal{B}(\mathcal{H}_1)(I-P)$. Thus $X \in \mathcal{T}(\mathcal{N}_1)$. It follows that $L_1X - XL_1 \in \mathcal{L}_1$. However, an easy calculation shows that $L_1X - XL_1 \in P\mathcal{B}(\mathcal{H}_1)(I-P)$, while it is also included in $\mathcal{D}(\mathcal{N}_1)$ by virtue of the fact that $\mathcal{L}_1 \subseteq \mathcal{D}(\mathcal{N})$. This is satisfied only when $L_1X - XL_1 = 0$, i.e. $AX - XB = 0$. By considering only rank one operators X, we see that every vector in $P\mathcal{H}_1$ is an eigenvector of A. Thus $A = \lambda P$ for some $\lambda \in \mathbb{C}$. It then follows easily that $B = \lambda(I-P)$ and so $L_1 = A + B = \lambda I$.

Now, we have shown that $S = 0$, i.e., that $(E - \overline{\overline{E}})(LT - TL)(E - \overline{\overline{E}}) = 0$. It follows by Lemma 2.4 that $LT - TL \in \mathcal{I} \vee \mathbb{A}_1$, that is, $L \in [\mathcal{T}(\mathcal{N}) : \mathcal{I} \vee \mathbb{A}_1]$. Finally, the equation $[\mathcal{T}(\mathcal{N}) : \mathcal{I} \vee \mathbb{A}_1] = (\mathcal{I} \vee \mathbb{A}_1) + \mathcal{C}_{\mathcal{I}}$ follows from the previous Proposition.

That (c) implies (d) is obvious.

To prove that (d) implies (a), let $L \in \mathcal{L}$, $T \in \mathcal{T}(\mathcal{N})^{-1}$. Write $L = K + D$ with $K \in \mathcal{J}$ and $D \in \mathcal{C}_{\mathcal{J}}$. Let $E \mapsto \widetilde{E}$ be the Erdos-Power function on $\mathcal{N}$ associated with $\mathcal{J}$. Now $T^{-1}KT - K \in \mathcal{J}$ as $\mathcal{J}$ is an ideal. Also, $(E - \widetilde{E})(T^{-1}DT - D)(E - \widetilde{E}) = 0$, since $(E - \widetilde{E})D(E - \widetilde{E}) = \lambda_E(E - \widetilde{E})$ and $X \mapsto (E - \widetilde{E})X(E - \widetilde{E})$ is an algebra homomorphism. Thus $T^{-1}DT - D \in \mathcal{J}$ by Lemma 2.4 and so $T^{-1}LT - L \in \mathcal{J}$.

The compressions of L and $T^{-1}LT$ to any finite-dimensional atom have the same trace, so $T^{-1}LT - L \in \mathcal{J}^\circ \subseteq \mathcal{L}$. Therefore $T^{-1}LT \in \mathcal{L}$. □

Recall that a *commutator* in an associative algebra $\mathcal{A}$ is an element of the form $[a,b] := ab - ba$ for some pair of elements $a, b \in \mathcal{A}$. An operator T is said to be a *square-zero* operator if $T^2 = 0$.

3.10. **Corollary.** *Let $\mathcal{T}(\mathcal{N})$ be a nest algebra. Each of the following is weakly dense in $\mathcal{T}(\mathcal{N})^\circ$, the set of all operators in $\mathcal{T}(\mathcal{N})$ with trace zero.*

(a) *The span of the commutators;*
(b) *The span of the nilpotents;*
(c) *The span of the quasi-nilpotents;*
(d) *The span of the square-zero operators.*

Proof. Let $\mathcal{L}$ be the weak closure of any of the sets above. Then $\mathcal{L}$ is a conjugation-invariant subspace and hence it is a Lie ideal. Let $\mathcal{I}$ be the corresponding atomic-diagonal disjoint ideal as in paragraph 3.1. As before, let Δ be the expectation on the atomic-diagonal and $\mathcal{U} = \mathcal{T}(\mathcal{N}) \cap \ker\,\Delta$, i.e., $\mathcal{U}$ is the maximal atomic-diagonal disjoint ideal.

Let $E \in \mathcal{N}$, $T \in \mathcal{T}(\mathcal{N})$, and let $S = ET(I-E)$. The operator S is evidently a square-zero operator, and hence both a nilpotent and a quasi-nilpotent. It is also a commutator since $S = [E, S]$. Thus $S \in \mathcal{I}$. It follows that $\mathcal{I} \supseteq \text{wk}-\text{cl span}\{P\mathcal{T}(\mathcal{N})(I-P) : P \in \mathcal{N}\}$. By Proposition 2.5, we conclude that $\mathcal{I} \supseteq \mathcal{U}$ and hence $\mathcal{I} = \mathcal{U}$.

Furthermore, for every atom $P \in \mathcal{N}$, the Lie ideal $\mathcal{L}$ includes all commutators and all quasi-nilpotents in $P\mathcal{B}(\mathcal{H})P = \mathcal{B}(P\mathcal{H}P)$. When $\dim \mathcal{H} = \infty$, every operator is a sum of commutators ([10]; Problem 234), and is a sum of square-zero operators [7]. When

$\dim \mathcal{H} < \infty$, it is well-known that the span of commutators (respectively, nilpotents, square-zero operators) is the set of all operators of trace zero. Thus $\mathcal{L}$ includes $(\mathcal{T}(\mathcal{N}))^0$, the zero trace part of $\mathcal{T}(\mathcal{N})$. The reverse inclusion is obvious. □

3.11. **Corollary.** *Let $\mathcal{T}(\mathcal{N})$ be a nest algebra. The linear span of the idempotents is weakly dense in $\mathcal{T}(\mathcal{N})$.*

Proof. Let $\mathcal{L}$ be the weak closure of the span of the idempotents. As in the proof of Corollary 3.10, for every $E \in \mathcal{N}$, $T \in \mathcal{T}(\mathcal{N})$, the operator $S := ET(I-E) \in \mathcal{L}$, since $S = E - (E - ET(I-E))$, a difference of two idempotents. On the other hand, the span of idempotents in $\mathcal{B}(\mathcal{H})$ is all of $\mathcal{B}(\mathcal{H})$ [23, 20]. Thus $\mathcal{L}$ includes $P\mathcal{B}(\mathcal{H})P$ for every atom P in $\mathcal{N}$ and also includes $\mathcal{T}(\mathcal{N}) \cap \ker \Delta$, and so $\mathcal{L} = \mathcal{T}(\mathcal{N})$. □

3.12. **Example.** Let us now consider two examples. In the first instance, we suppose $\{\mathcal{H}_j\}_{j=1}^5$ are finite dimensional Hilbert spaces, and that $\mathcal{H} = \oplus_{j=1}^5 \mathcal{H}_j$. Let $\mathcal{N} = \{\{0\}, \oplus_{j=1}^k \mathcal{H}_j, 1 \le k \le 5\}$ be our nest.

Consider the associative ideal $\mathcal{J}$ of $\mathcal{T}(\mathcal{N})$ given by:

$$\mathcal{J} = \{ \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\ & 0 & 0 & A_{24} & A_{25} \\ & & 0 & A_{34} & A_{35} \\ & & & A_{44} & A_{45} \\ & & & & 0 \end{bmatrix} : A_{ij} \in \mathcal{B}(\mathcal{H}_j, \mathcal{H}_i)\}.$$

Then $\mathcal{J}^\circ = \{A = [A_{ij}] \in \mathcal{J} : \mathrm{trace}(A_{11}) = \mathrm{trace}(A_{44}) = 0\}$, while

$$\mathcal{J} + \mathcal{C}_{\mathcal{J}} = \{ \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\ & \alpha I & 0 & A_{24} & A_{25} \\ & & \alpha I & A_{34} & A_{35} \\ & & & A_{44} & A_{45} \\ & & & & \beta I \end{bmatrix} : A_{ij} \in \mathcal{B}(\mathcal{H}_j, \mathcal{H}_i), \alpha, \beta \in \mathbb{C}\}.$$

By Theorem 3.9(d), every linear manifold $\mathcal{L}$ in $\mathcal{T}(\mathcal{N})$ satisfying $\mathcal{J}^\circ \subseteq \mathcal{L} \subseteq \mathcal{J} + \mathcal{C}_{\mathcal{J}}$ is a Lie ideal.

3.13. **Example.** The *Volterra nest* is a continuous nest on $\mathcal{H} = L^2([0,1]$ with respect to Lebesgue measure, defined as follows: For each $s \in [0,1]$, let

$$N_s = \{f \in L^2([0,1]) : f = 0 \ \text{ almost everywhere on } [s,1]\}$$

and set $\mathcal{N} = \{N_s : s \in [0,1]\}$. The corresponding nest algebra is known as the *Volterra nest algebra* and is denoted by $\mathcal{V}$. The diagonal and the core of the Volterra algebra coincide and are equal to the algebra of all multiplication operators M_g, $g \in L^\infty([0,1])$, where $M_g f = gf$.

By the Erdos-Power Theorem, the set of weakly closed associative ideals of $\mathcal{V}$ is in a one-to-one correspondence with the set of all increasing lower semi-continuous functions $\eta : [0,1] \to [0,1]$ such that $\eta(x) \le x$ for every x. Given an increasing, lower semi-continuous function η as above, the corresponding lie ideal $\mathcal{I}(\eta)$ is the set of all operators that map N_s into $N_{\eta(s)}$ for every s. Observe that the Volterra nest has no atoms, and so $\mathcal{I}(\eta)$ is automatically atomic diagonal disjoint. It is easy to see that $\mathcal{C}_{\mathcal{I}(\eta)}$ is the subalgebra of the core consisting of all multiplication operators M_g, where $g \in L^\infty([0,1])$ is constant on each connected component of the set $\{x \in [0,1] : \eta(x) < x\}$.

If $\mathcal{L}$ is a linear manifold in $\mathcal{V}$ such that

$$\mathcal{I}(\eta) \subseteq \mathcal{L} \subseteq \mathcal{I}(\eta) + \mathcal{C}_{\mathcal{I}(\eta)},$$

then it follows that $\mathcal{L}$ is a Lie ideal in $\mathcal{V}$. Conversely, by Theorem 3.9, every weakly closed Lie ideal in $\mathcal{V}$ is of this form.

It is natural to ask whether the associative ideals in Theorem 3.9 are unique. Our next result provides an answer.

3.14. **Proposition.** *Let $\mathcal{L}$ be a weakly closed Lie ideal in a nest algebra $\mathcal{T}(\mathcal{N})$ and let $\mathcal{I}$ and $\mathcal{J}$ be associative ideals satisfying the inclusions in parts (c) and (d) of Theorem 3.9. Let $\mathbb{A}_1$ also be as in that Theorem. Then*

(a) $\mathcal{I} = \mathcal{I}_{\mathcal{L}} := \mathrm{wk}-\mathrm{cl}\ \mathrm{span}\{ET(I-E) : E \in \mathcal{N}\}$.
(b) $\mathbb{A}_1 \supseteq \mathbb{A}_0^- := \{P \in \mathbb{A}_{\mathcal{I}} : P\mathcal{L}P \not\subseteq \mathbb{C}P\}$.
(c) $\mathbb{A}_1 \subseteq \mathbb{A}_0^+ := \mathbb{A}_0^- \cup$ *the set of all atoms of dimension one that are adjacent to* $\mathcal{I}$.
(d) $\mathcal{I} \vee \mathbb{A}_0^- \subseteq \mathcal{J} \subseteq \mathcal{I} \vee \mathbb{A}_0^+$.

Thus $\mathcal{I}$, $\mathcal{J}^\circ$, $[\mathcal{T}(\mathcal{N}) : \mathcal{I} \vee \mathbb{A}_1]$ and $\mathcal{J} + \mathcal{C}_{\mathcal{J}}$ are uniquely determined by $\mathcal{L}$, while $\mathcal{J}$ and $\mathbb{A}_1$ are uniquely determined by $\mathcal{L}$ modulo the one-dimensional atoms of $\mathcal{T}(\mathcal{N})$.

Proof. (a) We have

$$\mathcal{I} \subseteq (\mathcal{I} \vee \mathbb{A}_1)^\circ \subseteq \mathcal{L} \subseteq (\mathcal{I} \vee \mathbb{A}_1) + \mathcal{C}_{\mathcal{I}}.$$

For every $E \in \mathcal{N}$, we also have

$$E\mathcal{I}(I-E) \subseteq E\mathcal{L}(I-E) \subseteq E(\mathcal{I} \vee \mathbb{A}_1)(I-E) + E\mathcal{C}_{\mathcal{I}}(I-E).$$

However $E\mathcal{C}_{\mathcal{I}}(I-E) = 0$ and for every atom P, we have $E(PTP)(I-E) = 0$. Thus

$$E\mathcal{L}(I-E) = E\mathcal{I}(I-E).$$

By Proposition 3.2, we find that $\mathcal{I} = \mathcal{I}_{\mathcal{L}}$, proving the uniqueness of $\mathcal{I}$.

(b) Assume that $P \in \mathbb{A}_{\mathcal{I}}$ and $P\mathcal{L}P \not\subseteq \mathbb{C}P$. Since $\mathcal{L} \subseteq (\mathcal{I} \vee \mathbb{A}_1) + \mathcal{C}_{\mathcal{I}}$, then $P\mathcal{L}P \subseteq P\mathbb{A}_1P + \mathbb{C}P$. Thus $P \in \mathbb{A}_1$.

(c) Assume that $P \in \mathbb{A}_1$ and that $\dim P\mathcal{H} \geq 2$. Since $\mathbb{A}_1^\circ \subseteq (\mathcal{I} \vee \mathbb{A}_1)^\circ \subseteq \mathcal{L}$, we have $P\mathbb{A}_1^\circ P \subseteq P\mathcal{L}P$, and so $P\mathcal{L}P \not\subseteq \mathbb{C}P$, i.e., $P \in \mathbb{A}_0^+$.

(d) From the equation $\mathcal{J}^\circ \subseteq \mathcal{L} \subseteq \mathcal{J} + \mathcal{C}_{\mathcal{J}}$, we have, for any $P \in \mathcal{N}$,

$$P\mathcal{J}(I-P) = P\mathcal{J}^\circ(I-P) \subseteq P\mathcal{L}(I-P) \subseteq P\mathcal{J}(I-P).$$

Thus $P\mathcal{J}(I-P) = P\mathcal{L}(I-P)$, which implies that $\mathcal{J} = \mathcal{I} \vee \mathbb{B}$, where $\mathbb{B}$ is a subset of the atoms of $\mathcal{N}$. The results of (b) and (c) now imply (d). The rest of the assertions are now evident. $\square$

3.15. REMARK. In [8], it was shown that a linear manifold $\mathcal{L}$ in $\mathcal{B}(\mathcal{H})$ is invariant under conjugation by invertible operators if and only if it is invariant under conjugation by unitary operators, thereby providing an alternate description of Lie ideals in that algebra. Similar results also hold for other classes of selfadjoint operator algebras [14, 15].

However, the concept of unitary invariance is inappropriate for non selfadjoint algebras for several reasons. In a nest algebra $\mathcal{T}(\mathcal{N})$, the only elements in $\mathcal{T}(\mathcal{N})$ whose adjoints also lie in $\mathcal{T}(\mathcal{N})$ necessarily lie in the diagonal $\mathcal{D}(\mathcal{N})$. If the nest is not trivial, i.e., $\mathcal{T}(\mathcal{N}) \neq \mathcal{B}(\mathcal{H})$, then we observe that the diagonal $\mathcal{D}(\mathcal{N})$ is invariant under conjugation by diagonal unitaries but is obviously not a Lie ideal, indeed if $P \in \mathcal{N}$, $0 \neq P \neq I$, then it is evident that $[P, \mathcal{T}(\mathcal{N})] \not\subseteq \mathcal{D}(\mathcal{N})$

It is true that nest algebras may contain unitary operators which do not lie in $\mathcal{D}(\mathcal{N})$. For example, If $\mathcal{N}$ is an atomic nest that is order-isomorphic to the integers and with all atoms of dimension 1, then $\mathcal{T}(\mathcal{N})$ contains a bilateral shift. In fact, it was shown in [1, 3] that in many nest algebras, every element can by written as a linear combination of unitary operators in $\mathcal{T}(\mathcal{N})$. However, this is not of much help since if we consider any collection $\mathcal{U}$ of unitary operators on $\mathcal{H}$ such that $\mathcal{T}(\mathcal{N})$ is invariant under conjugation by $\mathcal{U}$, then upon taking adjoints, we see that $\mathcal{T}(\mathcal{N})^*$, and hence also $\mathcal{D}(\mathcal{N})$ is invariant under conjugation by $\mathcal{U}$ but is not a Lie ideal. Conversely, for every $P \in \mathcal{N}$, the set $\mathcal{L}_P := P\mathcal{T}(\mathcal{N})(I - P)$ is a Lie ideal (indeed it is an associative ideal). However by considering rank one operators in $\mathcal{L}_P$, it is evident that conjugation $X \mapsto U^*XU$ by a unitary operator U leaves $\mathcal{L}_P$ invariant if and only if U^* commutes with P. Therefore, if every $\mathcal{L}_P$ is invariant under conjugation by every unitary in $\mathcal{U}$, then we must have $\mathcal{U} \subseteq \mathcal{D}(\mathcal{N})$.

4. ALGEBRAS OF INFINITE MULTIPLICITY.

4.1. In Theorem 3.9 we saw that if $\mathcal{L}$ is a weakly closed linear manifold in a nest algebra $\mathcal{T}(\mathcal{N})$, then (i) $\mathcal{L}$ is a Lie ideal if and only if (ii) $\mathcal{L}$ is invariant under conjugation by invertibles. We shall now demonstrate that if $\mathcal{N}$ has no finite dimensional atoms, then (i) and (ii) are equivalent with no topological restrictions on the linear manifold $\mathcal{L}$. In fact, the results hold in even greater generality.

4.2. Let $\mathcal{A}$ be a weakly closed subalgebra of $\mathcal{B}(\mathcal{H})$. If $\mathcal{K}$ is a second (complex separable) Hilbert space, then the tensor product $\mathcal{A} \otimes \mathcal{B}(\mathcal{K})$ is defined as the weak operator closure of the span of all elementary tensors $A \otimes B$ acting on $\mathcal{H} \otimes \mathcal{K}$, where $A \otimes B$ is determined by the formula $(A \otimes B)(x \otimes y) = Ax \otimes By$. If $\mathcal{K}$ is identified with ℓ^2 and $\mathcal{H} \otimes \mathcal{K}$ with $\mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H} \oplus \cdots$, then it is not too hard to see that $\mathcal{A} \otimes \mathcal{B}(\mathcal{K})$ may be realised as the set of all infinite matrices of operators $[A_{ij}]$ where $A_{ij} \in \mathcal{A}$ and where $[A_{ij}]$ defines a bounded operator.

A weakly closed subalgebra of $\mathcal{B}(\mathcal{H})$ is said to be of *infinite multiplicity* if $\mathcal{A} \otimes \mathcal{B}(\mathcal{K})$ is isomorphic to $\mathcal{A}$. It is obvious that if T is invertible, then $T^{-1}\mathcal{A}T$ has infinite multiplicity if and only if $\mathcal{A}$ does. *Caveat:* This notion of multiplicity is different from other notions of multiplicity which are invariant under conjugations by unitary operators on $\mathcal{H}$, but not under similarity.

The next result is well-known. It readily follows from the Similarity Theorem ([2]; p.162).

4.3. **Proposition.** *A nest algebra $\mathcal{T}(\mathcal{N})$ has infinite multiplicity if and only if $\mathcal{N}$ has no finite dimensional atoms.*

4.4. Let $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ be a unital subalgebra. An operator $X \in \mathcal{A}$ is said to be *interpolating in* $\mathcal{A}$ if there exist $L, R \in \mathcal{A}$ so that $LXR = I$. These were studied by J. Orr [18] who obtained the following result when $\mathcal{T}(\mathcal{N})$ is a continuous nest algebra.

4.5. **Theorem.** (see [4]). *Let $\mathcal{T}(\mathcal{N})$ be a nest algebra of infinite multiplicity. Then the set of interpolating operators in $\mathcal{T}(\mathcal{N})$ is (norm) dense in $\mathcal{T}(\mathcal{N})$.*

4.6. Our goal is to show that if $\mathcal{A}$ is a weakly closed unital subalgebra of $\mathcal{B}(\mathcal{H})$, and if the set of interpolating operators is norm dense in $\mathcal{A}$, then a linear manifold $\mathcal{L}$ is a Lie ideal precisely if $\mathcal{L}$ is invariant under conjugation by invertibles. In essence, all of the tools required to prove this have individually been obtained elsewhere, but in different contexts. All that is required is to observe that the proofs given there carry over with only minor modifications to our setting. As such, we shall refer the reader to the original sources for the proofs of the following Proposition; but first we require a couple of definitions.

An operator A in a unital algebra $\mathcal{A}$ is called an *involution* if $A^2 = I$. (We remark that some authors refer to these as *symmetries*, while others reserve the term "symmetry" for self-adjoint involutions.) A *unipotent of order* k in $\mathcal{A}$ is an element of the form $1+N$, where $N^{k-1} \neq 0 = N^k$.

4.7. **Theorem.** *Let $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ be a weakly closed, unital algebra of infinite multiplicity. Then*

(a) *every $A \in \mathcal{A}$ is a sum of two commutators in $\mathcal{A}$;*
(b) *every $A \in \mathcal{A}$ is a sum of eight idempotents in $\mathcal{A}$;*
(c) *every $A \in \mathcal{A}$ is a sum of eight nilpotents of order 2 in $\mathcal{A}$;*
(d) *if the set of interpolating operators in $\mathcal{A}$ is dense, then every invertible element of $\mathcal{A}$ is a product of at most 28 involutions;*
(e) *if the set of interpolating operators in $\mathcal{A}$ is dense, then every invertible element of $\mathcal{A}$ is a product of at most 36 unipotents of order 2.*

Proof. (a) This is Proposition 2.6 of [16], and as mentioned there, the proof is identical to that of Halmos ([10]; Problem 234), where it is demonstrated for the special case where $\mathcal{A} = \mathcal{B}(\mathcal{H})$.

(b) and (c) follow from the results of Pearcy and Topping [20].

(d) This follows from Lemma 3.1 and Lemma 3.2 of [5], combined with the remark at the end of [4].

(e) An examination of the proofs of Lemmas 3.1 and 3.2 in [5] and Theorem 1.4 in [4] shows that every invertible is a product of 8 unipotents of order two, 2 involutions conjugate to $\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$, 3 invertibles of the form $\begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix}$, and 2 invertibles of the form $\begin{bmatrix} I & X \\ 0 & Y \end{bmatrix}$.

Note that by the proofs of Lemmas 3 and 4 of [9], every invertible element of the form $\begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix}$ can be written as a product of 4 unipotents of order two. Furthermore,

$$\begin{bmatrix} I & X \\ 0 & Y \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & Y \end{bmatrix} \begin{bmatrix} I & X \\ 0 & I \end{bmatrix},$$

and hence is a product of at most 5 unipotents of order 2. As for the invertible elements conjugate to $\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$, consider the following.

Since $-I \in \mathcal{A} \simeq \mathbb{M}_2(\mathcal{A})$, we can identify $\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$ with $\begin{bmatrix} I & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & -I \end{bmatrix}$. From the Corollary to Theorem 1 of [22], this latter operator is a product of three unipotent operators of order 2 in $\mathbb{M}_3(\mathbb{C}) \otimes I \subseteq \mathcal{A}$. Indeed, in $\mathbb{M}_2(\mathbb{C})$, we can write $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ as a product of three unipotent matrices U_1, U_2, U_3 which are necessarily of order 2. Let $V_i = \begin{bmatrix} 1 & 0 \\ 0 & U_i \end{bmatrix}$, $1 \leq i \leq 3$. Then each V_i is unipotent of order 2 in $\mathbb{M}_3(\mathbb{C})$, and

$$\begin{bmatrix} I & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & -I \end{bmatrix} \otimes I = (V_1 \otimes I)\,(V_2 \otimes I)\,(V_3 \otimes I)$$

is a product of three unipotents of order 2. Since unipotents of order 2 are invariant under conjugation, we have written each such invertible as a product of 3 unipotents of order two.

In conclusion, we deduce that every invertible in $\mathcal{A}$ is a product of at most 36 unipotents of order 2. □

4.8. **Theorem.** *Let $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ be a weakly closed unital algebra of infinite multiplicity and suppose that the interpolating operators are dense in $\mathcal{A}$. Let $\mathcal{L} \subseteq \mathcal{A}$ be a linear manifold. The following are equivalent:*

(a) *$\mathcal{L}$ is a Lie ideal;*
(b) *$\mathcal{L}$ is invariant under conjugation by all invertibles;*
(c) *$\mathcal{L}$ is invariant under conjugation by involutions;*
(d) *$\mathcal{L}$ is invariant under conjugation by unipotents of order 2.*

Proof. To prove that (a) implies (b), we argue as in [8]. if S is an involution in $\mathcal{A}$ and $M \in \mathcal{L}$, then

$$S^{-1}MS = M - \frac{1}{2}[S,[S,M]] \in \mathcal{L}.$$

Since every invertible in $\mathcal{A}$ is a product of at most 28 involutions by Theorem 4.7, we are done.

To prove that (b) implies (a), we again use a calculation from [8]. If $E \in \mathcal{A}$ is an idempotent, then $S = E + i(I - E)$ is invertible and

$$[E,M] = (S^{-1}MS - SMS^{-1})/2i \in \mathcal{L}$$

for all $M \in \mathcal{L}$. Since, by Theorem 4.7 (b), each element of $\mathcal{A}$ is a sum of eight idempotents in $\mathcal{A}$, we get that $\mathcal{L}$ is a Lie ideal.

Finally the equivalence of (b), (c) and (d) is clear in view of Theorem 4.7. □

4.9. **Corollary.** *If $\mathcal{N}$ is a nest with no finite-dimensional atoms, then a linear manifold $\mathcal{L}$ in $\mathcal{T}(\mathcal{N})$ is a Lie ideal if and only if it is invariant under conjugation.*

5. An example.

5.1. This section is devoted to the characterisation of the Lie ideals of a particular algebra of infinite multiplicity. As before, all Hilbert spaces are assumed separable.

Let $\mathcal{T}_n$ denote the algebra of all $n \times n$ upper triangular matrices over $\mathbb{C}$. We shall denote the standard matrix units of $\mathcal{T}_2$ by E_{11}, E_{12} and E_{22}. In considering the algebra $\mathcal{T}_2 \otimes \mathcal{T}_2 \subseteq \mathcal{T}_4$, we shall denote the matrix units by $\{F_{ij} : 1 \leq i \leq j \leq 4,\ (i,j) \neq (2,3)\}$. We shall also denote the algebra of $n \times n$ diagonal matrices by $\mathcal{D}_n$.

The algebra we shall consider is $\mathcal{A} = \mathcal{B}(\mathcal{H}) \otimes (\mathcal{T}_2 \otimes \mathcal{T}_2)$. It is clear that $\mathcal{A}$ is a weakly closed unital operator algebra of infinite multiplicity. We shall first show that $\mathcal{A}$ possesses a dense set of interpolating operators. By Theorem 4.8, it would then follow that the Lie ideals of $\mathcal{A}$ coincide with those linear manifolds of $\mathcal{A}$ which are invariant under conjugation by invertibles. The second step will be to characterise all of the Lie ideals of $\mathcal{A}$.

5.2. **Lemma.** *Let $\mathcal{Q} \subseteq \mathcal{T}_n$ be a unital algebra which includes the diagonal of $\mathcal{T}_n$. Let $\mathcal{R} \subseteq \mathcal{B}(\mathcal{H})$ be a weakly closed unital algebra of infinite multiplicity with a dense set of interpolating operators. Then $\mathcal{R} \otimes \mathcal{Q}$ is a weakly closed unital operator algebra having infinite multiplicity and a dense set of interpolating operators.*

Proof. The only statement which is not completely apparent is the last - namely that $\mathcal{R} \otimes \mathcal{Q}$ has a dense set of interpolating operators. Let $[A_{ij}] \in \mathcal{R} \otimes \mathcal{Q} \subseteq \mathcal{R} \otimes \mathcal{T}_n$. Let $\epsilon > 0$. For $1 \leq i \leq n$, choose $A'_{ii} \in \mathcal{R}$ such that A'_{ii} is interpolating and $\|A_{ii} - A'_{ii}\| < \epsilon$. If $i \neq j$, let $A'_{ij} = A_{ij}$, and set $A' = [A'_{ij}]$. Choose $L_{ii}, R_{ii} \in \mathcal{R}$ so that $L_{ii}A'_{ii}R_{ii} = I$, $1 \leq i \leq n$. Let $L = L_{11} \oplus \cdots \oplus L_{nn}$ and $R = R_{11} \oplus \cdots \oplus R_{nn}$. Then $LA'R = I + N$, where $N \in \mathcal{R} \otimes \mathcal{Q}$ is nilpotent (of order at most n). Thus $LA'R$ is invertible in $\mathcal{R} \otimes \mathcal{Q}$, and so A' is interpolating in $\mathcal{R} \otimes \mathcal{Q}$. Since $\|A' - A\| = \max_{1 \leq i \leq n} \|A'_{ii} - A_{ii}\| < \epsilon$, it follows that $\mathcal{R} \otimes \mathcal{Q}$ has a dense set of interpolating operators. □

5.3. **Lemma.** *Let $\mathcal{R}$ be a unital algebra and $\mathcal{Q} = \mathcal{R} \otimes \mathcal{T}_2$.*

(a) *$\mathcal{K}$ is an ideal of $\mathcal{Q}$ if and only if* $\mathcal{K} = \begin{bmatrix} \mathcal{K}_1 & \mathcal{K}_2 \\ 0 & \mathcal{K}_4 \end{bmatrix}$, *where*

(i) *$\mathcal{K}_1, \mathcal{K}_2$, and $\mathcal{K}_4$ are ideals of $\mathcal{R}$;*

(ii) *$\mathcal{K}_2 \supseteq \mathcal{K}_1 + \mathcal{K}_4$.*

(b) $[\mathcal{Q}, \mathcal{K}] = \begin{bmatrix} [\mathcal{R}, \mathcal{K}_1] & \mathcal{K}_2 \\ 0 & [\mathcal{R}, \mathcal{K}_4] \end{bmatrix}$.

Proof. Elementary. □

As an easy consequence, we obtain the following:

5.4. **Proposition.** *Let $\mathcal{R}$ be a unital algebra and $\mathcal{Q} = \mathcal{R} \otimes (\mathcal{T}_2 \otimes \mathcal{T}_2)$. Then $\mathcal{K}$ is an ideal of $\mathcal{Q}$ if and only if*

$$\mathcal{K} = \begin{bmatrix} \mathcal{K}_{11} & \mathcal{K}_{12} & \mathcal{K}_{13} & \mathcal{K}_{14} \\ 0 & \mathcal{K}_{22} & 0 & \mathcal{K}_{24} \\ 0 & 0 & \mathcal{K}_{33} & \mathcal{K}_{34} \\ 0 & 0 & 0 & \mathcal{K}_{44} \end{bmatrix},$$

where

(i) *each $\mathcal{K}_{ij}$ is an ideal of $\mathcal{R}$;*

(ii) *$\mathcal{K}_{ij} \supseteq \mathcal{K}_{ii} + \mathcal{K}_{jj}$ for $i < j$.*

(iii) *$\mathcal{K}_{14} \supseteq \mathcal{K}_{12} + \mathcal{K}_{13} + \mathcal{K}_{24} + \mathcal{K}_{34}$.*

We recall that if $\mathcal{U}$ and $\mathcal{V}$ and ideals in an algebra $\mathcal{A}$, then $[\mathcal{U},\mathcal{V}]$ denotes the linear span of the set of commutators $uv - vu$; for $u \in \mathcal{U}$ and $v \in \mathcal{V}$.

5.5. **Lemma.** *Let $\mathcal{K} \subseteq \mathcal{B}(\mathcal{H})$ be an ideal. Then the ideal $\mathcal{J}$ generated by $[\mathcal{B}(\mathcal{H}),\mathcal{K}]$ is $\mathcal{K}$ itself.*

Proof. Let P be an infinite rank projection with infinite dimensional kernel, and let V be a partial isometry such that $VV^* = P$ and $V^*V = (I-P)$. It follows that $VP = 0$. For example, $V = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}$ relative to the decomposition $\mathcal{H} = P\mathcal{H} \oplus (P\mathcal{H})^{\perp}$, and where we identify $P\mathcal{H}$ with $(P\mathcal{H})^{\perp}$.

Let $K \in \mathcal{K}$. Then $PK(I-P) = [PK(I-P),(I-P)] \in [\mathcal{K},\mathcal{B}(\mathcal{H})] \subseteq \mathcal{J}$. Similarly, $(I-P)KP \in \mathcal{J}$. Also $PKP = [PKV, V^*]P \in [\mathcal{K},\mathcal{B}(\mathcal{H})]\mathcal{B}(\mathcal{H}) \subseteq \mathcal{J}$, and similarly $(I-P)K(I-P) \in \mathcal{J}$. Thus $K = PKP + (I-P)K(I-P) + PK(I-P) + (I-P)KP \in \mathcal{J}$.

Hence $\mathcal{K} \subseteq \mathcal{J}$. The reverse inclusion is clear. □

We shall make use of the following result from [8] about the relationship between Lie ideals and associative ideals in $\mathcal{B}(\mathcal{H})$

5.6. **Theorem.** ([8]). *Let $\mathcal{L}$ be a Lie ideal in $\subseteq \mathcal{B}(\mathcal{H})$. Then there exists an associative ideal $\mathcal{J}$ such that*

$$[\mathcal{B}(\mathcal{H}),\mathcal{J}] \subseteq \mathcal{L} \subseteq \mathcal{J} + \mathbb{C}I.$$

Next, we make an observation regarding the relationship between $\mathcal{L}$ and $\mathcal{J}$.

5.7. **Lemma.** *Let $\mathcal{L}$ be a Lie ideal in $\mathcal{B}(\mathcal{H})$ and let $\mathcal{J}$ be an associative ideal that satisfies the inclusions in Theorem 5.6. Then $[\mathcal{B}(\mathcal{H}),\mathcal{L}] = [\mathcal{B}(\mathcal{H}),\mathcal{J}]$.*

Proof. First note that $[\mathcal{B}(\mathcal{H}),\mathcal{L}] \subseteq [\mathcal{B}(\mathcal{H}),\mathcal{J}+\mathbb{C}I] = [\mathcal{B}(\mathcal{H}),\mathcal{J}]$.

For the reverse inclusion, first recall that $\mathcal{B}(\mathcal{H})$ is linearly spanned by its projections [7]. Thus it suffices to show that if $P = P^2 \in \mathcal{B}(\mathcal{H})$ and $J \in \mathcal{J}$, then $[P,J] \in [\mathcal{B}(\mathcal{H}),\mathcal{L}]$. However

$$[P,J] = [P,[P,[P,J]]].$$

Now $[P,J] \in \mathcal{J}$, and so $[P,[P,J]] \in [\mathcal{B}(\mathcal{H}),\mathcal{J}] \subseteq \mathcal{L}$. Thus $[P,[P,[P,J]]] \in [\mathcal{B}(\mathcal{H}),\mathcal{L}]$, i.e., $[P,J] \in [\mathcal{B}(\mathcal{H}),\mathcal{L}]$ as required. □

With the results we already have, we are now in a position to prove the uniqueness of the associative ideal that corresponds to a Lie ideal in $\mathcal{B}(\mathcal{H})$. Although we shall make no use of this in the foregoing, it may be of independent interest.

5.8. **Corollary.** *If $\mathcal{L}$ is a Lie ideal in $\mathcal{B}(\mathcal{H})$, then the corresponding associative ideal in Theorem 5.6 is uniquely determined by $\mathcal{L}$.*

Proof. This follows immediately from 5.5 and 5.7.

5.9. **Lemma.** *Suppose $\mathcal{R}$ is a unital algebra and $\mathcal{Q} = \mathcal{R} \otimes \mathcal{T}_2$. Let $\mathcal{L} \subseteq \mathcal{Q}$ be a Lie ideal. For $1 \leq i \leq j \leq 2$, let $\mathcal{M}_{ij} = \{A \in \mathcal{R} : A \otimes E_{ij} \in (I \otimes E_{ii})\mathcal{L}(I \otimes E_{jj})\}$, i.e., $\mathcal{M}_{ij}$ is the set of all elements of $\mathcal{R}$ that appear as entry in the (i,j) position of some element in $\mathcal{L}$. Then*

(a) *$\mathcal{M}_{11}$ and $\mathcal{M}_{22}$ are Lie ideals of $\mathcal{R}$ and*

$$\mathcal{L} \supseteq \begin{bmatrix} [\mathcal{R}, \mathcal{M}_{11}] & \mathcal{M}_{12} \\ 0 & [\mathcal{R}, \mathcal{M}_{22}] \end{bmatrix}.$$

(b) *$\mathcal{M}_{12}$ is an ideal of $\mathcal{R}$, and $\mathcal{M}_{12} \supseteq [\mathcal{R}, \mathcal{M}_{11}] + [\mathcal{R}, \mathcal{M}_{22}]$.*

Proof. (a) First observe that if $\begin{bmatrix} L_1 & L_2 \\ 0 & L_4 \end{bmatrix} \in \mathcal{L}$, then $\left[\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} L_1 & L_2 \\ 0 & L_4 \end{bmatrix}\right] = \begin{bmatrix} 0 & L_2 \\ 0 & 0 \end{bmatrix} \in \mathcal{L}$. It follows that $\begin{bmatrix} L_1 & 0 \\ 0 & L_4 \end{bmatrix} \in \mathcal{L}$, and hence

$$\left[\begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} L_1 & 0 \\ 0 & L_4 \end{bmatrix}\right] = \begin{bmatrix} [X, L_1] & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{L},$$

. Similarly

$$\begin{bmatrix} 0 & 0 \\ 0 & [Y, L_4] \end{bmatrix} \in \mathcal{L}.$$

Thus $\mathcal{M}_{11}$ and $\mathcal{M}_{22}$ are Lie ideals of $\mathcal{R}$, and

$$\mathcal{L} \supseteq \begin{bmatrix} [\mathcal{R}, \mathcal{M}_{11}] & \mathcal{M}_{12} \\ 0 & [\mathcal{R}, \mathcal{M}_{22}] \end{bmatrix},$$

as claimed.

(b) If $T \in \mathcal{M}_{12}$, then by part (a) , we have $\begin{bmatrix} 0 & T \\ 0 & 0 \end{bmatrix} \in \mathcal{L}$. Then

$$\left[\left[\begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & T \\ 0 & 0 \end{bmatrix}\right], \begin{bmatrix} 0 & 0 \\ 0 & Y \end{bmatrix}\right] = \begin{bmatrix} 0 & XTY \\ 0 & 0 \end{bmatrix} \in \mathcal{L}$$

for all $X, Y \in \mathcal{R}$. Hence $\mathcal{M}_{12}$ is an ideal of $\mathcal{R}$.

Finally, if $X \in [\mathcal{R}, \mathcal{M}_{11}]$ and $Y \in [\mathcal{R}, \mathcal{M}_{22}]$, then $\begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & Y \end{bmatrix} \in \mathcal{L}$ by (a), and hence

$$\left[\begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}\right] = \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} \text{ and } \left[\begin{bmatrix} 0 & 0 \\ 0 & Y \end{bmatrix}, \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}\right] = \begin{bmatrix} 0 & Y \\ 0 & 0 \end{bmatrix}.$$

Thus X, $Y \in \mathcal{M}_{12}$, completing the proof. □

5.10. **Proposition.** *Let $\mathcal{L}$ be a Lie ideal in $\mathcal{Q} := \mathcal{B}(\mathcal{H}) \otimes \mathcal{T}_2$. Then there exists an assocative ideal $\mathcal{K}$ of $\mathcal{Q}$ such that*

$$(\dagger) \qquad [\mathcal{K}, \mathcal{Q}] \subseteq \mathcal{L} \subseteq \mathcal{K} + \mathcal{D}_{\mathcal{K}},$$

where $\mathcal{D}_{\mathcal{K}}$ is the subspace of $I \otimes \mathcal{D}_2$, the diagonal of $\mathcal{Q}$, determined by :

$$(\ddagger) \qquad \text{if } \begin{bmatrix} 0 & \mathcal{B}(\mathcal{H}) \\ 0 & 0 \end{bmatrix} \subseteq \mathcal{K}, \text{ then } \mathcal{D}_{\mathcal{K}} = I \otimes \mathcal{D}_2; \text{ otherwise } \mathcal{D}_{\mathcal{K}} = \mathbb{C}I.$$

Conversely, if $\mathcal{K}$ is an ideal of $\mathcal{Q}$ and $\mathcal{L}$ is a subspace of $\mathcal{Q}$ satisfying $(\dagger)$, then $\mathcal{L}$ is a Lie ideal.

Proof. Let $\mathcal{M}_{ij}$ be as in the preceding lemma. Thus $\mathcal{M}_{11}$ and $\mathcal{M}_{22}$ are Lie ideals of $\mathcal{B}(\mathcal{H})$ and

$$\mathcal{L} \supseteq \begin{bmatrix} [\mathcal{B}(\mathcal{H}), \mathcal{M}_{11}] & \mathcal{M}_{12} \\ 0 & [\mathcal{B}(\mathcal{H}), \mathcal{M}_{22}] \end{bmatrix}.$$

By [8], we can find ideals $\mathcal{K}_{11}$ and $\mathcal{K}_{22}$ of $\mathcal{B}(\mathcal{H})$ so that

$$[\mathcal{B}(\mathcal{H}), \mathcal{K}_{ii}] \subseteq \mathcal{M}_{ii} \subseteq \mathcal{K}_{ii} + \mathbb{C}I, \quad i = 1, 2.$$

Furthermore, by Lemma 5.7, $[\mathcal{B}(\mathcal{H}), \mathcal{M}_{ii}] = [\mathcal{B}(\mathcal{H}), \mathcal{K}_{ii}]$, $i = 1, 2$. Thus

$$\mathcal{L} \supseteq \begin{bmatrix} [\mathcal{B}(\mathcal{H}), \mathcal{K}_{11}] & \mathcal{M}_{12} \\ 0 & [\mathcal{B}(\mathcal{H}), \mathcal{K}_{22}] \end{bmatrix}.$$

Let $\mathcal{K} = \begin{bmatrix} \mathcal{K}_{11} & \mathcal{M}_{12} \\ 0 & \mathcal{K}_{22} \end{bmatrix}$. Then by Lemma 5.9 (b), $\mathcal{M}_{12}$ is an ideal of $\mathcal{B}(\mathcal{H})$ and $\mathcal{M}_{12} \supseteq [\mathcal{B}(\mathcal{H}), \mathcal{M}_{11}] + [\mathcal{B}(\mathcal{H}), \mathcal{M}_{22}] = [\mathcal{B}(\mathcal{H}), \mathcal{K}_{11}] + [\mathcal{B}(\mathcal{H}), \mathcal{K}_{22}]$.

Since $\mathcal{M}_{12}$ is an ideal, then by Lemma 5.5, we have that $\mathcal{M}_{12} \supseteq \mathcal{K}_{11} + \mathcal{K}_{22}$, and so Lemma 5.3(a) implies that $\mathcal{K}$ is an ideal of $\mathcal{Q}$. By Lemma 5.3(b),

$$[\mathcal{Q}, \mathcal{K}] = \begin{bmatrix} [\mathcal{B}(\mathcal{H}), \mathcal{K}_{11}] & \mathcal{M}_{12} \\ 0 & [\mathcal{B}(\mathcal{H}), \mathcal{K}_{22}] \end{bmatrix} \subseteq \mathcal{L}.$$

It is now clear that $\mathcal{L} \subseteq \mathcal{K} + I \otimes \mathcal{D}_2$. We will show that if $\mathcal{M}_{12} \neq \mathcal{B}(\mathcal{H})$, then $\mathcal{L} \subseteq \mathcal{K} + \mathbb{C}I$.

If $\mathcal{M}_{12} \neq \mathcal{B}(\mathcal{H})$, then $\mathcal{M}_{12}$ is included in the ideal of compact operators and so must $\mathcal{K}_{11}$ and $\mathcal{K}_{22}$, as they are included in $\mathcal{M}_{12}$. Now suppose that $\begin{bmatrix} K_1 + \alpha_1 I & M \\ 0 & K_2 + \alpha_2 I \end{bmatrix} \in \mathcal{L}$ with $K_1 \in \mathcal{K}_{11}$ and $K_2 \in \mathcal{K}_{22}$, $M \in \mathcal{M}_{12}$, and α_1, $\alpha_2 \in \mathbb{C}$. Then

$$\begin{bmatrix} 0 & (\alpha_2 - \alpha_1)I + K_1 - K_2 \\ 0 & 0 \end{bmatrix} = \left[\begin{bmatrix} K_1 + \alpha_1 I & 0 \\ 0 & K_2 + \alpha_2 I \end{bmatrix}, \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \right] \in \mathcal{L}.$$

Therefore $(\alpha_1 - \alpha_2)I$ is compact, and so $\alpha_1 = \alpha_2$.

The converse is straightforward. □

For the following Theorem, recall that we let E_{ij} denote the standard matrix units of $\mathcal{T}_2$, $1 \leq i \leq j \leq 2$, and we let F_{ij} denote the standard matrix units of $\mathcal{T}_2 \otimes \mathcal{T}_2$. We identify $\mathcal{T}_2$ with a subspace of $\mathcal{T}_2 \otimes \mathcal{T}_2$ by identifying the matrix unit E_{11} with $(F_{11} + F_{22})$, E_{22} with $F_{33} + F_{44}$, and E_{12} with $F_{13} + F_{24}$.

5.11. **Theorem.** *Let $\mathcal{Q} = \mathcal{B}(\mathcal{H}) \otimes \mathcal{T}_2$, and $\mathcal{R} = \mathcal{B}(\mathcal{H}) \otimes (\mathcal{T}_2 \otimes \mathcal{T}_2) \simeq \mathcal{Q} \otimes \mathcal{T}_2$. Let $\mathcal{L} \subseteq \mathcal{R}$ be a Lie ideal. Then there exists an associative ideal $\mathcal{K}$ of $\mathcal{R}$ such that*

$$(\lozenge) \qquad [\mathcal{K}, \mathcal{R}] \subseteq \mathcal{L} \subseteq \mathcal{K} + \mathcal{D}_{\mathcal{K}},$$

where $\mathcal{D}_{\mathcal{K}}$ is a subspace of $\subseteq \{\sum_{i=1}^{4} \alpha_i I \otimes F_{ii} : \alpha_i \in \mathbb{C}, 1 \leq i \leq 4\}$, the diagonal of $\mathcal{R}$, and is determined by the condition:

$$(\blacklozenge) \qquad \alpha_i = \alpha_j \textit{ if } \mathcal{K}_{ij} \neq \mathcal{B}(\mathcal{H}) \textit{ for } 1 \leq i < j \leq 4, \ (i,j) \neq (2,3),$$

where $\mathcal{K}_{ij}$ are as in Propositon 5.4.

Conversely, if $\mathcal{K}$ is an associative ideal of $\mathcal{R}$ and $\mathcal{L}$ is a linear manifold in $\mathcal{R}$ satisfying the above condition $\lozenge$, then $\mathcal{L}$ is a Lie ideal in $\mathcal{R}$.

Proof. Let $W_i = I \otimes E_{ii}$, $i = 1, 2$. For all i, j, let $\mathcal{M}_{ij} = \{X \in \mathcal{Q} : X \otimes E_{ij} \in W_i \mathcal{L} W_j\}$. By Lemma 5.9, $\mathcal{M}_{11}$ and $\mathcal{M}_{22}$ are Lie ideals of $\mathcal{Q}$ and

$$\mathcal{L} \supseteq \begin{bmatrix} [\mathcal{Q}, \mathcal{M}_{11}] & \mathcal{M}_{12} \\ 0 & [\mathcal{Q}, \mathcal{M}_{22}] \end{bmatrix}.$$

Since $\mathcal{M}_{11}$ is a Lie ideal of $\mathcal{Q} = \mathcal{B}(\mathcal{H}) \otimes \mathcal{T}_2$, then by Lemma 5.9, we can find ideals $\mathcal{K}_{11}$, $\mathcal{K}_{12}$ and $\mathcal{K}_{22}$ of $\mathcal{B}(\mathcal{H})$ so that $\mathcal{J}_1 = \begin{bmatrix} \mathcal{K}_{11} & \mathcal{K}_{12} \\ 0 & \mathcal{K}_{22} \end{bmatrix}$ is an ideal of $\mathcal{Q}$ such that

$$[\mathcal{J}_1, \mathcal{Q}] \subseteq \mathcal{M}_{11} \subseteq \mathcal{J}_1 + \{\sum_{i=1}^{2} \alpha_i I \otimes F_{ii}\},$$

where $\alpha_1 = \alpha_2$ if $\mathcal{K}_{12} \neq \mathcal{B}(\mathcal{H})$.

Similarly, there exist ideals $\mathcal{K}_{33}$, $\mathcal{K}_{34}$ and $\mathcal{K}_{44}$ of $\mathcal{B}(\mathcal{H})$ so that the ideal $\mathcal{J}_2 = \begin{bmatrix} \mathcal{K}_{33} & \mathcal{K}_{34} \\ 0 & \mathcal{K}_{44} \end{bmatrix}$ in $\mathcal{Q}$ satisfies

$$[\mathcal{J}_2, \mathcal{Q}] \subseteq \mathcal{M}_{11} \subseteq \mathcal{J}_2 + \{\sum_{i=3}^{4} \alpha_i I \otimes F_{ii}\},$$

where $\alpha_3 = \alpha_4$ if $\mathcal{K}_{34} \neq \mathcal{B}(\mathcal{H})$.

Let $\mathcal{K} = \begin{bmatrix} \mathcal{J}_1 & \mathcal{M}_{12} \\ 0 & \mathcal{J}_2 \end{bmatrix}$. Then $\mathcal{K}$ is an ideal in $\mathcal{R}$ and

$$[\mathcal{Q}, \mathcal{M}_{11}] = \begin{bmatrix} [\mathcal{B}(\mathcal{H}), \mathcal{K}_{11}] & \mathcal{K}_{12} \\ 0 & [\mathcal{B}(\mathcal{H}), \mathcal{K}_{22}] \end{bmatrix} = [\mathcal{Q}, \mathcal{J}_1]$$
$$[\mathcal{Q}, \mathcal{M}_{22}] = \begin{bmatrix} [\mathcal{B}(\mathcal{H}), \mathcal{K}_{33}] & \mathcal{K}_{34} \\ 0 & [\mathcal{B}(\mathcal{H}), \mathcal{K}_{44}] \end{bmatrix} = [\mathcal{Q}, \mathcal{J}_2].$$

Thus $\mathcal{L} \supseteq \begin{bmatrix} [\mathcal{Q}, \mathcal{J}_1] & \mathcal{M}_{12} \\ 0 & [\mathcal{Q}, \mathcal{J}_2] \end{bmatrix} = [\mathcal{R}, \mathcal{K}]$.

It is clear that $\mathcal{L} \subseteq \mathcal{K} + \{\sum_{i=1}^{4} \alpha_i I \otimes F_{ii} : \alpha_i \in \mathbb{C}, 1 \leq i \leq 4\}$. A calculation similar to that in the previous Proposition shows that ($\blacklozenge$) holds.

Once again, the converse implication is a straightforward calculation. $\square$

6. Triangular UHF algebras.

6.1. Suppose that $\{p_n\}$ is an increasing sequence of positive integers such that for each $n \geq 1$, $p_n | p_{n+1}$. For each such n, consider a C^*-algebra $\mathbb{A}_n$ that is star-isomorphic to to $\mathbb{M}_{p_n}$ and a *-homomorphisms $\phi_n : \mathbb{A}_n \to \mathbb{A}_{n+1}$. The C^*-algebra inductive limit $\mathbb{A}$ of the system $\{(\mathbb{A}_n, \phi_n)\}$ is called a *uniformly hyperfinite* or *UHF* algebra. Alternatively, $\mathbb{A}$ is a UHF algebra if there exists an increasing sequence $\{\mathbb{A}_n\}$ of full matrix algebras whose union is dense in $\mathbb{A}$.

Let $\mathfrak{D}$ be a maximal abelian self-adjoint subalgebra (i.e. a masa) of a UHF algebra $\mathbb{A}$, and let $\mathcal{C}$ be any subset of $\mathbb{A}$. We define the *normalizer* of $\mathfrak{D}$ in $\mathcal{C}$ as the set

$$\mathcal{N}_{\mathfrak{D}}(\mathcal{C}) = \{w \in \mathcal{C} | w \text{ is a partial isometry}, w\mathfrak{D}w^* \subseteq \mathfrak{D},\ w^*\mathfrak{D}w \subseteq \mathfrak{D}\}.$$

$\mathfrak{D}$ is said to be a *canonical* masa if there exists an increasing sequence $\{\mathbb{A}_n\}$ of full matrix algebras whose union is dense in $\mathbb{A}$ such that $\mathfrak{D}_n := \mathfrak{D} \cap \mathbb{A}_n$ is a masa in $\mathbb{A}_n$ and $\mathcal{N}_{\mathfrak{D}_n}(\mathbb{A}_n) \subseteq \mathcal{N}_{\mathfrak{D}_{n+1}}(\mathbb{A}_{n+1})$ for all $n \geq 1$.

Recall that a *triangular UHF algebra* is a closed subalgebra $\mathcal{Q}$ of a UHF algebra $\mathbb{A}$ such that $\mathcal{Q} \cap \mathcal{Q}^*$ is a canonical masa in $\mathbb{A}$. Triangular UHF algebras will be called TUHF algebras for short.

6.2. **Definition.** A TUHF algebra $\mathcal{Q}$ is said to be *strongly maximal in factors* if it can be written as the Banach algebra direct limit of a system

$$\mathcal{Q}_1 \overset{\varphi_1}{\to} \mathcal{Q}_2 \overset{\varphi_2}{\to} \mathcal{Q}_3 \overset{\varphi_3}{\to} \mathcal{Q}_4 \cdots , \tag{1}$$

where $\mathcal{Q}_n$ is isometrically isomorphic to some full upper triangular matrix algebra $\mathcal{T}_{p_n}$ and $\varphi_n : \mathcal{Q}_n \to \mathcal{Q}_{n+1}$ is an embedding, i.e. the restriction of a C^*-isomorphism, so that the extension of φ_n carries $\mathcal{N}_{\mathfrak{D}_n}(\mathbb{A}_n)$ into $\mathcal{N}_{\mathfrak{D}_{n+1}}(\mathbb{A}_{n+1})$.

We refer to [21] and [19] for general facts about such algebras. In particular, it can be shown that $\mathcal{Q} + \mathcal{Q}^* = \mathbb{A}$.

As with UHF algebras, we may identify $\mathcal{Q}_n \simeq \mathcal{T}_{p_n}$ with its image in $\mathbb{A}$, and thus view $\mathcal{Q}$ as the closure in $\mathbb{A}$ of the union of the increasing sequence $\{\mathcal{T}_{p_n}\}$.

6.3. **Proposition.** *Suppose $X = [x_{ij}] \in \mathcal{T}_n$ is invertible and $x_{ii} = 1$ for all $1 \le i \le n$. Then X is a product of unipotent of order 2 in $\mathcal{T}_n$.*

Proof. We argue by induction on n. If $n = 2$, then X is already a unipotent of order 2, so there is nothing to prove. Suppose next that the result holds for $n \le k$. Let $X = [x_{ij}] \in \mathcal{T}_{k+1}$ with $x_{ii} = 1$ for all i. We can then write $X = \begin{bmatrix} 1 & Y \\ 0 & Z \end{bmatrix}$, where $Y \in \mathbb{M}_{1k}$ and $Z \in \mathcal{T}_k$ with $z_{ii} = 1$ for all i. Then

$$X = \begin{bmatrix} 1 & YZ^{-1} \\ 0 & I_k \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & Z \end{bmatrix}.$$

The first matrix on the right-hand side of the equation is clearly unipotent of order two, while the induction step guarantees that Z, and hence the second matrix on the right-hand side of the equation is a product of unipotents of order two. This completes the induction step and proves the Proposition. ☐

We note that the above condition is also necessary for X to be a product of unipotents.

6.4. **Theorem.** *Let $\mathcal{Q}$ be a TUHF algebra that is strongly maximal in factors and let $\mathcal{L}$ be a norm-closed subspace of $\mathcal{Q}$. The following are equivalent:*

(a) *$\mathcal{L}$ is a Lie ideal;*
(b) *$\mathcal{L}$ is invariant under conjugation by invertible elements of $\mathcal{Q}$;*

Proof. In view of Theorem 1.4, we need only prove that (a) implies (b) Suppose that $\mathcal{L}$ is a closed Lie ideal in $\mathcal{Q}$. By [13], we can find a closed, associative diagonal-disjoint ideal $\mathcal{K}$ and a C^*-subalgebra $\mathcal{D}_{\mathcal{K}}$ of the diagonal such that $\mathcal{K} \subseteq \mathcal{L} \subseteq \mathcal{K} + \mathcal{D}_{\mathcal{K}}$ Thus if $M \in \mathcal{L}$, we can find unique elements $K \in \mathcal{K}$ and $D \in \mathcal{D}_{\mathcal{K}}$ so that $M = D + K$. If $X \in \mathcal{Q}^{-1}$, then $X^{-1}MX = X^{-1}DX + X^{-1}KX$. Since $\mathcal{K}$ is an ideal, it is obvious that $X^{-1}KX \in \mathcal{K} \subseteq \mathcal{L}$, and so it suffices to show that $X^{-1}DX \in \mathcal{L}$ for all $X \in \mathcal{Q}^{-1}$.

In fact, since $\mathcal{L}$ is closed, it suffices to prove that $X^{-1}DX \in \mathcal{L}$ whenever $X \in \cup_{n=1}^{\infty} \mathcal{Q}_n^{-1}$, which is dense in $\mathcal{Q}^{-1}$. Suppose therefore that $X = [x_{ij}] \in \mathcal{Q}_n^{-1} \simeq \mathcal{T}_{k_n}^{-1}$ for some $n \ge 1$. Write $X = FY$, where $F = \mathrm{diag}\{x_{ii}\}_{i=1}^n$ and $Y = [y_{ij}] \in \mathcal{T}_{k_n}^{-1}$ with $y_{ii} = 1$ for all i. Then

$X^{-1}DX = Y^{-1}F^{-1}DFY = Y^{-1}DY$, as D and F lie in the diagonal of $\mathcal{Q}$ and this is an abelian algebra.

In general, if $T \in \mathcal{L}$ and $W = I + N$, with N nilpotent of order 2, then

$$\begin{aligned} W^{-1}TW &= (I - N)T(I + N) = T + (TN - NT) - NTN \\ &= T + [T, N] + \frac{1}{2}[[T, N], N]. \end{aligned}$$

Since $[T, N]$ and $[[T, N], N]$ are in $\mathcal{L}$ by virtue of the fact that $\mathcal{L}$ is a Lie ideal, it follows that $W^{-1}TW \in \mathcal{L}$.

By Proposition 6.3, we have that $Y = W_1 W_2 \cdots W_r$, a product of such unipotents of order 2. Let

$$D_j = W_j^{-1} \cdots W_1^{-1} D W_1 \cdots W_j = W_j^{-1} D_{j-1} W_j.$$

The argument of the preceding paragraph shows that $D_j \in \mathcal{L}$, $(1 \leq j \leq r)$. Since $D_r = Y^{-1}DY$, we are done. $\square$

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