Math 435 **Assignment #2** **Due on 6 October 2017**

- Solutions should be correct, complete, clear, and neatly presented to get full marks.
- You may discuss the homework questions with your colleagues but you must write the solutions in your own words and should understand the solutions and should be able to explain them.
- It is possible that only a proper subset of the following questions will be marked.

Notation: m denotes Lebesgue measure, m^* Lebesgue outer measure and μ an arbitrary measure.

(1) The lim inf and lim sup of a sequence of sets $\{A_n\}$ is defined (Bass, ex 2.9, p.16) by
$\liminf A_n = \cup_{k=1}^{\infty} \cap_{j=k}^{\infty} A_j$ and $\limsup A_n = \cap_{k=1}^{\infty} \cup_{j=k}^{\infty} A_j$.
Assume that (X, Σ, μ) is a measure space and $A_n \subseteq X$.
   (a) If A_n are all measurable, prove that $\limsup A_n$ and $\liminf A_n$ are measurable and that $\mu(\liminf A_n) \leq \liminf \mu(A_n)$.
   (b) Give an example where $\mu(\liminf A_n) < \liminf \mu(A_n)$.

(2) Let E be a (Lebesgue) nonmeasurable set. Show that there exists a $\mathbf{G}_\delta$ set G such that $E \subset G$, $m^*(E) = m^*(G)$ and $m^*(G \cap E^c) \neq 0$.

(3) Let E be a (Lebesgue) measurable set with $m(E) > 0$. Prove that there exists an interval I such that $m(E \cap I) > .99(m(I))$.

(4) Let f be a Lipschitz continuous function (Bass p. 128) Show that f maps every set of (Lebesgue) measure zero onto a set of measure zero.
Hint: *First consider the image of intervals. The image of an interval is an interval but the endpoints may not be mapped to the endpoints as exemplified by* sin *on* $[0, 2\pi]$.

(5) Let (X, Σ, μ) be a measure space and f a real-valued function defined on X. If the set $\{x \in X \ : \ f(x) < r\}$ is measurable for every *rational* number r, prove that f is measurable.

(6) Let φ be the Cantor function and $f(x) = \varphi(x) + x$.
   (a) Prove that f maps the Cantor set C onto a set of measure 1.
   Hint: *Consider the image of the complement of the Cantor set in* $[0, 1]$.
   (b) Prove that there is a measurable set A (a subset of C) such that $f(A)$ is not measurable.
   (c) (Optional): Prove that A is not a Borel set.