Additive rank one preserving mappings on triangular matrix algebras

Jeremy Bell[1] and **A. R. Sourour**[2]

Department of Mathematics and Statistics, University of Victoria,
Victoria, British Columbia, Canada V8W 3P4

Corresponding author: sourour@math.uvic.ca
A. R. Sourour
Department of Mathematics and Statistics, University of Victoria,
Victoria, British Columbia, Canada V8W 3P4

Short running title: Rank One Preserving Maps

Abstract

We classify surjective additive maps on the space of block upper triangular matrices that preserve matrices of rank one as well as linear maps preserving matrices of rank one on fairly general subspaces of matrices.

AMS Subject Classification: Primary 15A04, Secondary 15A03.

Key words: Triangular matrix algebras, Rank-one preservers.

1. Research supported by an NSERC undergraduate student research award.

2. Research supported by an NSERC research grant.

Email: jbell@math.uvic.ca (J. Bell), sourour@math.uvic.ca (A. R. Sourour)

1 Introduction

Characterizing linear maps on spaces of matrices or operators that preserve certain subsets or properties have been an active area of research for quite a while. Of these so called linear preserver problems, one of the most basic is arguably the rank-one preservers. Indeed several other questions about preservers may be reduced to, or solved with the help of, rank-one preservers. This has been observed in [10] and [14]. For example, preserving commutativity ([4], [15], [17]), spectrum ([7], [13]) or invertibility [16] involve rank-one preservers. Classifying isomorphisms of several types of operator algebras is frequently accomplished by exploiting the fact that they preserve rank one operators; see, e.g. [5; Chapter 17].

The linear rank one preservers on the space of all $n \times n$ matrices was characterized by Marcus and Moyls [11]. They show that every such map is a composition of a left multiplication L_A by an invertible matrix A, a right multiplication R_B by an invertible matrix B, and possibly the transpose map. Related results may be found in [2], [3], [6], [8], [9] and [12]. More recently, Omladic and Semrl [14] characterized surjective additive maps on the space of finite rank operators on real or complex Banach spaces. In case of finite dimensional spaces, they show that every such a map is a composition of the three types of maps described above and a fourth type induced by an automorphism of the underlying field.

The main purpose of this paper is to characterize additive rank one preserving maps on algebras of block upper triangular matrices (§§5, 7). In addition to the four types of maps described above, we identify a fifth type which we may occur. Furthermore, we also identify linear rank one preserving maps on fairly general subspaces of matrices (§3).

Let us now fix some notation and terminology. By $\mathbf{M}_{mn}(\mathbb{F})$, we denote the space of all $m \times n$ matrices over an arbitrary field $\mathbb{F}$, and as usual $\mathbf{M}_n = \mathbf{M}_{nn}$. Given two vectors $u \in \mathbb{F}^m$ and $v \in \mathbb{F}^n$, we shall denote by $u \otimes v$ the $m \times n$ matrix uv^t, which we may associate with the operator $z \mapsto (v^t z)u$ from $\mathbb{F}^n$ onto $\mathbb{F}^m$. It is obvious that a matrix A has rank one if and only if $A = u \otimes v$ for nonzero u and v. The standard basis for $\mathbb{F}^n$ is denoted by $\{e_k\}_{k=1}^n$, i.e. $e_1 = (1, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0)$, etc. the matrix units $e_i \otimes e_j$ are denoted by E_{ij}.

A map φ from a space $\mathcal{S}_1$ of matrices into a space $\mathcal{S}_2$ of matrices is said to *preserve matrices of rank one* if $\varphi(T)$ is of rank one whenever T has rank one. It is said *preserve rank one matrices in both directions* when $\varphi(T)$ is of rank one if and only if T has rank one. A *left multiplication* L_A is the map $T \mapsto AT$ for a fixed matrix A. *Right multiplications* R_B are defined analogously.

We make use of a particular permutation matrix J given by

$$J = \begin{bmatrix} 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 1 & 0 \\ \vdots & & & & \\ 0 & 1 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \end{bmatrix} \tag{1}$$

i.e $J = [\delta_{i,n+1-i}]$, where δ is the Kronecker delta symbol. If T^t denotes the transpose of T, then it is straightforward to verify that

$$T^+ := JT^tJ \tag{2}$$

maps the algebra of upper triangular matrices onto itself and preserves rank-one matrices. We observe that $T \mapsto T^+$ may also be described as the transpose with respect to the anti-diagonal, i.e., the "diagonal" that contains the positions $(i, 1+n-i)$.

We will also find it useful to define a *space of rank one matrices* to be a subspace S of matrices with the property that all non-zero matrices in S have rank one.

2. Preliminaries

We prove a lemma which will be used frequently in the following sections.

Lemma 2.1. *Assume that x_1, x_2, u are nonzero vectors in $\mathbb{F}^n$ and that y_1, y_2, v are nonzero vectors in $\mathbb{F}^m$ and that each of the three linear transformations:*

$$C_1 = u \otimes v + x_1 \otimes y_1, \quad C_2 = u \otimes v + x_2 \otimes y_2, \quad C_3 = u \otimes v + x_1 \otimes y_1 + x_2 \otimes y_2 + x_1 \otimes y_2$$

is of rank one, then:

(i) If x_1, x_2 are linearly independent and y_1, y_2 are linearly independent then $u \otimes v = x_2 \otimes y_1$.

(ii) If x_1, x_2 are linearly independent but y_1, y_2 are linearly dependent then v is a scalar multiple of y_1.

(iii) If x_1, x_2 are linearly dependent but y_1, y_2 are linearly independent then u is a scalar multiple of x_1.

(iv) If both pairs $\{x_1, x_2\}$ and $\{y_1, y_2\}$ are linearly dependent then u is a scalar multiple of x_1 or v is a scalar multiple of y_1.

Proof. (i) If u, x_1 are linearly independent, then $v = c_1 y_1$ for some $c_1 \in \mathbb{F}$, since rank $C_1 = 1$. Now v, y_2 are linearly independent and so $u = c_2 x_2$ for some $c_2 \in \mathbb{F}$, since rank $C_2 = 1$. Now $C_3 = x_1 \otimes (y_1 + y_2) + x_2 \otimes (c_1 c_2 y_1 + y_2)$ has rank one, implying that $y_1 + y_2$ and $c_1 c_2 y_1 + y_2$ are linearly dependent, hence $c_1 c_2 = 1$ and $u \otimes v = x_2 \otimes y_1$. On the other hand if $u = c_1 x_1$, then $v = c_2 y_2$ since rank $C_2 = 1$, and hence $C_3 = (c_1 c_2 + 1) x_1 \otimes y_2 + x_1 \otimes y_1 + x_2 \otimes y_2$ which is never of rank one.

Parts (ii), (iii), (iv) are quite easy to verify. ∎

3. Linear maps

In this section, we characterize rank one preserving linear maps on fairly general subspaces of matrices.

Theorem 3.1. *Let $\mathfrak{L}$ be a subspace of $\mathbf{M}_{mn}(\mathbb{F})$ satisfying the following conditions:*

(a) *$\mathfrak{L}$ contains $x_0 \otimes \mathbb{F}^n$ for some $x_0 \in \mathbb{F}^m$,*

(b) *$\mathfrak{L}$ contains $\mathbb{F}^m \otimes y_0$ for some $y_0 \in \mathbb{F}^n$,*

(c) *$\mathfrak{L}$ is spanned by its rank one matrices.*

Let $\varphi : \mathfrak{L} \longrightarrow \mathbf{M}_{kl}(\mathbb{F})$ be a linear mapping preserving rank one matrices. Then either

(i) *$m \leq k$, $n \leq l$, and there exists a $k \times m$ matrix A of rank m and an $n \times l$ matrix B rank n such that:*

$$\varphi(T) = ATB \text{ for every } T \in \mathfrak{L};$$

or

(ii) $m \leq l$, $n \leq k$, and there exists a $k \times n$ matrix A of rank n and an $m \times l$ matrix B rank m such that:

$$\varphi(T) = AT^tB \text{ for every } T \in \mathfrak{L};$$

or

(iii) $\varphi(\mathfrak{L})$ is a space of rank one matrices.

Before proving Theorem 3.1, we state two immediate corollaries.

Corollary 3.2. *With the same notation as above, if φ preserves rank one, then either φ preserves every rank, i.e., rank $\varphi(T)$ = rank T for every $T \in \mathfrak{L}$, or $\varphi(\mathfrak{L})$ is a space of rank one matrices.*

Corollary 3.3. *With the same notation as above, if φ preserves rank one in both directions, then:*

(a) φ is of the form (i) or (ii) of Theorem 3.1;

(b) φ is injective;

(c) φ preserves every rank.

Proof of Theorem 3.1. The image under φ of $(x_0 \otimes \mathbb{F}^n)$ is a vector space of rank one matrices. Then $\varphi(x_0 \otimes \mathbb{F}^n) = V \otimes v_0$ or $u_0 \otimes W$ for some subspace V of $\mathbb{F}^k$ and a vector

$v_0 \in \mathbb{F}^l$ or a subspace W of $\mathbb{F}^l$ and a vector $u_0 \in \mathbb{F}^k$. Replacing φ by the map $\psi(T) = \varphi(T^t)$ if necessary, we may assume with no loss of generality that $\varphi(x_0 \otimes \mathbb{F}^n) = u_0 \otimes W$. Since the kernel of φ contains no matrices of rank one, then $\dim W = n$. Consequently $l \geq n$ and $\varphi(x_0 \otimes y) = u_0 \otimes g(y)$ for some injective linear transformation $g : \mathbb{F}^n \longrightarrow \mathbb{F}^l$, i.e. $\varphi(x_0 \otimes y) = u_0 \otimes B^t y$ for an $n \times l$ matrix B of rank n.

Similarly, $\varphi(\mathbb{F}^m \otimes y_0)$ is a space of rank one matrices and hence takes one of the two forms mentioned above. We consider two cases.

Case 1. $\varphi(\mathbb{F}^m \otimes y_0) \subseteq u_1 \otimes \mathbb{F}^l$. Therefore $\varphi(x \otimes y_0) = u_1 \otimes h(x)$ for an injective linear transformation h. Since $\varphi(x_0 \otimes y_0) = u_0 \otimes w = u_1 \otimes w'$ for nonzero vectors w and w', then u_0 and u_1 are linearly dependent. But u_1 is only determined up to a multiplicative scalar, hence we may assume that $u_0 = u_1$.

We show that the image under φ of every rank one matrix in $\mathfrak{L}$ is contained in $u_0 \otimes \mathbb{F}^l$, and consequently $\varphi(\mathfrak{L}) \subseteq u_0 \otimes \mathbb{F}^l$. Assume, to the contrary, that there exist nonzero vectors x, y, u, v such that $\varphi(x \otimes y) = u \otimes v$ and $\{u_0,\ u\}$ are linearly independent. Let $K_1 = x \otimes y$, and

$$K_2 = (x + x_0) \otimes y, \quad K_3 = x \otimes (y + y_0) \text{ and } K_4 = (x + x_0) \otimes (y + y_0)$$

and let $C_j = \varphi(K_j); 1 \leq j \leq 4$. Thus C_1, C_2, C_3, C_4 are all of rank one, $C_1 = u \otimes v$,

$$C_2 = u \otimes v + u_0 \otimes g(y), \;\; C_3 = u \otimes v + u_0 \otimes h(x) \text{ and } C_4 = u \otimes v + u_0 \otimes (h(x) + g(y) + g(y_0)).$$

Since u and u_0 are linearly independent, we conclude that $v = \alpha g(y) = \beta h(x) = \gamma g(y_0)$ for nonzero scalars α, β and γ. It follows that $\varphi((\beta x - \gamma x_0) \otimes y_0) = 0$ contradicting the rank one preserving property. This establishes that $\varphi(\mathfrak{L}) \subseteq u_0 \otimes \mathbb{F}^l$, a space of rank one matrices.

Case 2. $\varphi(\mathbb{F}^m \otimes y_0) \subseteq \mathbb{F}^k \otimes v_0$. As before, we have that $\varphi(x \otimes y_0) = Ax \otimes v_0$, for a $k \times m$ matrix of rank m, i.e., an injective linear transformation from $\mathbb{F}^m$ into $\mathbb{F}^k$. Furthermore, $u_0 \otimes B^t y_0 = \varphi(x_0 \otimes y_0) = Ax_0 \otimes v_0$. After absorbing a constant in u_0 and v_0 if necessary, we may assume that $Ax_0 = u_0$ and $B^t y_0 = v_0$.

Now consider an arbitrary rank one matrix $K_1 = x \otimes y \in \mathfrak{L}$. Let

$$K_2 = (x + x_0) \otimes y, \quad K_3 = x \otimes (y + y_0) \text{ and } K_4 = (x + x_0) \otimes (y + y_0)$$

and let $C_j = \varphi(K_j); 1 \leq j \leq 4$. Thus C_1, C_2, C_3, C_4 are all of rank one. If $C_1 = u \otimes v$, then

$$C_2 = u \otimes v + u_0 \otimes B^t y, \;\; C_3 = u \otimes v + Ax \otimes v_0 \text{ and } C_4 = u \otimes v + u_0 \otimes B^t y + Ax \otimes v_0 + u_0 \otimes v_0.$$

If u_0, Ax are linearly independent and $v_0, B^t y$ are linearly independent, then by Lemma 2.1, we conclude that

$$\varphi(x \otimes y) = Ax \otimes B^t y.$$

On the other hand, if $Ax = cu_0$ for a scalar c, then $\varphi((x - cx_0) \otimes y_0) = Ax \otimes v_0 - cu_0 \otimes v_0 = 0$. Thus $(x - cx_0) \otimes y_0$ is not of rank one, and so $x = cx_0$. In this case, we also get $\varphi(x \otimes y) = cu_0 \otimes B^t y = Ax \otimes B^t y$. A similar argument proves the same conclusion when $B^t y$ and v_0 are linearly dependent. Therefore $\varphi(x \otimes y) = A(x \otimes y)B$, for every rank one matrix $x \otimes y \in \mathfrak{L}$. ¿From condition (c) we conclude that $\varphi(T) = ATB$ for every $T \in \mathfrak{L}$. ▮

We will now give a couple of examples to illustrate that conditions (a) and (b) in Theorem 3.1 cannot be removed. It is quite easy to construct examples violating condition (c).

Example 3.4. Let $\mathfrak{L}$ be the subspace of $\mathbf{M}_3$ spanned by $E_{11}, E_{12}, E_{22}, E_{23}, E_{33}$ and define $\varphi : \mathfrak{L} \longrightarrow \mathbf{M}_3$ by

$$\varphi\left(\begin{bmatrix} a_{11} & a_{12} & 0 \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}\right) = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{12} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

Example 3.5. Let $\mathfrak{L} = \mathbf{M}_2 \oplus \mathbf{M}_2$, identified as usual with a subspace of $\mathbf{M}_4$ and define $\varphi : \mathfrak{L} \longrightarrow \mathbf{M}_4$ by

$$\varphi\left(\begin{bmatrix} a_1 & b_1 & 0 & 0 \\ c_1 & d_1 & 0 & 0 \\ 0 & 0 & a_2 & b_2 \\ 0 & 0 & c_2 & d_2 \end{bmatrix}\right) = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 & c_2 & d_2 \\ c_1 & d_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

In either example, it is clear that φ preserves rank one matrices, and that the image of φ is not a space of rank one matrices. It is easy to verify that there do not exist matrices A and B, invertible or not, such that $\varphi(T) = ATB$ or $\varphi(T) = AT^t B$.

4. Triangular algebras

For every finite sequence of positive integers $n_1, n_2, \ldots n_k$, satisfying $n_1 + n_2 + \ldots + n_k = n$, we associate an algebra $\mathcal{T}(n_1, n_2, \ldots n_k)$ consisting of all $n \times n$ matrices of the form

$$A = \begin{bmatrix} A_{11} & A_{12} & \ldots & A_{1k} \\ 0 & A_{22} & \ldots & A_{2k} \\ \vdots & & & \\ 0 & 0 & \ldots & A_{kk} \end{bmatrix} \tag{3}$$

where A_{ij} is an $n_i \times n_j$ matrix. We call such an algebra a *block upper triangular algebra.*

Associated with such an algebra $\mathfrak{A}$ is its chain of invariant subspaces

$$\{0\} = V_0 \subset V_1 \subset \ldots \subset V_k = \mathbb{F}^n,$$

i.e.,

$$V_j = \text{span}\{e_i :\ 1 \le i \le n_1 + n_2 + \ldots + n_j\}.$$

It is evident that the triangular algebra $\mathfrak{A}$ is the set of all operators leaving every V_j invariant. We also make use of the subspaces

$$W_j = \text{span}\{e_i : n_1 + n_2 + \ldots + n_{j-1} \le i \le n_1 + n_2 + \ldots + n_j\}.$$

These are the subspaces corresponding to the diagonal blocks. We have $V_j = W_1 \oplus \ldots \oplus W_j$, in particular $\mathbb{F}^n = W_1 \oplus \ldots \oplus W_k$

A special case of block upper triangular algebras is the algebra $\mathcal{T}_n(\mathbb{F})$ of upper triangular matrices. In this case $W_j = \text{span}(e_j)$ and $V_j = \text{span}\{e_1 \cdots e_j\}$.

Our first lemma is a standard fact and quite easy to prove. First, we fix some notation. For a subspace V of $\mathbb{F}^n$, we denote, as usual, its orthogonal complement with respect to the standard dot product by $V^\perp$. Thus, we have $V_j^\perp = W_{j+1} \oplus \ldots \oplus W_k$.

Lemma 4.1. *Let $\mathfrak{A}$ be a block upper triangular algebra, and let $\{V_j : 1 \le j \le k\}$ be its chain of invariant subspaces. The rank one matrix $x \otimes y \in \mathfrak{A}$ if and only if $x \in V_j$ and $y \in V_{j-1}^\perp$ for an index j; $(1 \le j \le k)$.*

Proof. Omitted.

Lemma 4.2. *Let A, B be invertible matrices in $M_n(\mathbb{F})$, and $\mathfrak{A}$ be a block upper triangular algebra in $M_n(\mathbb{F})$. If the mapping $T \mapsto ATB$ maps $\mathfrak{A}$ into $\mathfrak{A}$, then $A, B \in \mathfrak{A}$*

Proof. Take any $x \in V_j$, then $x \otimes z \in \mathfrak{A}$ for every $z \in V_{j-1}^\perp$, and so $A(x \otimes z)B = Ax \otimes B^t z \in \mathfrak{A}$. As B is invertible, the space $\{B^t z \mid\ z \in V_{j-1}^\perp\}$ has the same dimension as $V_{j-1}^\perp$, and so $Ax \in V_j$. This shows that A leaves every V_j invariant, and thus, $A \in \mathfrak{A}$. By a similar argument, we also have $B \in \mathfrak{A}$. ∎

Lemma 4.3. *Let A, B be invertible matrices in $M_n(\mathbb{F})$, and $\mathfrak{A}$ and $\mathfrak{B}$ be block upper triangular algebras in $M_n(\mathbb{F})$. If the mapping φ defined by $\varphi(T) = ATB$ maps $\mathfrak{A}$ onto $\mathfrak{B}$, then $\mathfrak{A} = \mathfrak{B}$, and $A, B \in \mathfrak{A}$*

Proof. First we observe that φ^{-1} exists and that $\varphi^{-1}(T) = A^{-1}TB^{-1}$. Denote the chain of invariant subspaces of $\mathfrak{A}$ (respectively, $\mathfrak{B}$) by $\{V_j : 1 \le j \le k\}$ (respectively, $\{V'_j : 1 \le j \le k'\}$) .

Now, take $x \in V_1$ we know that $x \otimes \mathbb{F}^n \in \mathfrak{A}$. As B is invertible, we have $Ax \otimes \mathbb{F}^n = A(x \otimes \mathbb{F}^n)B$ and so $Ax \otimes \mathbb{F}^n \in \mathfrak{B}$. Therefore $Ax \in V'_1$ and hence $AV_1 \subseteq V'_1$. The same argument applied to φ^{-1} gives us $A^{-1}V'_1 \subseteq V_1$. Therefore dim $V_1 =$ dim V'_1.

Now take $x \in V_2$. Then $x \otimes V_1^\perp \in \mathfrak{A}$ and so $Ax \otimes B^t V_1^\perp \in \mathfrak{B}$. As dim $(B^t V_1^\perp) =$ dim $V_1^\perp =$ dim $V_1'^\perp$, we conclude that $Ax \in V'_2$. Thus $AV_2 \subseteq V'_2$ and using φ^{-1} we get that $A^{-1}V'_2 \subseteq V_2$ and hence that dim $V_2 =$ dim V'_2.

Continuing in this fashion, one sees that the dimension of every block in $\mathfrak{A}$ agrees with the dimension of the corresponding block in $\mathfrak{B}$. Thus $\mathfrak{A} = \mathfrak{B}$. Now it follows from lemma 4.2 that $A, B \in \mathfrak{A}$. ▮

We are now in position to apply the results of §3 to triangular algebras.

Theorem 4.4. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be block upper triangular algebras in $\mathbf{M}_n(\mathbb{F})$ and $\mathbf{M}_m(\mathbb{F})$ respectively, and let $\varphi : \mathfrak{A} \longrightarrow \mathfrak{B}$ be a surjective linear mapping preserving rank one matrices. Then $m = n$, $\mathfrak{B} = \mathfrak{A}$ or $\mathfrak{A}^+$ and*

$$\varphi(T) = ATB \quad \text{or} \quad \varphi(T) = AT^+B$$

where A and B are invertible matrices in $\mathfrak{B}$ and $T \mapsto T^+$ is the transpose relative to the anti-diagonal as in equation (2). Consequently φ is bijective and preserves every rank.

Proof. By Theorem 3.1, we have two cases.

Case 1: $m \geq n$ and $\varphi(T) = ATB$, where A and B are matrices of size $m \times n$ and $n \times m$ respectively, and each has rank n. As the map φ is onto, there exists $T_0 \in \mathfrak{A}$ such that $AT_0B = I_m$, the $m \times m$ identity matrix. But this is possible only when $n \geq m$. So we have $n = m$, and the result now follows from Lemma 4.2.

Case 2: $m \geq n$ and there exist matrices C and D of rank n such that $\varphi(T) = CT^tD$. As in case 1, we get that $n = m$. We may now write $\varphi(T) = CJT^+JD$, where J is the matrix in equation (1). Let $A = CJ$, $B = JD$, and $\psi(T) = CJTJD$, a map from $\mathfrak{A}^+$ to $\mathfrak{B}$. Now it follows from lemma 4.2 that $\mathfrak{A}^+ = \mathfrak{B}$ and that $A,\ B \in \mathfrak{B}$. ▮

Remarks. 1. The second form of φ may be written as $\varphi(T) = (ATB)^+$, where A and B are now in $\mathfrak{A}$, rather than $\mathfrak{B}$.

2. Both forms of φ, may be present when $\mathfrak{A} = \mathfrak{A}^+$, i.e., when the sizes $n_1, \ldots n_k$ of the diagonal blocks satisfy $n_j = n_{k-j+1}$.

3. Even for such highly structured spaces of matrices as triangular algebras it is possible to have a rank one preserving map whose range is a space of rank one matrices as the following example illustrates.

Example 4.5. Define $\varphi : \mathcal{T}_3(\mathbb{F}) \longrightarrow \mathcal{T}_3(\mathbb{F})$ by

$$\varphi\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}\right) = \begin{bmatrix} a_{11} + a_{22} + a_{33} & a_{12} + a_{23} & a_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Then φ preserves rank one.

5. Surjective additive maps

In this section, we characterize surjective additive, rather than linear, mappings on triangular algebras that preserve rank one. As in [OS], the proofs are quite a bit more delicate than the corresponding proof for the linear case. As to the form of such maps, in addition to left multiplications, right multiplications and "transposing", other "elementary" rank one preservers appear, which we will presently describe.

5.1. Assume that $c \mapsto \tilde{c}$ is an automorphism of $\mathbb{F}$, and $C = [c_{ij}] \in \mathbf{M}_{mn}(\mathbb{F})$. We denote the matrix $[\tilde{c}_{ij}]$ by $\tilde{C}$. The map $C \mapsto \tilde{C}$ preserves every rank.

5.2. Let each of $f_1, f_2, \dots f_n$ be an additive mapping from $\mathbb{F}$ to $\mathbb{F}$ such that f_1 is bijective, and let $\mathbf{f} = (f_1, f_2, \dots f_n)$. Define a mapping $\hat{\mathbf{f}}$ on a triangular algebra $\mathfrak{A} = \mathcal{T}(n_1 \dots n_k)$, with $n_1 = 1$, by

$$\hat{\mathbf{f}}\left(\begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ 0 & c_{22} & \dots & c_{2n} \\ \vdots & & & \\ 0 & 0 & \dots & c_{nn} \end{bmatrix}\right) = \begin{bmatrix} f_1(c_{11}) & f_2(c_{11}) + c_{12} & \dots & f_n(c_{11}) + c_{1n} \\ 0 & c_{22} & \dots & c_{2n} \\ \vdots & & & \\ 0 & 0 & \dots & c_{nn} \end{bmatrix}$$

i.e.,

$$\hat{\mathbf{f}}(c_{11}E_{11}) = \sum_{j=1}^{n} f_j(c_{11})E_{1j}; \quad \hat{\mathbf{f}}(cE_{ij}) = cE_{ij} \text{ if } (i,j) \neq (1,1),$$

This is a surjective additive mapping on $\mathfrak{A}$ and it preserves rank one matrices, but only when $n_1 = 1$.

5.3. For $\mathbf{f}$ and $f_1, f_2, \dots f_n$ as above, define a mapping $\check{\mathbf{f}}$ on a triangular algebra $\mathfrak{A} = \mathcal{T}(n_1 \dots n_k)$, with $n_k = 1$, by

$$\check{\mathbf{f}}\left(\begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ 0 & c_{22} & \dots & c_{2n} \\ \vdots & & & \\ 0 & 0 & \dots & c_{n-1n} \\ 0 & 0 & \dots & c_{nn} \end{bmatrix}\right) = \begin{bmatrix} c_{11} & c_{12} & \dots & f_n(c_{nn}) + c_{1n} \\ \vdots & & & \\ 0 & 0 & \dots & f_2(c_{nn}) + c_{n-1n} \\ 0 & 0 & \dots & f_1(c_{nn}) \end{bmatrix}$$

i.e., $\check{\mathbf{f}}(C) = (\hat{\mathbf{f}}(C^+))^+$. Again this is an additive mapping on $\mathfrak{A}$ preserving rank one matrices, but only when $n_k = 1$.

Next we recall a classical definition.

Definition 5.4. A mapping φ from a vector space V to a vector space W, is said to be *semilinear* if it is additive and if there exists an automorphism $f : \mathbb{F} \longrightarrow \mathbb{F}$ such that $\varphi(\lambda v) = f(\lambda)\varphi(v)$ for every $\lambda \in \mathbb{F}$ and $v \in V$.

Now, we state the main theorem in this section. The case $n = 2$ and $\mathfrak{A} = \mathcal{T}_2 = \mathcal{T}(1,1)$, the upper triangular 2×2 matrices is exceptional as will be seen in Example 6.2. We should point out however that the theorem is valid for $\mathfrak{A} = \mathbf{M}_2 = \mathcal{T}(2)$.

Theorem 5.5. *Let $\mathfrak{A} = \mathcal{T}(n_1 \dots n_k)$ be a block upper triangular algebra in $\mathbf{M}_n(\mathbb{F})$, such that $\mathfrak{A} \neq \mathcal{T}_2(\mathbb{F})$. Let $\varphi : \mathfrak{A} \longrightarrow \mathfrak{A}$ be a surjective additive mapping that preserves rank one matrices. Then φ is a composition of some or all of the following maps:*

(i) Left multiplication by an invertible matrix in $\mathfrak{A}$.

(ii) Right multiplication by an invertible matrix in $\mathfrak{A}$.

(iii) The map $C \mapsto \tilde{C}$, induced by a field automorphism $a \mapsto \tilde{a}$ of $\mathbb{F}$.

(iv) The map $\hat{\mathbf{f}}$ defined in 5.2 above, but only when $n_1 = 1$.

(v) The map $\check{\mathbf{f}}$ defined in 5.3 above, but only when $n_k = 1$.

(vi) The transpose with respect to the antidiagonal $T \mapsto T^+$. This is present only when $\mathfrak{A} = \mathfrak{A}^+$, i.e., $n_j = n_{k-j+1}$ for every j.

Thus the restriction of φ to the space

$$\mathfrak{M} := \{[c_{ij}] \in \mathfrak{A} \ : \ c_{11} = 0 \text{ if } n_1 = 1, \quad \text{and } c_{nn} = 0 \text{ if } n_k = 1\}$$

is semilinear. In particular, if $n_1 \geq 2$ and $n_k \geq 2$, then φ is semilinear.

Before proving Theorem 5.5, we state a couple of corollaries.

Corollary 5.6. *If φ is as in Theorem 5.5, then:*

(a) φ is injective;

(b) φ preserves every rank, i.e., rank $\varphi(T)$ = rank T, for every $T \in \mathfrak{A}$.

Corollary 5.7. *Let $\mathfrak{A}$ be a block upper triangular algebra in $\mathbf{M}_n(\mathbb{R})$, over the field of real numbers, and assume that each of the first and last diagonal block has size at least 2×2. Then every additive surjective rank one preserving mapping $\varphi : \mathfrak{A} \longrightarrow \mathfrak{A}$ is linear.*

Proof of Corollary 5.7. It is well-known that the identity is the only automorphism of the field of real numbers; see, e.g., [1; p. 58] ∎

The proof of Theorem 5.5 will be accomplished via several lemmas. We find it convenient to deal with mappings between slightly different triangular algebras $\mathfrak{A}$ and $\mathfrak{B}$ having the same dimension, and show that $\mathfrak{B}$ must $= \mathfrak{A}$ or $\mathfrak{A}^+$, in addition to the conclusions of Theorem 5.5.

Lemma 5.8. *Let $\mathfrak{A} = \mathcal{T}(n_1 \dots n_k)$ and $\mathfrak{B} = \mathcal{T}(m_1 \dots m_\ell)$ be block upper triangular algebras in $\mathbf{M}_n(\mathbb{F})$ such that* $\dim \mathfrak{A} = \dim \mathfrak{B} \geq 3$. *If $\varphi : \mathfrak{A} \longrightarrow \mathfrak{B}$ is a surjective additive mapping that preserves rank one matrices, then there exist nonzero vectors $u_0,\ v_0 \in \mathbb{F}^n$, and injective additive mappings $g, h : \mathbb{F}^n \longrightarrow \mathbb{F}^n$ such that either:*

$$\varphi(e_1 \otimes y) = u_0 \otimes g(y) \text{ and } \varphi(x \otimes e_n) = h(x) \otimes v_0, \text{ for } x, y \in \mathbb{F}^n. \tag{5.8.1}$$

or

$$\varphi(e_1 \otimes y) = g(y) \otimes u_0 \text{ and } \varphi(x \otimes e_n) = v_0 \otimes h(x), \text{ for } x, y \in \mathbb{F}^n. \tag{5.8.2}$$

Proof. The image under φ of $(x_0 \otimes \mathbb{F}^n)$ is an additive group of rank one matrices. It follows that $\varphi(e_1 \otimes \mathbb{F}^n) = u_0 \otimes G$ or $G \otimes u_0$ for some $u_0 \in \mathbb{F}^n$ and an additive subgroup G of $\mathbb{F}^n$. It follows easily that there exists an injective additive mapping $g : \mathbb{F}^n \longrightarrow \mathbb{F}^n$, such that

(a) $\varphi(e_1 \otimes y) = u_0 \otimes g(y)$; or (b) $\varphi(e_1 \otimes y) = g(y) \otimes u_0$.

Similarly

(a)$'$ $\varphi(x \otimes e_n) = h(x) \otimes v_0$; or (b)$'$ $\varphi(x \otimes e_n) = v_0 \otimes h(x)$.

for some $v_0 \in \mathbb{F}^n$ and an injective additive mapping $h : \mathbb{F}^n \longrightarrow \mathbb{F}^n$.

To prove the lemma, we must show that it is not possible to have equations (a) and (b)$'$ satisfied simultaneously. The impossibility of (a)$'$ together with (b) may be proved similarly. Towards this end, assume that $\varphi(e_1 \otimes y) = u_0 \otimes g(y)$ and $\varphi(x \otimes e_n) = v_0 \otimes h(x)$. Thus $u_0 \otimes g(e_n) = \varphi(e_1 \otimes e_n) = v_0 \otimes h(e_1)$. It follows that u_0 and v_0 are linearly dependent, and since each is determined up to a multiplicative scalar, we may assume with no loss of generality that $u_0 = v_0$. We now proceed exactly as in the proof of Theorem 3.1 (case 1), to conclude that $\varphi(\mathfrak{A}) \subseteq u_0 \otimes \mathbb{F}^n$, contradicting surjectivity. ∎

5.9. If φ satisfies equations (5.8.2), we may replace φ by the map $\varphi^+ : \mathfrak{A} \longrightarrow \mathfrak{B}^+$ defined by $\varphi^+(T) = (\varphi(T))^+$. In view of this, we shall henceforth assume that φ satisfies equations (5.8.1) in addition to the hypotheses of lemma 5.8.

For the following lemma, we recall that $\{V_0, V_1 \dots, V_k\}$ is the chain of invariant subspaces of $\mathfrak{A}$.

Lemma 5.10. *Let $\mathfrak{L}_1 := e_1 \otimes \mathbb{F}^n \cup \mathbb{F}^n \otimes e_n$ and $\mathfrak{L}_2 := u_0 \otimes \mathbb{F}^n \cup \mathbb{F}^n \otimes v_0$. Let $\xi \otimes \eta$ be a rank one matrix in $\mathfrak{A}$, with $\xi \in V_j$, $\eta \in V_{j-1}^\perp$ such that $\varphi(\xi \otimes \eta) \notin \mathfrak{L}_2$, Then:*

(i) $\varphi(\xi \otimes \eta) = h(\xi) \otimes g(\eta)$.

(ii) $\varphi(x \otimes y) = h(x) \otimes g(y)$ *for every $x \in V_j$ and $y \in V_{j-1}^\perp$.*

(iii) For every $c \in \mathbb{F}$, there exists $\tilde{c} \in \mathbb{F}$, independent of x and y, such that $h(cx) = \tilde{c}h(x)$ and $g(cy) = \tilde{c}g(y)$ for every $x \in V_j$ and $y \in V_{j-1}^\perp$.

Furthermore, if $\mathfrak{B} \neq \mathcal{T}_2(\mathbb{F})$, then there exists an invariant subspace V_j, with $\dim V_j \geq 2$ *and* $\dim V_{j-1}^{\perp} \geq 2$*, such that the conclusions of parts (ii) and (iii) hold.*

Proof. (i) Consider $K_1 = \xi \otimes \eta$,

$$K_2 = (\xi + e_1) \otimes \eta, \quad K_3 = \xi \otimes (\eta + e_n) \text{ and } K_4 = (\xi + e_1) \otimes (\eta + e_n)$$

and proceed exactly as in the proof of Theorem 3.1 (case 2).

(ii) For any $x \in V_j$, we have

$$\varphi(x \otimes \eta) = \varphi(\xi \otimes \eta) + \varphi((x - \xi) \otimes \eta).$$

If $\varphi(x - \xi) \otimes \eta \notin \mathfrak{L}_2$, the by part (i), we have $\varphi((x - \xi) \otimes \eta) = h(x - \xi) \otimes g(\eta)$ and so

$$\varphi(x \otimes \eta) = h(\xi) \otimes g(\eta) + h(x - \xi) \otimes g(\eta) = h(x) \otimes g(\eta).$$

On the other hand, if $\varphi(x - \xi) \otimes \eta \in \mathfrak{L}_2$, then $\varphi(x \otimes \eta) \notin \mathfrak{L}_2$, and again by applying part (i) directly to $x \otimes \eta$, we get $\varphi(x \otimes \eta) = h(x) \otimes g(\eta)$. For an $x \in V_j$ and $y \in V_{j-1}^{\perp}$, we may repeat the above argument using the equation

$$\varphi(x \otimes y) = \varphi(x \otimes \eta) + \varphi(x \otimes (y - \eta))$$

to reach the desired conclusion.

(iii) Assume that $x \in V_j$ and $y \in V_{j-1}^{\perp}$ and $c \in \mathbb{F}$. First we observe that $\dim V_j \geq 2$ and $\dim V_{j-1}^{\perp} \geq 2$, since otherwise, $\xi \otimes \eta \in \mathfrak{L}_1$, and hence $\varphi(\xi \otimes \eta) \in \mathfrak{L}_2$. It follows that $e_2 \in V_j$ and $e_{n-1} \in V_{j-1}^{\perp}$ and $x \otimes e_{n-1}$ and $e_2 \otimes y \in \mathfrak{A}$. So

$$h(cx) \otimes g(e_{n-1}) = \varphi(cx \otimes e_{n-1}) = \varphi(x \otimes ce_{n-1}) = h(x) \otimes g(ce_{n-1}).$$

Therefore there exists a scalar $\tilde{c}$ such that $g(ce_{n-1}) = \tilde{c}e_{n-1}$ and consequently $h(cx) = \tilde{c}h(x)$. Considering $\varphi(x \otimes cy)$ yields that $g(cy) = \tilde{c}g(y)$.

As to the last assertion of the lemma, we note that since φ is surjective, there exists a rank one matrix in $\mathfrak{A}$ whose image under φ is not in $\mathfrak{L}_2$. The result now readily follows. ∎

For the purpose of the next lemma, we define the *support* of an $n \times n$ matrix $C = [c_{ij}]$, to be the subset of indices (i, j) for which $c_{ij} \neq 0$ and we denote it by supp C. For a collection $\mathcal{C}$ of matrices, we define supp $\mathcal{C}$ by supp $\mathcal{C} = \cup\{\text{supp } C : C \in \mathcal{C}\}$. We denote the cardinality of a set X by card X. We also recall the matrix units $E_{ij} = e_i \otimes e_j$.

We continue to assume that φ is as in 5.8, and satisfies equation (5.8.1). We also assume that $\mathfrak{B} \neq \mathcal{T}_2(\mathbb{F})$, which implies that $\mathfrak{A} \neq \mathcal{T}_2(\mathbb{F})$.

Lemma 5.11. *Let $\mathbb{A}$ be the support of $\mathfrak{A}$, and let $\mathbb{J}$ be the the subset of all (i, j) for which $\varphi(E_{ij}) \notin \mathfrak{L}_2$. Then:*

(i) $\{\varphi(E_{ij}) \ : \ (i,j) \in \mathbb{J}\}$ *are linearly independent.*

(ii) Span $\{\varphi(E_{ij}) \ : \ (i,j) \in \mathbb{J}\}$ *is disjoint from* $\mathfrak{L}_2$.

(iii) $\mathbb{J}$ *consists of all indices* $(i,j) \in \mathbb{A}$ *for which* $i \geq 2$ *and* $j \leq n-1$.

(iv) If $x \otimes y \notin \mathfrak{L}_1$, *then* $\varphi(x \otimes y) \notin \mathfrak{L}_2$.

(v) If $x \otimes y \in \mathfrak{M}$, *then* $\varphi(x \otimes y) = h(x) \otimes g(y)$.

(vi) $u_0 \in W_1$ *and* $v_0 \in W_k$, *in particular when* $n_1 = 1$, *we may take* $u_0 = e_1$ *and when* $n_k = 1$, *we may take* $v_0 = e_n$

Proof. Define $\mathbb{A}_0 = \{(i,j) \in \mathbb{A} : i \geq 2 \text{ and } j \leq n-1\}$. It follows from Lemma 5.10 that if $T = E_{ij}$ with $(i,j) \in \mathbb{J}$, then $\varphi(\text{span}\{T\}) \subseteq \text{span}\{\varphi(T)\}$, and so

$$\varphi(\text{span}\{E_{ij} \ : \ (i,j) \in \mathbb{J}\}) \subseteq \text{span}\{\varphi(E_{ij}) \ : \ (i,j) \in \mathbb{J}\}.$$

Every matrix in $\mathfrak{A}$ may be written as $L+E$ where $L \in \mathfrak{L}_1$ and $E \in \text{span}\{E_{ij} \ : \ (i,j) \in \mathbb{J}\}$. So

$$\mathfrak{B} = \varphi(\mathfrak{A}) \subseteq \varphi(\mathfrak{L}_1) + \varphi(\text{span}\{E_{ij} \ : \ (i,j) \in \mathbb{J}\}) \subseteq \mathfrak{L}_2 + \text{span}\{\varphi(E_{ij}) \ : \ (i,j) \in \mathbb{J}\}.$$

Thus,

$$\dim \mathfrak{B} \leq \dim \mathfrak{L}_2 + \text{card } \mathbb{J} \leq (2n-1) + \text{card } \mathbb{J} \leq (2n-1) + \text{card } \mathbb{A}_0 = \dim \mathfrak{A}.$$

But $\dim \mathfrak{B} = \dim \mathfrak{A}$, so all of the above inequalities become equalities. Assertions (i)-(iv) and (vi) are immediate.

To prove (v), we first notice that the conclusion has been established for $x \otimes y \notin \mathfrak{L}_1$ in Lemma 5.10 together with part (iv). If $x \otimes y \in \mathfrak{L}_1$, then x is a scalar multiple of e_1 or y is a scalar multiple of e_n. In the former case, we have $e_2 \otimes y \in \mathfrak{A}$ and $\notin \mathfrak{L}_1$. Then

$$\varphi(x \otimes y) = \varphi(e_2 \otimes y) + \varphi((x-e_2) \otimes y) = h(e_2) \otimes g(y) + h(x-e_2) \otimes g(y) = h(x) \otimes g(y).$$

A similar calculation establishes the result when y is a scalar multiple of e_n. ∎

Lemma 5.12. *There exists an automorphism* $c \mapsto \tilde{c}$ *of* $\mathbb{F}$ *such that* $\varphi(cT) = \tilde{c}\varphi(T)$ *for every* $T \in \mathfrak{M}$ *and* $c \in \mathbb{F}$, *where* $\mathfrak{M}$ *is defined in the statement of Theorem 5.5.*

Proof. Define

$\mathcal{U}_g := \text{span}\{e_2, \ldots e_n\}$ or $\mathbb{F}^n$ according as $n_1 = 1$ or not,

and

$\mathcal{U}_h := \text{span}\{e_1, \ldots e_{n-1}\}$ or $\mathbb{F}^n$ according as $n_k = 1$ or not.

Take $x \in \mathcal{U}_h$ and $y \in \mathcal{U}_g$ with $x \otimes y \in \mathfrak{A}$. By lemma 5.11, we have $\varphi(x \otimes y) = h(x) \otimes g(y)$. We also know that $e_2 \otimes y \in \mathfrak{A}$ and $x \otimes e_{n-1} \in \mathfrak{A}$. Exactly as in Lemma 5.10 (iii), this leads to the existence of a function $c \mapsto \tilde{c}$ such that $\varphi(cR) = \tilde{c}\varphi(R)$ for every rank one $R \in \mathfrak{M}$, and as $\mathfrak{M}$ is the span of its rank one matrices, $\varphi(cT) = \tilde{c}\varphi(T)$ for every $T \in \mathfrak{M}$. It remains to show that $c \mapsto \tilde{c}$ is an automorphism. Additivity and surjectivity are obvious. The map is multiplicative as $\widetilde{c_1 c_2}\varphi(T) = \varphi(c_1 c_2 T) = \tilde{c_1}\varphi(c_2 T) = \tilde{c_1}\tilde{c_2}\varphi(T)$. Finally the map is injective since its kernel is obviously not all of $\mathbb{F}$, and being an ideal, it must then be $\{0\}$. ∎

Lemma 5.13. *There exists invertible matrices A and B in $\mathbf{M}_n(\mathbb{F})$ and a surjective rank one preserving mapping $\varphi_2 : \mathfrak{A} \longrightarrow A^{-1}\mathfrak{B}B^{-1}$ such that:*

(i) φ is a composition of φ_2, the multiplication operators L_A, R_B, and the mapping $C \mapsto \tilde{C}$ induced by a field automorphism.

(ii) The restriction of φ_2 to $\mathfrak{M}$ is the identity mapping.

Proof. Define φ_1 by $\varphi_1(\tilde{C}) = \varphi(C)$. Then φ_1 is a surjective rank one preserving mapping from $\mathfrak{A}$ to $\mathfrak{B}$ and $\varphi_1|\mathfrak{M}$ is linear. Furthermore, there exist injective linear mappings h_1 (respectively, g_1), from $\mathcal{U}_h$ (respectively, $\mathcal{U}_g$) into $\mathbb{F}^n$ such that $\varphi_1(x \otimes y) = h_1(x) \otimes g_1(y)$ for all $x \otimes y \in \mathfrak{M}$. (Here $\mathcal{U}_h$ and $\mathcal{U}_g$ are the spaces defined in the proof of the preceding lemma). It is now easy to find invertible $n \times n$ matrices A and B such that $h_1(x) = Ax$ for $x \in \mathcal{U}_h$ and $g_1(y) = B^t y$ for $y \in \mathcal{U}_g$. Define $\varphi_2(T) = A^{-1}\varphi_1(T)B^{-1}$. Both assertions of the lemma are easily verified. ∎

Lemma 5.14. $\mathfrak{A} = \mathfrak{B}$ *and* $A, B \in \mathfrak{A}$.

Proof. We will first show that $\mathfrak{A} = A^{-1}\mathfrak{B}B^{-1}$. As $\varphi_2(e_1 \otimes \mathcal{U}_g) = e_1 \otimes \mathcal{U}_g$, we get that $\varphi_2(e_1 \otimes \mathbb{F}^n) \subseteq e_1 \otimes \mathbb{F}^n$. Similarly, $\varphi_2(\mathbb{F}^n \otimes e_n) \subseteq \mathbb{F}^n \otimes e_n$. Since $\mathfrak{A} = \mathfrak{M} + \text{span } E_{11} + \text{span } E_{nn}$ we see that

$$A^{-1}\mathfrak{B}B^{-1} = \varphi_2(\mathfrak{A}) \subseteq e_1 \otimes \mathbb{F}^n + \mathbb{F}^n \otimes e_n \subseteq \mathfrak{A}.$$

Since $\dim A^{-1}\mathfrak{B}B^{-1} = \dim \mathfrak{B} = \dim \mathfrak{A}$, it follows that $\mathfrak{A} = A^{-1}\mathfrak{B}B^{-1}$.

The function $T \mapsto ATB$ maps the triangular algebra $\mathfrak{A} = A^{-1}\mathfrak{B}B^{-1}$ onto the triangular algebra $\mathfrak{B}$. So, by Lemma 4.2, we get that the two algebras are equal and that A and $B \in \mathfrak{A}$. ∎

Lemma 5.15. *The mapping φ_2 of Lemma 5.13 is a composition of mappings $\hat{\mathbf{f}}$ and $\breve{\mathbf{g}}$ defined in 5.2 and 5.3. The former is present only when $n_1 = 1$ and the latter is present only when $n_k = 1$.*

Proof. First assume that $n_1 = 1$. As before , we have $\varphi_2(e_1 \otimes y) = e_1 \otimes g_2(y)$ for an injective additive mapping g_2 on $\mathbb{F}^n$. In particular $\varphi_2(c_{11}E_{11}) = \sum_{j=1}^n f_j(c_{11})E_{1j}$ for additive maps $f_1, f_2, \dots f_n$ on $\mathbb{F}$. Next we show that f_1 is bijective. Surjectivity is obvious.

To prove bijectivity, assume that $f_1(c) = 0$. Then $\varphi_2(cE_{11} - \sum_{j=2}^n f_j(c)E_{1j}) = 0$. Thus $c = 0$, since otherwise φ_2 annihilates a rank one matrix. This proves injectivity. If $n_1 \geq 2$, then $\varphi_2|(\text{span } E_{11})$ is the identity, and we may take $\mathbf{f} = (id, 0, \dots, 0)$, where id is the identity.

The case $n_k = 1$ is dealt with similarly, showing the existence of an additive mapping $\mathbf{g} = (g_n, \dots, g_1)$, such that $\varphi_2(c_{nn}E_{nn}) = \sum_{j=1}^n g_j(c_{nn})E_{jn}$. It is now straightforward to verify that φ_2 is a composition of $\hat{\mathbf{f}}$ and $\breve{\mathbf{g}}$. ∎

Conclusion: Theorem 5.5 now follows from Lemmas 5.8 to 5.15. When φ satisfies equations (5.8.1), we have written as a composition of maps of the form (i) -(v). In the alternative case, the map $\varphi^+ : T \mapsto (\varphi(T))^+$ is a composition of the same maps. ∎

6. Counter-examples

In this section we give examples related to section 5. We provide examples to illustrate that the hypothesis of surjectivity in Theorem 5.5 is indispensable, to show that $\mathcal{T}_2$ is indeed an exceptional case and to show that unlike the linear case, the condition that φ preserves rank one matrices in both directions does not imply surjectivity and does not imply that φ has a form resembling that of Theorem 5.5.

We will find the following facts useful:

(i) $\mathbb{R} \cong \mathbb{R}^d$ as additive groups for all positive integers d. This is true as each of $\mathbb{R}, \mathbb{R}^d$ is isomorphic to $\mathbb{Q}^{\mathbf{c}}$ where $\mathbf{c}$ is the cardinality of the continuum.

(ii) A field $\mathbb{F}$ may be isomorphic to a proper subfield of itself. Indeed if $\mathbb{K}$ is an arbitrary field and $\mathbf{t}$ is an indeterminate, then $\mathbf{t} \mapsto \mathbf{t}^3$ induces an isomorphism between $\mathbb{K}(\mathbf{t})$ and $\mathbb{K}(\mathbf{t}^3)$. For a concrete example, we have $\mathbb{Q}(\pi) \cong \mathbb{Q}(\pi^3)$.

In each of the following examples, we have an additive rank one preserving mapping that is not of the form of Theorem 5.5 and also the range is not included in a space of rank one matrices. Example 4.5 provides a linear map with range included in a space of rank one matrices.

Examples 6.1. Let $f : \mathbb{R}^4 \longrightarrow \mathbb{R}$ be an additive-group isomorphism and then define mappings $\varphi : \mathcal{T}_3(\mathbb{R}) \longrightarrow \mathcal{T}_3(\mathbb{R})$ and $\psi : \mathcal{T}_3(\mathbb{R}) \longrightarrow \mathcal{T}_2(\mathbb{R})$ by

$$\varphi\left(\begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & a_4 & a_5 \\ 0 & 0 & a_6 \end{bmatrix}\right) = \begin{bmatrix} a_1 & 0 & f(a_2, a_3, a_4, a_5) \\ 0 & 0 & 0 \\ 0 & 0 & a_6 \end{bmatrix}$$

$$\psi\left(\begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & a_4 & a_5 \\ 0 & 0 & a_6 \end{bmatrix}\right) = \begin{bmatrix} a_1 & f(a_2, a_3, a_4, a_5) \\ 0 & a_6 \end{bmatrix}$$

The mapping φ is not surjective, while ψ is bijective but maps $\mathcal{T}_m$ onto $\mathcal{T}_n$ where $m \neq n$.

The next example shows that $\mathcal{T}_2(\mathbb{R})$ is indeed exceptional even when φ preserves rank one matrices in both directions in addition to being bijective.

Example 6.2. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be an additive-group isomorphism and define $\varphi : \mathcal{T}_2(\mathbb{R}) \longrightarrow \mathcal{T}_2(\mathbb{R})$ by

$$\varphi\left(\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}\right) = \begin{bmatrix} a_{11} & f(a_{12}) \\ 0 & a_{22} \end{bmatrix}$$

In the linear case, we have seen that surjectivity, together with preserving rank one, is equivalent to preserving rank one in both directions. Our next examples illustrate the difference in the additive case. They show that preserving rank one in both directions does not give us the forms in Theorem 5.5. We give three examples for three different types of matrix algebras, namely $\mathcal{T}_n$, block triangular algebras, and the full matrix algebra M_n.

Examples 6.3. Let $\mathbb{F}$ be a field which is isomorphic to a proper subfield $\mathbb{F}'$ such that $[\mathbb{F} : \mathbb{F}'] \geq 3$. (As usual $[\mathbb{F} : \mathbb{F}']$ is the dimension of $\mathbb{F}$ as a vector space over $\mathbb{F}'$.) Let θ be an isomorphism from $\mathbb{F}$ to $\mathbb{F}'$. Let π_1, π_2 be elements of $\mathbb{F}$ such that $1, \pi_1, \pi_2$ are linearly independent over $\mathbb{F}'$. For instance, we may take $\mathbb{F} = \mathbb{Q}(\pi)$, $\mathbb{F}' = \mathbb{Q}(\pi^3)$, $\pi_1 = \pi$, $\pi_2 = \pi^2$ and $\theta(r(\pi)) = r(\pi^3)$ for every rational expression $r(x) \in \mathbb{Q}(x)$. Define $\varphi : \mathcal{T}_4(\mathbb{F}) \longrightarrow \mathcal{T}_4(\mathbb{F})$ and $\psi : \mathcal{T}(1,2)(\mathbb{F}) \longrightarrow \mathcal{T}(1,2)(\mathbb{F})$ and $\chi : M_3(\mathbb{F}) \longrightarrow M_3(\mathbb{F})$ by

$$\varphi\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}\right) = \begin{bmatrix} \theta(a_{11}) & \theta(a_{12}) & \theta(a_{13}) + \pi_1\theta(a_{33}) & \theta(a_{14}) + \pi_1\theta(a_{34}) \\ 0 & \theta(a_{22}) & \theta(a_{23}) + \pi_2\theta(a_{33}) & \theta(a_{24}) + \pi_2\theta(a_{34}) \\ 0 & 0 & \theta(a_{33}) & \theta(a_{34}) \\ 0 & 0 & 0 & \theta(a_{44}) \end{bmatrix}$$

$$\psi\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix}\right) = \begin{bmatrix} \theta(a_{11}) & \theta(a_{12}) + \pi_1\theta(a_{32}) & \theta(a_{13}) + \pi_1\theta(a_{33}) \\ 0 & \theta(a_{22}) + \pi_2\theta(a_{32}) & \theta(a_{23}) + \pi_2\theta(a_{33}) \\ 0 & 0 & 0 \end{bmatrix}$$

$$\chi\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\right) = \begin{bmatrix} f(a_{11}) + f(a_{31})\pi_1 & f(a_{12}) + f(a_{32})\pi_1 & f(a_{13}) + f(a_{33})\pi_1 \\ f(a_{21}) + f(a_{31})\pi_2 & f(a_{22}) + f(a_{32})\pi_2 & f(a_{23}) + f(a_{33})\pi_2 \\ 0 & 0 & 0 \end{bmatrix}$$

Using the fact that a nonzero matrix has rank one if and only if every 2×2 submatrix has zero determinant, it is straightforward to verify that each of the three maps preserves rank one matrices in both directions and is injective.

7. Additive maps preserving rank one in both directions

As seen in Examples 6.3, additive maps that preserve rank one in both directions need

not be of a form resembling the forms described in Theorem 5.5. We show, however, that the only "obstruction" is the fact that the field $\mathbb{F}$ may be isomorphic to a proper subfield of itself. When this is not the case, we obtain a form for such maps that is nearly identical to the form of Theorem 5.5. The details are slightly less delicate than the proofs in §5.

Remark: Each of the following fields is not isomorphic to a proper subfield of itself. Finite fields and their algebraic closures; the field of (real or complex) algebraic numbers; finite extensions of $\mathbb{Q}$, and the field $\mathbb{R}$ of real numbers. All this is easy to verify. For the field $\mathbb{R}$, we may also refer to [1; p. 58].

Of course, every prime field, i.e. $\mathbb{Q}$ or $\mathbb{Z}_p$ (for a prime p), is not isomorphic to a proper subfield of itself, but in this case additive maps are automatically linear.

Definition 7.1. A mapping φ from a vector space V to a vector space W, is said to be *quasi-linear* if it is additive and if there exists a nonzero ring-endomorphism $\theta : \mathbb{F} \longrightarrow \mathbb{F}$ such that $\varphi(\lambda v) = \theta(\lambda)\varphi(v)$ for every $\lambda \in \mathbb{F}$ and $v \in V$. It should be noted that the map θ is an isomorphism from $\mathbb{F}$ onto a subfield of itself.

In the following theorem, we again assume that $\mathfrak{A} \neq \mathcal{T}_2$. Example 6.2 shows that this is necessary.

Theorem 7.2 *Let $\mathbb{F}$ be a field that is not isomorphic to a proper subfield of itself, let $\mathfrak{A} = \mathcal{T}(n_1 \dots n_k)$ be a block upper triangular algebra in $\mathbf{M}_n(\mathbb{F})$ such that $\mathfrak{A} \neq \mathcal{T}_2(\mathbb{F})$. Let $\varphi : \mathfrak{A} \longrightarrow \mathfrak{A}$ be an additive mapping that preserves rank one matrices in both directions. Then φ is a composition of some or all of the maps (i) - (iv) of Theorem 5.5, except that the map f_1 (in 5.2 and 5.3) is only required to be injective rather than bijective.*

The restriction of φ to the space

$$\mathfrak{M} := \{[c_{ij}] \in \mathfrak{A} \ : \ c_{11} = 0 \text{ if } n_1 = 1, \quad \text{and } c_{nn} = 0 \text{ if } n_k = 1\}$$

is semilinear. In particular, if $n_1 \geq 2$ and $n_k \geq 2$, then φ is semilinear.

Furthermore, φ is injective and it preserves every rank.

Again, we prove Theorem 7.2 via several lemmas.

Lemma 7.3. (Cf. Lemma 5.8). *Let $\mathfrak{A} = \mathcal{T}(n_1 \dots n_k)$ be a block upper triangular algebra in $\mathbf{M}_n(\mathbb{F})$ and let $\varphi : \mathfrak{A} \longrightarrow \mathbf{M}_n(\mathbb{F})$ be an additive mapping that preserves rank one matrices in both directions. Then there exist nonzero vectors $u_0,\ v_0 \in \mathbb{F}^n$, and injective* *additive mappings* $g, h : \mathbb{F}^n \longrightarrow \mathbb{F}^n$ *such that* φ *satisfies equations* 5.8.1 *or equations* 5.8.2.

Proof. The vectors u_0, v_0 and the mappings g, h are established exactly as in the first paragraph of the proof of Lemma 5.8. Upon examining the remainder of the proof of Lemma 5.8, we see that it suffices to show that it is not possible to have $\varphi(e_1 \otimes y) = u_0 \otimes g(y)$ and $\varphi(x \otimes e_n) = u_0 \otimes h(x)$.

If this is the case, then consider $T_{x,y} := e_1 \otimes y + x \otimes e_n$; with each of the pairs $\{x,\ e_1\}$ $\{y,\ e_n\}$ linearly independent. Each $T_{x,y}$ has rank two and $\varphi(T_{x,y}) = u_0 \otimes (h(x) + g(y))$. Thus $h(x) + g(y) = 0$, since otherwise the image of a rank two matrix has rank one. Upon replacing x by $e_1 + x$, we get that $h(e_1) = 0$. But then the image of $e_1 \otimes e_n$ is zero, a contradiction. ∎

The next lemma replaces lemmas 5.10 and 5.11.

Lemma 7.4. *Let* $\mathfrak{L}_1$ *and* $\mathfrak{L}_2$ *be as in Lemma 5.10 and let* $x \otimes y$ *be a rank one matrix in* $\mathfrak{A}$*, but not in* $\mathfrak{L}_1$*. Then:*

(i) $\varphi(x \otimes y) \notin \mathfrak{L}_2$.

(ii) $h(x)$, u_0 *are linearly independent and* $g(y)$, v_0 *are linearly independent.*

(iii) $\varphi(x \otimes y) = h(x) \otimes g(y)$.

proof. (i) If $\varphi(x \otimes y) \in \mathfrak{L}_2$, then $\varphi(x \otimes y) = u_0 \otimes w$ or $w \otimes v_0$. In the former case, we consider the rank two matrices $K_z := x \otimes y + e_1 \otimes z$, where z and y are linearly independent. $\varphi(K_z) = u_0 \otimes (w + g(z))$. Thus $w + g(z) = 0$ for every z that is not a scalar multiple of y. This contradicts the injectivity of g. If $\varphi(K_1) = w \otimes v_0$, a similar calculation leads to a contradiction.

(ii) Consider $K_1 = x \otimes y$,

$$K_2 = (x + e_1) \otimes y, \quad K_3 = x \otimes (y + e_n) \ \text{ and } \ K_4 = (x + e_1) \otimes (y + e_n)$$

and let $C_j = \varphi(K_j); 1 \le j \le 4$. Thus C_1, C_2, C_3, C_4 are all of rank one. If $C_1 = u \otimes v$, then

$$C_2 = u \otimes v + u_0 \otimes g(y), \quad C_3 = u \otimes v + h(x) \otimes v_0 \ \text{ and } \ C_4 = u \otimes v + u_0 \otimes g(y) + h(x) \otimes v_0 + u_0 \otimes v_0.$$

If $u_0, h(x)$ are linearly dependent or $v_0, g(y)$ are linearly dependent, then by Lemma 2.1, we conclude that $\varphi(x \otimes y) \in \mathfrak{L}_2$, contradicting (i)

(iii) Applying lemma 2.1 again to C_1, C_2, C_3, C_4, we now conclude, in view of, (ii) that $\varphi(x \otimes y) = h(x) \otimes g(y)$. ∎

Lemma 7.5. *Assume that φ is as above and that $\mathfrak{A} \neq \mathcal{T}_2(\mathbb{F})$. Then there exists a ring homomorphism $c \mapsto \tilde{c}$ from $\mathbb{F}$ into $\mathbb{F}$ such that $\varphi(cT) = \tilde{c}\varphi(T)$ for every $T \in \mathfrak{M}$ and $c \in \mathbb{F}$. Thus the restriction of φ to $\mathfrak{M}$ is quasi-linear. In particular if $\mathbb{F}$ is not isomorphic to a proper field of itself, then $c \mapsto \tilde{c}$ is an automorphism of $\mathbb{F}$ and $\varphi|\mathfrak{M}$ is semilinear.*

Proof. First consider a rank one matrix $x \otimes y \notin \mathfrak{L}_1$, then by Lemma 7.4 (iii) we know that $\varphi(x \otimes y) = h(x) \otimes g(y)$. This may now be extended to all matrices $e_1 \otimes y$ (similarly $x \otimes e_n$) that are in $\mathfrak{M}$, via the equation $\varphi(e_1 \otimes cy) = \varphi(e_2 \otimes cy) + \varphi((e_1 - e_2) \otimes cy)$. Thus, for all $x \otimes y \in \mathfrak{M}$ we have $\varphi(x \otimes y) = h(x) \otimes g(y)$.

Now exactly as in the proofs of Lemma 5.10(iii) and Lemma 5.13, we see that $h(cx) = \tilde{c}h(x)$ and $g(cy) = \tilde{c}g(y)$, for a mapping $c \mapsto \tilde{c}$ from $\mathbb{F}$ into itself. Thus $\varphi(cT) = \tilde{c}\varphi(T)$ for every $T \in \mathfrak{M}$. We see that $c \mapsto \tilde{c}$ is additive, multiplicative and injective as in Lemma 5.13. Thus $\varphi|\mathfrak{M}$ is quasi-linear. Also, $c \mapsto \tilde{c}$ is surjective if $\mathbb{F}$ is not isomorphic to a proper subfield of itself, yielding $c \mapsto \tilde{c}$ and automorphism and $\varphi|\mathfrak{M}$ semilinear. ∎

Lemma 7.6. *Assume that φ and $\mathfrak{A}$ are as above. Assume further that $\mathbb{F}$ is not isomorphic to a proper subfield of itself. Then φ is a composition of a mapping $C \mapsto \tilde{C}$ induced by a field automorphism and an additive mapping $\varphi_1 : \mathfrak{A} \longrightarrow \mathbf{M}_n(\mathbb{F})$ that preserves rank one in both directions with $\varphi_1|\mathfrak{M}$ linear.*

Proof. Define $\varphi_1(\tilde{C}) = \varphi(C)$. The results are easily verified. ∎

We continue to make the same assumptions as in Lemma 7.6.

Lemma 7.7. *There exists invertible matrices A and B in $\mathbf{M}_n(\mathbb{F})$ and a rank one preserving mapping $\varphi_2 : \mathfrak{A} \longrightarrow \mathbf{M}_n(\mathbb{F})$ such that:*

(i) $\varphi_1(T) = A\varphi_2(T)B$

(ii) The restriction of φ_2 to $\mathfrak{M}$ is the identity mapping.

Proof. Exactly as in Lemma 5.14. ∎

Lemma 7.8. *The mapping φ_2 of Lemma 7.7 is a composition of mappings $\hat{\mathbf{f}}$ and $\breve{\mathbf{g}}$ defined in 5.2 and 5.3 with f_1 and g_1 injective. The former is present only when $n_1 = 1$ and the latter is present only when $n_k = 1$.*

Proof. This is exactly as in Lemma 5.16 except that in this case f_1 and g_1 need not be surjective. ∎

(7.9) Conclusion: Lemmas 7.3 to 7.8 constitute a proof for Theorem 7.2.

References

[1] J. Aczel and J. Dhombres, *Functional Equations in Several Variables*, Encyclopedia Math. Appl. 31, Cambridge U.P., 1989.

[2] L. Beasley, Rank-k-preservers and preservers of sets of ranks, *Linear Algebra Appl.* 55:11-17 (1983).

[3] P. Botta, Linear maps preserving rank less than or equal to one, *Linear and Multilinear Algebra* 20:197-201 (1987).

[4] G. H. Chan and M. H. Lim, Linear transformations on symmetric matrices that preserve commutativity, *Linear Algebra Appl.* 47:11-22 (1982).

[5] K. R. Davidson, *Nest Algebras*, Pitman Research Notes in Mathematics, vol. 191, Longman Scientific and Technical, London and New York, 1988.

[6] J. C. Hou, Rank-preserving linear maps on $B(X)$, *Sci. China Ser. A* 32:929-940 (1989).

[7] A. A. Jafarian and A. R. Sourour, Spectrum-preserving linear maps, *J. Funct. Anal.* 66:255-261 (1986).

[8] T. J. Laffey and R. Loewy, Linear transformations which do not increase rank, *Linear and Multilinear Algebra.* 26:181-186 (1990).

[9] R. Loewy, Linear transformations which preserve or decrease rank, *Linear Algebra Appl.* 121:151-161 (1989).

[10] M. Marcus, Linear transformations on matrices, *J. Nat. Bureau Standards* 75B:107-113 (1971).

[11] M. Marcus and B. N. Moyls, Transformations on tensor product spaces, *Pacific J. Math.* 9:1215-1221 (1959).

[12] H. Minc, Linear transformations on matrices: Rank 1 preservers and determinant preservers, *Linear and Multilinear Algebra* 4:265-272 (1977).

[13] M. Omladič and P. Šemrl, Spectrum-preserving additive maps, *Linear Algebra Appl.* 153:67-72 (1991).

[14] M. Omladič and P. Šemrl, Additive mappings preserving operators of rank one, *Linear Algebra Appl.* 182:239-256 (1993).

[15] H. Radjavi, Commutativity-preserving operators on symmetric matrices, *Linear Algebra Appl.* 61:219-224 (1984).

[16] A. R. Sourour, Invertibility preserving linear maps on $\mathcal{L}(\mathbf{X})$, *Trans. Amer. Math. Soc.* 348:14-30 (1996).

[17] W. Watkins, Linear maps that preserve commuting pairs of matrices, *Linear Algebra Appl.* 14:29-35 (1976).