SUBSHIFTS OF MULTI-DIMENSIONAL
SHIFTS OF FINITE TYPE

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Abstract. We show that every shift of finite type \( X \) with positive entropy has proper subshifts of finite type with entropy strictly smaller than the entropy of \( X \), but with entropy arbitrarily close to the entropy of \( X \). Consequently, \( X \) contains an infinite chain of subshifts of finite type which is strictly decreasing in entropy.

Introduction

For a dynamical system \( X \), one can ask what are the subsystems of \( X \), and what are the possible values of entropies of subsystems of \( X \)? In the case of a one-dimensional irreducible shift of finite type, the subshifts of finite type in \( X \) are characterized by the Krieger embedding theorem ([5], Theorem 10.1.1). But for higher dimensional shifts of finite type, the answers are not known. So we ask some weaker questions: are there infinitely many subshifts of \( X \), and if so, do the entropies of those subshifts take on infinitely many values? If the topological entropy of \( X \) is positive, the answer to both these questions is yes. In [4], a related question of embedding multi-dimensional subshifts is considered.

In §2, we prove the first of these statements using elementary methods (Corollary 2.7). We then prove a stronger result (which implies both statements), using a theorem due to Ornstein and Weiss on recurrence for stationary random fields. Given a shift of finite type \( X \), with positive entropy, we show that there exists a proper subshift of finite type contained in \( X \), with entropy arbitrarily close to and less than that of \( X \) (see Theorem 2.9).

In §3, we discuss continuous, shift-commuting maps (or codes) between higher dimensional shifts of finite type, and generalize some known results for shifts in one dimension (Theorem 3.2).

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1. Background

Let $\mathcal{A}$ be a finite set, $d \in \mathbb{N}$, and let $\Sigma = \mathcal{A}^{\mathbb{Z}^d}$ be the space of all maps $\mathbb{Z}^d \to \mathcal{A}$. Elements of $\mathcal{A}$ are called symbols, and points in $\Sigma$ can be thought of as infinite $d$-dimensional arrays of symbols. We give $\Sigma$ the product topology, where $\mathcal{A}$ has the discrete topology. If $x \in \Sigma$ and $S \subseteq \mathbb{Z}^d$, we let $x_S$ denote the restriction of $x$ to $S$. (If $S = \{a\}$ is a singleton, we simply write $x_a$.) A configuration on $S$ is a map $E : S \to \mathcal{A}$. For any subset $X$ of $\Sigma$, a configuration $E : S \to \mathcal{A}$ is said to be allowed for $X$ (or simply allowed) if there exists $x \in X$ such that $x_S = E$. We say that $E$ occurs in $x$. If $B : S \to \mathcal{A}$ is any configuration, then we write $[B]$ for $\{x \in X : x_S = B\}$. These sets are also known as cylinder sets. If $E$ and $F$ are configurations and $S$ and $T$ are disjoint subsets of the domains of $E$ and $F$ respectively, then the configuration $E_S F_T$ is the map $S \cup T \to \mathcal{A}$ which agrees with $E$ on $S$ and with $F$ on $T$.

For each $a \in \mathbb{Z}^d$, the shift map $\sigma_a : \Sigma \to \Sigma$ is defined by $(\sigma_a(x))_c = x_{c+a}$. Clearly, $\sigma_a$ is a homeomorphism, and $\sigma_a \sigma_b = \sigma_b \sigma_a$ for all $a, b \in \mathbb{Z}^d$. Thus $\mathbb{Z}^d$ acts on $\Sigma$ by homeomorphisms. A closed, non-empty subset $X$ of $\Sigma$ which is invariant under $\sigma_a$, for all $a \in \mathbb{Z}^d$, is called a $d$-dimensional shift space (or simply a shift space). If $Y \subseteq X$ is closed, non-empty and invariant under $\sigma_a$, for all $a \in \mathbb{Z}^d$, we say that $Y$ is a subshift of $X$.

The $k$-cube with lowest corner at the origin is the set $\Lambda(k)$ consisting of all $b = (b_1, \ldots, b_d) \in \mathbb{Z}^d$ such that $0 \leq b_i < k$, for $1 \leq i \leq d$. For $a \in \mathbb{Z}^d$, the set $a + \Lambda(k)$ is the $k$-cube with lowest corner at $a$. The $2k-1$-cube centered at the origin is the set $\Lambda(2k-1)$ consisting of all $b = (b_1, \ldots, b_d) \in \mathbb{Z}^d$ such that $|b_i| < k$. For any $S \subseteq \mathbb{Z}^d$, the border of $S$, denoted $\partial S$, is the set of $a \in S$ such that there exists $b$ in the complement of $S$ with $\|a - b\| = 1$, where $\| \cdot \|$ is the usual norm on $\mathbb{R}^d$. The border of an allowed configuration $B = x_S$ is defined to be $\partial(B) = x_{\partial S}$.

A $k$-block is an allowed configuration on $\Lambda(k)$. We write $B_k(X)$ for the set of $k$-blocks and $\mathcal{B}(X)$ for $\bigcup_{k=1}^{\infty} B_k(X)$. Elements of $\mathcal{B}(X)$ are called allowed blocks. If $B$ is a $k$-block and $x \in X$, we say that $B$ occurs in $x$ with lowest corner at $a \in \mathbb{Z}^d$ if $B(j) = x(a+j)$ for $j \in \Lambda(k)$. When this holds, we will abuse notation and write $B = x_{a+\Lambda(k)}$. If $N \geq k$, we say that $B$ occurs in $x_{b+\Lambda(N)}$ if there exists $a \in \mathbb{Z}^d$ such that $a + \Lambda(k) \subseteq b + \Lambda(N)$ and $B = x_{a+\Lambda(k)}$. If $E = x_{\Lambda(N)}$ is an $N$-block, $x \in X$, we say that $B$ occurs in $E$ if $B$ occurs in $x_{\Lambda(N)}$.

A shift space $X$ is a shift of finite type if there is a finite set $S \subseteq \mathbb{Z}^d$ and a non-empty subset $P \subseteq \mathcal{A}^S$ such that

$$X = \{x \in \Sigma : x_{S+a} \in P \text{ for every } a \in \mathbb{Z}^d\}.$$ 

We may think of $P$ as a finite set of allowed finite configurations. $X$ is a matrix shift if there is a collection of $d$ transition matrices $A_1, \ldots, A_d$, each indexed by $\mathcal{A}(X)$, such that

$$X = X(A_1, \ldots, A_d) = \{x \in \Sigma : A_i(x_a, x_{a+e_i}) = 1 \text{ for all } a \in \mathbb{Z}^d, 1 \leq i \leq d\}.$$ 

where $e_i$ is the $i$'th standard basis vector. Any shift of finite type is topologically conjugate to a matrix shift. A proof of this for the case $d = 1$ is given in [5], Prop. 2.3.9 (3), and the proof for higher dimensions is similar. For the remainder of this paper, we will assume that all shifts of finite type are presented as a matrix shifts.
If \( X \) is a shift space, and \( E \) is an allowed block for \( X \), then we let \( X \setminus E \) denote the set of points in \( X \) which do not contain an occurrence of \( E \). Formally, 
\[
X \setminus E = X \setminus \bigcup_{a \in \mathbb{Z}^d} \sigma_a([E]).
\]
If \( X \setminus E \neq \emptyset \), then it is a subshift of \( X \), and if \( X \) is a shift of finite type, so is \( X \setminus E \).

Let \( X \) be a \( d \)-dimensional shift space and \( N \) a positive integer. Define a map 
\[
\Gamma_N : X \to \mathcal{B}_N(X)^{\mathbb{Z}^d} \text{ by } (\Gamma_N(x))_a = x_{N\mathbf{a} + \Lambda(N)}.
\]
The image of \( \Gamma_N \) is a subshift of \( \mathcal{B}_N(X)^{\mathbb{Z}^d} \), called the \( N \)'th power of \( X \), and denoted \( X^N \). Any point \( x \in X \) corresponds naturally under \( \Gamma_N \) to a point \( \bar{x} = \Gamma_N(x) \in X^N \), and for any \( \mathbf{a} \in \mathbb{Z}^d \), we have \( \Gamma_N \sigma^N_{\mathbf{a}} = \sigma_{\mathbf{a}} \Gamma_N \). An allowed \( k \)-block \( E \) for \( X^N \) corresponds to an allowed \( Nk \)-block for \( X \) (which we also denote by \( E \), abusing notation).

The topological entropy of a \( d \)-dimensional shift space \( X \) is defined to be
\[
h(X) = \lim_{k \to \infty} \frac{1}{k^d} \log |\mathcal{B}_k(X)|.
\]
It is easy to verify that \( h(X^N) = Nh(X) \).

The measure-theoretic entropy of a shift space \( X \) with respect to a \( \mathbb{Z}^d \)-invariant measure \( \mu \) is
\[
h_\mu(X) = \lim_{k \to \infty} \frac{1}{k^d} \sum_{B \in \mathcal{B}_k} -\mu[B] \log \mu[B].
\]

A shift space \( X \) is irreducible if for any allowed blocks \( U \) and \( V \), there is a point \( x \in X \) and disjoint sets of coordinates \( S \) and \( T \) such that \( x_S = U \) and \( x_T = V \).

We make use at many points of the Variational Principle which is stated below. For a proof of the one-dimensional version, see [8].

**Theorem 1.1.** Let \( G = \mathbb{Z}^d \) act on a compact topological space \( X \) by homeomorphisms. Then
\[
h(X) = \sup_{\mu \in \mathcal{M}(X)} h_\mu(X)
\]
where \( \mathcal{M}(X) \) is the set of \( G \)-invariant measures on \( X \). Further if the action of \( G \) is expansive, then the supremum is attained on a non-empty compact set of measures.

A measure of maximal entropy for \( X \) is a shift-invariant measure \( \mu \) such that \( h_\mu(X) = h(X) \). Such measures always exist in the case when \( X \) is a shift space as then the action of \( \mathbb{Z}^d \) on \( X \) is necessarily expansive.
2. Positive Entropy and Subshifts

In this section, we show that if $X$ is a shift of finite type, and $h(X) > 0$, then $X$ contains infinitely many subshifts of finite type, all of positive entropy. The key observation for all that follows is contained in the following simple lemma.

**Lemma 2.1.** Let $X$ be a shift space and let $\mu$ be an invariant probability measure for $X$. If $Y$ is a subshift of $X$ such that $h(Y) < h(\mu)$ (or if $Y$ is empty and $0 < h(\mu)$), then for any positive integer $M$, there exist $M$ distinct blocks, all having the same border, which are not allowed in $Y$, but are allowed in the support of $\mu$ (and hence have positive $\mu$-measure).

**Proof.** First, suppose that $Y \neq \emptyset$. Let $\bar{X}$ denote the support of $\mu$. Then

$$h(Y) < h(\mu) = h(\bar{X}) \leq h(\bar{X}).$$

Arguing by contradiction, suppose that there is a positive integer $K$ such that for any configuration $H$, which is the border of an allowed block for $\bar{X}$, there are at most $K$ blocks in $B(\bar{X}) \setminus B(Y)$ with border $H$. Since $|\partial(\Lambda(n))| \leq 2^d n^{d-1}$, it would then follow that $|B_n(\bar{X}) \setminus B_n(Y)| \leq K |\mathcal{A}|^{2^d n^{d-1}}$, where $\mathcal{A}$ is the symbol set for $X$. Therefore

$$|B_n(\bar{X})| = |B_n(Y)| + |B_n(\bar{X}) \setminus B_n(Y)| \leq |B_n(Y)| \cdot K |\mathcal{A}|^{2^d n^{d-1}}.$$

Consequently,

$$h(\bar{X}) \leq \lim_{n \to \infty} \left[ \frac{1}{n^d} \log |B_n(Y)| + \frac{1}{n^d} \log K |\mathcal{A}|^{2^d n^{d-1}} \right] = h(Y),$$

since the limit of the second term in the sum is 0. Since $h(Y) < h(\bar{X})$, this is a contradiction.

The proof for the case $Y = \emptyset$ is similar. $\square$

**Lemma 2.2.** Let $X$ be a matrix shift of finite type. Let $B, C$ be distinct blocks such that $\partial(B) = \partial(C)$. For any $N$-block $E$ in which $C$ occurs, there is an $N$-block $F$ in which $B$ occurs, but $C$ does not occur, such that $\partial(E) = \partial(F)$.

**Proof.** Put an arbitrary order on $\mathcal{A}$, the symbol set of $X$. This extends to a lexicographic order on $B_m(X)$ for any $m$. Specifically, for $d = 2$, this is defined as follows: $G < H$ if there exists a pair $r, s$ such that $G_{rs} < H_{rs}$, $G_{rj} = H_{rj}$ for $0 \leq j < s$ and $G_{ij} = H_{ij}$ for $0 \leq i < r$ and $0 \leq j \leq m - 1$.

Let $B$ be a block in which $C$ occurs. Assume $C < B$. Then changing any occurrence of $C$ in $E$ to a $B$ (which can be done since $\partial(B) = \partial(C)$ and $X$ is a matrix shift of finite type), produces a new block $E_1$, with $E < E_1$. Since $\partial(B) = \partial(C)$, we have that $\partial(E_1) = \partial(E)$. If $C$ occurs in $E_1$, we can repeat this procedure, to obtain a block $E_2$, with $E_1 < E_2$. So we obtain a sequence $E < E_1 < E_2 < \ldots$, in which no $E_i$ can be repeated and $\partial(E_i) = \partial(E)$. Since there are only finitely many $N$-blocks, we must eventually reach a block $E_k = F$ in which $C$ does not occur. Since $F$ was obtained from $E_{k-1}$ by changing a $C$ to a $B$, it follows that $B$ occurs in $F$. If $B < C$, then a similar argument, with inequalities reversed, proves the result. $\square$
Corollary 2.3. Let $X$ be a shift of finite type. If $h(X) > 0$, then $X$ contains a proper subshift of finite type. In particular, $X$ is not minimal.

Proof. By recoding, we may assume that $X$ is a matrix shift of finite type. By Lemma 2.1, there exist two distinct blocks $B, C$ such that $\partial(B) = \partial(C)$. It follows from Lemma 2.2 and compactness of $X$ that $X \setminus C$ is nonempty, so $X \setminus C$ is a proper subshift of finite type contained in $X$. \Box

Lemma 2.4. If $X$ is a shift of finite type, and $Y$ is a subshift of $X$ which is not of finite type, then $X$ contains infinitely many subshifts of finite type.

Proof. Since $Y$ is a proper subshift of $X$, there is a block $B_1$ which is allowed for $X$ but not for $Y$. Let $Y_1 = X \setminus B_1$. Then $Y_1$ is a shift of finite type, since $X$ is one. Also, $X \supseteq Y_1 \supseteq Y$ since $Y$ is not a shift of finite type. Repeating this argument, we obtain an infinite strictly decreasing sequence $X \supseteq Y_1 \supseteq Y_2 \supseteq \ldots$ of subshifts of finite type contained in $X$. \Box

In Corollary 2.13, we will show that every shift of finite type of positive entropy contains a subshift which is not of finite type.

Definition 1.

Given a $k$-block $B$ and a point $x \in X$, let $f_n(B, x)$ denote the number of occurrences of $B$ in $x_{\Lambda(n)}$; that is, the number of points $a \in \Lambda(n - k)$ such that $x_{a + \Lambda(k)} = B$. A block $B$ is positively recurrent in $x$ if

$$\liminf_{n \to \infty} \frac{f_n(B, x)}{n^d} > 0.$$ 

If such an $x$ exists, we say that $B$ is positively recurrent.

It can be shown that a block $B$ is positively recurrent if and only there exists an ergodic invariant measure $\mu$ on $X$ such that $\mu(B) > 0$.

Lemma 2.5. Let $X$ be a shift space, and let $x \in X$. If $C$ is a $k$-block which is positively recurrent in $x$, then there exists $N \in \mathbb{Z}^+$ such that for any $a \in \mathbb{Z}^d$, there exists $a' \equiv a \pmod{N\mathbb{Z}^d}$ such that $C$ occurs in $x_{a' + \Lambda(N)}$.

Proof. For simplicity, we will give the proof for the case $d = 2$, the proof of the general case being similar. Arguing by contradiction, suppose that for all $N \in \mathbb{Z}^+$, there exists $a_N = (a_N^x, a_N^y) \in \mathbb{Z}^d$ such that for any $a' \equiv a_N \pmod{N\mathbb{Z}^d}$, we have that $C$ does not occur in $x_{a' + \Lambda(N)}$. The condition that $C$ does not occur in $x_{a' + \Lambda(N)}$ implies that the lowest corner of $C$ must lie in a horizontal strip of width $k - 1$ below one of the $H_r$ or a vertical strip of width $k - 1$ to the left of one of the $V_s$.

We consider $x_{\Lambda(N)}$, the $N$-block of $x$ with lowest corner at the origin. At most one of the vertical lines of the form $V_s$, and at most one of the horizontal lines of the form $H_r$ can intersect $\Lambda(N)$. The vertical and horizontal strips of width $k - 1$ about these lines cover at most $2N(k - 1)$ of the coordinates of $\Lambda(N)$. It follows that the number of occurrences of $C$ in $x_{\Lambda(N)}$ can be at most $2N(k - 1)$. Since $\frac{2N(k - 1)}{N^2} \to 0$ as $N \to \infty$, this contradicts the fact that $C$ is positively recurrent in $x$. \Box

If $E$ is a block and $a \in \mathbb{Z}^d$, we say that $E$ occurs in $y \in X$ with lowest corner in $a + N\mathbb{Z}^d$ if there exists $a' \equiv a \pmod{N\mathbb{Z}^d}$ such that $E$ occurs in $y$ with lowest corner at $a'$.
Theorem 2.6. Let $X$ be a shift of finite type. If $Y$ is a subshift of $X$ and $h(Y) < h(X)$ (or if $Y$ is empty and $0 < h(X)$), then there exists a shift of finite type $Z$ such that $Y \subseteq Z \subseteq X$ and $h(Z) > 0$.

Proof. We may assume that $X$ is a matrix shift of finite type. Assume that $Y$ is nonempty, the proof in the case $Y = \emptyset$ being similar. Let $\mu$ be an ergodic measure of maximal entropy for $X$, and let $\bar{X}$ denote the support of $\mu$. Then $h(Y) < h(X) = h_\mu(X) = h_\mu(\bar{X})$. By Lemma 2.1, there exist three distinct blocks $B, C$ and $D$ in $\mathcal{B}(\bar{X}) \setminus \mathcal{B}(Y)$, such that $\partial(B) = \partial(C) = \partial(D)$. We may assume that $C < B < D$, were $<$ is the lexicographic ordering on blocks, as in the proof of Lemma 2.2. Since $C \in \mathcal{B}(\bar{X})$ we have $\mu[C] > 0$, and it follows from the ergodic theorem that there exists a point $x \in \bar{X}$ in which $C$ is positively recurrent. Let $a_1, a_2, \ldots a_{N^d}$ be a complete set of representatives of $\mathbb{Z}^d/\mathbb{N}\mathbb{Z}^d$. By Lemma 2.5, there exists $N \in \mathbb{Z}^+$ such that for each $a_i$, there exists $a_i' \equiv a_i \pmod{\mathbb{N}\mathbb{Z}^d}$ such that $C$ occurs in $x_{a_i' + \Lambda(N)}$. Let $x_{a_i' + \Lambda(N)} = E_i \in \mathcal{B}(\bar{X})$. By Lemma 2.2, for each $i$ there exists a block $F_i$, in which $B$ occurs but $C$ does not, such that $\partial(E_i) = \partial(F_i)$. If $E_i = F_j$, then we can choose $F_i = F_j$.

Let $W$ denote the set of points in $X$ in which $E_i$ does not occur with lowest corner in $\mathbb{N}\mathbb{Z}^d$, for $1 \leq i \leq N^d$. Now $W$ may not be shift invariant, but it is invariant under $\sigma_a$, for $a \in \mathbb{N}\mathbb{Z}^d$, and is therefore a subshift of $X^N$. Let

\begin{equation}
Z' = \bigcup_{a \in \mathbb{Z}^d} \sigma_a(W).
\end{equation}

Clearly $Z'$ is shift-invariant. Since $\sigma_a(W) = \sigma_b(W)$ if $a \equiv b \pmod{\mathbb{N}\mathbb{Z}^d}$, the union in (2.1) can be taken over a complete set of representatives of $\mathbb{Z}^d/\mathbb{N}\mathbb{Z}^d$, so $Z'$ is a finite union of closed sets, and therefore closed. Consequently, $Z'$ is a subshift of $X$. Since $C$ occurs in $E_i$ for each $i$, it follows that $E_i$ is not an allowed block for $Y$, and so $Y \subset W \subset Z'$. If $Z'$ is not a shift of finite type (see Example 1), then by the proof of Lemma 2.4, there is a shift of finite type $Z$ with $Z \subseteq Z \subseteq X$ and we show in the final paragraph below that $h(Z) > 0$. So suppose that $Z'$ is a shift of finite type, and let $Z = Z'$.

We show that $Y \neq Z$, by showing that $W$ contains a point in which $B$ occurs. Let $y \in X$ be a point in which some $E_k$, occurs with lowest corner in $\mathbb{N}\mathbb{Z}^d$. Let $z$ be the point obtained by changing all occurrences of $E_i$ in $y$, with lowest corner in $\mathbb{N}\mathbb{Z}^d$, to $F_i$, for $1 \leq i \leq N^d$. (This can be done, since $X$ is a matrix shift of finite type, $\partial(E_i) = \partial(F_i)$, and any two such occurrences do not overlap.) Since $C$ does not occur in any $F_i$, but does occur in every $E_i$, we have $F_i \neq E_j$, for all $i, j$. It follows that $E_i$ does not occur in $z$ with lowest corner in $\mathbb{N}\mathbb{Z}^d$, for all $i$, so that $z \in W \subset Z$. On the other hand, $F_k$ occurs in $z$, since $E_k$ occurs in $y$. Since $F_k$ contains a $B$, it follows that $z$ is not in $Y$.

Next, we show that $Z \neq X$, by showing that $x$ is not in $Z$. Note that $\sigma_{-a}(W)$ consists of the set of points in which $E_i$ does not occur with lowest corner in $a + \mathbb{N}\mathbb{Z}^d$, for $1 \leq i \leq N^d$. By the choice of the blocks $E_i$, for every $a \in \mathbb{Z}^d$, there is an $i$ such that $E_i$ occurs in $x$ with lowest corner in $a + \mathbb{N}\mathbb{Z}^d$. It follows that $x$ is not in $\sigma_{-a}(W)$ for any $a \in \mathbb{Z}^d$, so $x$ is not in $Z$.

Finally, we show that $h(Z) > 0$, by showing that $h(W) > 0$, where $W$ is considered as a subshift of $X^N$. Note that in $X^N$, the blocks $E_i$ and $F_i$ correspond to one-blocks, and $W$ corresponds to the set of points which contain no occurrences of
$E_i$, for $1 \leq i \leq N^d$. Let $E_m$ be the largest of the $E_i$, in the lexicographic ordering. By the proof of Lemma 2.2, $E_m < F_m$, and $B$ occurs in $F_m$. Now, replace any single occurrence of $B$ in $F_m$ with $D$, to produce a block $G$. Then $\partial(F_m) = \partial(G)$ and we have $E_i \leq E_m < F_m < G$, for $1 \leq i \leq N^d$. Since $E_m \in B(X)$, we have $\mu[E_m] > 0$. It follows from the ergodic theorem that there is a point $y \in X^N$ in which $E_m$ is positively recurrent. Now, replace all occurrences of $E_i$ in $y$ with $F_i$, for $1 \leq i \leq N^d$, to obtain a point $z \in W$. Clearly, $F_m$ is positively recurrent in $z$. Therefore there is an $\epsilon > 0$ such that for all sufficiently large $k$, the number of occurrences of $F_m$ in $z_{\Lambda(k)}$ is at least $\epsilon k^d$. Since $G \neq E_i$ for all $i$, and $\partial(G) = \partial(F_m)$, any occurrence of $F_m$ in $z_{\Lambda(k)}$ can be replaced with $G$, and the resulting block is allowed in $W$. It follows that $B_k(W) \geq 2^{\epsilon k^d}$ for sufficiently large $k$. Therefore, $h(W) \geq \epsilon \log 2 > 0$. Now, since $W \subset Z$, we have $h(Z) > 0$. \qed

The reader may wonder why we could not prove Theorem 2.6 by simply letting $Z = X \setminus C$, which is clearly a proper subshift of $X$. The reason is that $X \setminus C$ might equal $Y$, which would be the case if for every point $x \in X$, $C$ occurs in $x$ if and only if $B$ occurs in $x$. Later, we show that given a pair $B, C$, as in Theorem 2.6, we can extend $B$ and $C$ to a pair of larger blocks, $\bar{B}$ and $\bar{C}$, with $\partial(B) = \partial(C)$, such that $Y \neq X \setminus \bar{C}$ (see the remark preceding Corollary 2.13).

**Corollary 2.7.** Let $X$ be a shift of finite type. If $Y$ is a subshift of $X$ and $h(Y) < h(X)$ (or if $Y$ is empty and $0 < h(X)$), then there exists an infinite chain of subshifts of finite type (ordered by inclusion) $Z_i$, with $Y \subset Z_i \subset X$ and $h(Z_i) > 0$ for each $i$.

**Proof.** Assume that $Y$ is non-empty, the proof in the case $Y = \emptyset$ being similar. By Theorem 2.6, there exists a subshift of finite type $Z$, with $Y \subset Z \subset X$ and $h(Z) > 0$. Proceeding inductively, suppose that we have found a sequence of subshifts of finite type $Y = Z_1 \subset Z_2 \subset \cdots \subset Z_n = X$, and $h(Z_i) > 0$ for $2 \leq i \leq n$. Since $h(Y) < h(X)$, there exists an $i$ with $h(Z_i) < h(Z_{i+1})$. Then we can apply Theorem 2.6 to obtain a subshift of finite type $Z'$, with $Z_i \subset Z' \subset Z_{i+1}$ and $h(Z') > 0$. The corollary now follows by induction. \qed

If $X$ is a shift of finite type, and $h(X) > 0$, then Corollary 2.7 implies that $X$ contains infinitely many subshifts of finite type, all having positive entropy.

**Example 1.** We give an example to show that the subshift $Z'$ constructed in Theorem 2.6 may not be of finite type. Let $X$ be the full 2-dimensional shift on two symbols, 0 and 1. Let $E$ and $F$ be two distinct $N$-blocks whose borders consist of all 1s.

As in the proof of Theorem 2.6, let $W$ denote the set of points which contain no occurrence of $E$ with lowest corner in $\mathbb{Z}^2/N\mathbb{Z}^2$. Let

\[ Z' = \bigcup_{a \in \mathbb{Z}^2} \sigma_a(W). \]

Then $Z'$ is not a shift of finite type. To see this, let $a_1, a_2, \ldots, a_{N^2}$ be a complete set of representatives for $\mathbb{Z}^2/N\mathbb{Z}^2$. Then for any positive integer $M$, it is easy to construct a point $y_M$ with the following properties:

1. For any $a \in \mathbb{Z}^2$, $y_M$ contains an occurrence of $E$ with lowest corner in $a + N\mathbb{Z}^2$.
2. Any two $E$s in $y_M$ occur at a distance at least $2M$ apart.
3. All symbols in $y_M$ outside an occurrence of $E$ are 0.
Then $y_M$ is not in $Z'$ by property (1), but it is easy to see that every $M$-block in $y_M$ is allowed in $Z'$. The details are left to the reader. This shows that for each $M$, there exists a point, all of whose $M$-blocks are allowed in $Z'$, but such that the point itself does not belong to $Z'$. Consequently, $Z'$ is not a shift of finite type.

In what follows, we work for simplicity in the two-dimensional case ($d = 2$). All of the material generalizes in a straightforward manner to higher dimensions. We will show that by using recurrence arguments due to Ornstein and Weiss that we may control the entropy of $X \setminus C$ and hence create a sequence of such subshifts of $X$ of finite type with entropy close to $X$.

**Proposition 2.8.** Every shift space $X$ contains an entropy minimal subshift $Y$ with the property that $h(Y) = h(X)$.

**Proof.** Let $B_1, B_2, \ldots$ be a list of all possible finite blocks listed in order of increasing size. Set $X_0 = X$ and inductively set

$$X_n = \begin{cases} X_{n-1} \setminus B_n & \text{if } h(X_{n-1} \setminus B_n) = h(X_{n-1}) \\ X_{n-1} & \text{otherwise.} \end{cases}$$

Then $X_n$ is a decreasing sequence of compact shift-invariant subsets of $X$. It follows that the limit $Y = \bigcap_{n=0}^{\infty} X_n$ is also a compact shift-invariant subset of $X$. Since topological entropy is upper semi-continuous for shift spaces (see [5]) and $h(X_n) = h(X)$, it follows that $h(Y) = h(X)$. To show that $Y$ is entropy minimal, suppose for a contradiction that $Y$ has a proper subshift of equal entropy. Then there certainly exists a block $B$ such that $[B] \cap Y \neq \emptyset$ and $h(Y \setminus B) = h(Y)$. The block $B$ must occur in the list as $B_n$ for some $n$ and since $\emptyset \neq [B] \cap Y \subseteq [B] \cap X_n$, we conclude that $[B] \cap X_n \neq \emptyset$. It follows from the definition of $X_n$ that $h(X_n \setminus B_n) < h(X_n)$ from which we deduce a contradiction as follows: $h(Y \setminus B_n) \leq h(X_n \setminus B_n) < h(X_n) = h(X)$. The entropy minimality of $Y$ follows. □

Note: It is not known whether the space $Y$ constructed above is itself necessarily a shift of finite type.

**Theorem 2.9.** Let $X$ be a subshift of finite type with positive topological entropy. Then for all $\epsilon > 0$, there exists a proper subshift $Y$ of $X$ which is also a subshift of finite type with the property that $h(X) - \epsilon < h(Y) < h(X)$.

We make use in the proof of the theorem of three key results. The first is the Variational Principle, Theorem 1.1.

The second key result is due to Ornstein and Weiss ([6]) and gives a characterization of measure-theoretic entropy in terms of return time.

The return time of the (centered) central $2k - 1$-block of $x$, $x_{\Lambda(2k-1)}$ is defined to be

$$R_k(x) = \inf\{n > 0: x_{\Lambda(2k-1)} \text{ occurs in } x_{\Lambda(2n-1)}, \text{ other than at 0}\}.$$

**Theorem 2.10.** (Ornstein and Weiss) If $\mu$ is an ergodic stationary random field, then

$$\lim_{k \to \infty} \frac{d \log R_k(x)}{(2k-1)^d} = h_\mu(X)$$

for $\mu$-almost every $x$. 
The final result is a lemma due to Burton and Steif giving an important property of measures of maximal entropy for subshifts of finite type. The proof is in [1], where it is stated for a special class of shifts of finite type. The proof however applies verbatim in the general setting.

**Lemma 2.11.** (Burton and Steif) If \( \mu \) is a measure of maximal entropy for a matrix shift, then for any finite set \( G \subset \mathbb{Z}^d \), the conditional distribution of \( \mu \) on \( G \) given the configuration on \( \partial G \) is \( \mu \)-a.s. uniform over all configurations on \( G \) which extend the configuration on \( \partial G \).

The following lemma is then central in the proof of Theorem 2.9.

**Lemma 2.12.** If \( \mu \) is an ergodic invariant measure on \( X \) and \( B \) is a configuration on \( \Lambda(2n - 1) \) such that \( \mu[B] > 0 \), then there exists a configuration \( \tilde{B} \) on \( \Lambda(2N - 1) \) for some \( N > n \) extending \( B \) (in the sense that \( \tilde{B}|_{\Lambda(2n - 1)} = B \)) such that

(i) \( \mu[\tilde{B}] > 0 \);

(ii) If \( \tilde{D} = \tilde{B}_{\Lambda(2N - 1) \setminus \Lambda(2n - 1)} D_{\Lambda(n)} \) and \( \tilde{E} = \tilde{B}_{\Lambda(2N - 1) \setminus \Lambda(2n - 1)} E_{\Lambda(n)} \) are any two configurations (not necessarily distinct) on \( \Lambda(2N - 1) \) which agree with \( \tilde{B} \) on \( \Lambda(2N - 1) \setminus \Lambda(2n - 1) \), then if \( \tilde{E} \) and \( \tilde{D} \) occur in \( x \in \Sigma \), then they either occur at the same point \( a \in \mathbb{Z}^d \) or they occur at two points \( a, b \in \mathbb{Z}^d \) with the property that \( \|a - b\|_\infty \geq N + n - 1 \).

The following figure illustrates the closest that two blocks, \( \tilde{D} \) and \( \tilde{E} \), as described in the lemma above may occur in a single point \( x \) of \( X \). The small squares are the central \( 2n - 1 \)-blocks of \( \tilde{D} \) and \( \tilde{E} \).

![Figure 1. Two padded blocks with minimum separation](image)

**Proof.** Let \( \epsilon \) be less than \( \mu[B]/8 \). Since \( \mu \) is an ergodic invariant measure on \( X \), Theorem 2.10 applies so that

\[
\lim_{k \to \infty} \frac{d \log R_k(x)}{(2k - 1)^d} = h_\mu(X)
\]

for \( \mu \)-almost every \( x \). It follows that for almost every \( x \), there exists an \( n(x) \) such that for \( k > n(x) \), \( d \log R_k(x)/(2k - 1)^d \geq h_\mu(X)/2 \). This may be rewritten

\[
\log R_k(x) \geq \exp \left( \frac{h(2k - 1)^d}{2d} \right).
\]

In particular, there exists an \( m(x) > n(x) \) such that \( \log R_k(x) \geq 10k \) for each \( k \geq m(x) \). Letting \( m(x) \) be the minimum such number, the function \( m(x) \) becomes
measurable and integer-valued. It follows that there exists an $M$ such that $m(x) \leq M$ for $x$ belonging to a set of measure at least $1 - \epsilon$. Write $S$ for $\{x: m(x) \leq M\}$. The complement of this set has measure at most $\epsilon$.

Consider the arrangement of blocks shown in Figure 2. The (smaller) central block is a centered square of side $2n - 1$ placed at the origin and the surrounding (marker) squares are of side $2M - 1$ with centers at $a_1, a_2, \ldots, a_8$. The surrounding larger squares are separated by distances greater than $2n - 1$. The frame containing all of the squares is the subset $F = \Lambda(2N - 1)$ of $\mathbb{Z}^2$. The frame is constructed so as to ensure that $2N < 10M$. We will call the central part the core.

Now forming

$$A = [B] \cap \bigcap_{i=1}^8 \sigma_{a_i}(S) = [B] \setminus \left( \bigcup_{i=1}^8 \sigma_{a_i}(S^c) \right),$$

we see that $\mu(A) > 0$. We let $x$ be any point in $A$ and set $\tilde{B} = x|_F$. The block $\tilde{B}$ now has the property that none of its marker squares are repeated within $\tilde{B}$. We then show that this set has the required properties. By elementary measure theory, for almost all $x \in A$, the set $\tilde{B}$ as defined above has positive measure so we can ensure that (i) is satisfied.

To show that (ii) is satisfied, suppose that $\tilde{D}$ and $\tilde{E}$ are two configurations as in the statement of the lemma. Suppose also that they occur in a point $x$ at positions $a$ and $b$ with $0 < \|a - b\|_{\infty} < N + n + 1$. It follows that in $x$, the core of $\tilde{D}$ overlaps the frame of $\tilde{E}$. We will then show that this contradicts the recurrence properties of points in $A$. To establish the contradiction, we distinguish three modes of overlapping of $\tilde{D}$ and $\tilde{E}$ as follows:

Case (1): The corner of the frame of $\tilde{E}$ lies within the core of $\tilde{D}$;

Case (2): The corners of the frame of $\tilde{E}$ lie outside the core of $\tilde{D}$, but the core of $\tilde{D}$ is not entirely contained within the frame of $\tilde{E}$;

Case (3): The core of $\tilde{D}$ lies entirely within the frame of $\tilde{E}$.

These possibilities are illustrated in Figures 3(1), 3(2) and 3(3).
We will show that in each of these cases, the intersection is such that there is a marker square of \( \tilde{E} \) entirely contained within the frame of \( \tilde{D} \), but not intersecting the core of \( \tilde{D} \). This will provide the required contradiction.

In cases (1) and (2), we assume (without loss of generality) that the top of \( \tilde{E} \) lies above the bottom of the core of \( \tilde{D} \) and the left side of \( \tilde{E} \) lies to the left of the right of the core of \( \tilde{D} \). This is as shown in Figure 4. The top left middle marker square of \( \tilde{E} \) is then completely contained within the frame of \( \tilde{D} \), but does not intersect the core of \( \tilde{D} \).

The case (3) is similar. A typical configuration is shown in Figure 5. Here (assuming again that the top left corner of \( \tilde{E} \) lies in the top left quadrant of \( \tilde{D} \)), we see that both of the marker squares lying in the top left quadrant of \( \tilde{E} \) are entirely within the frame of \( \tilde{D} \). Since the gap between the marker squares is greater than \( 2n - 1 \), it follows that at least one of them does not intersect the core of \( \tilde{D} \).
The marker square in question now occurs twice in $\tilde{D}$ with neither occurrence overlapping the core. It follows that the marker square in question occurs twice in $\tilde{B}$ and this gives the required contradiction. \qed

Proof of Theorem 2.9. Let $X$ be a subshift of finite type with positive topological entropy. Then by Proposition 2.8, $X$ has a subshift $X_0$ with the property that $h(X_0) = h(X)$ and that $X_0$ is entropy minimal. That is if $Y$ is any proper subshift of $X_0$, then $h(Y) < h(X_0) = h(X)$.

Since any subshift is expansive, it follows that the set of measures of maximal entropy is a non-empty compact convex set. Let $\mu$ be any ergodic measure of maximal entropy on $X_0$. Then $\mu$ may also be regarded as an invariant measure on $X$. It is also a measure of maximal entropy on $X$. The support of $\mu$ is $X_0$ (as otherwise the support of $\mu$ is a subshift of $X_0$ of entropy at least $h_{\mu}(X_0) = h(X_0)$ by the variational principle).

Now by Lemma 2.1, since $h_{\mu}(X_0) > 0$, there exist $n$-blocks $B$ and $C$ of positive measure with the property that $\partial B = \partial C$ and $\mu[B] > 0$. We now apply Lemma 2.12 to get an extension $\tilde{B}$ of $B$ with the required properties. We then define $\tilde{C} = \tilde{B}_{\lambda(2N-1)} \setminus \tilde{B}_{\lambda(2n-1)} C_{\lambda(2n-1)}$. The block $\tilde{C}$ is an allowed $N$-block because all the adjacent pairs outside the core are allowed as they occur in $\tilde{B}$; the adjacent pairs in $\tilde{C}$ consisting of one symbol from the core and one from outside the core are allowed as they occur in $\tilde{B}$ (because $\partial B = \partial C$); and the adjacent pairs in the core of $\tilde{C}$ are allowed as they occur in $C$. Since $\partial B = \partial \tilde{C}$ and $\tilde{C}$ is an allowed block, it follows from Lemma 2.11 that $\mu[\tilde{C}] = \mu[\tilde{B}] > 0$.

Next, we show that $X \setminus \tilde{C}$ is non-empty. We note that now if we take a configuration $x$ in $X$ and replace $\tilde{C}$s with $\tilde{B}$s, then we no longer have the difficulty which was present in the context of Lemma 2.2. Changing a $\tilde{C}$ to a $\tilde{B}$ cannot produce any new $\tilde{C}$s because if it did, the new $\tilde{C}$ would have to overlap the core of the replacement $\tilde{B}$ which contradicts Lemma 2.12. It follows that we can take a point $x \in X$ and simultaneously replace all $\tilde{C}$ blocks by $\tilde{B}$ blocks to get a new point $x' \in X$ with no $\tilde{C}$s, but $\tilde{B}$s in each of the places where $\tilde{C}$ previously occurred. It follows that $X \setminus \tilde{C}$ is non-empty.

This argument can be modified to show that $X_0 \setminus \tilde{C}$ is non-empty as follows: Since the support of $\mu$ is $X_0$, for every $x \in X_0$ we have $\mu[x|_{\lambda(2k-1)}] > 0$ for each $k$. By Lemma 2.11, replacing the $\tilde{C}$s by $\tilde{B}$s in $x|_{\lambda(2k-1)}$ gives a point $x'$ such that $\mu[x'|_{\lambda(2k-1)}] > 0$. Since $\mu$ is concentrated on $X_0$, it follows that $x' \in X_0$.

Since $\mu[\tilde{C}] > 0$, $X_0 \setminus \tilde{C}$ is a proper subshift of $X_0$. It follows from the entropy minimality of $X_0$ that $h(X_0 \setminus \tilde{C}) < h(X)$.

We now seek a lower bound for the entropy of $X_0 \setminus \tilde{C}$ and will then produce a subshift of finite type which is a proper subshift of $X$ which has the required entropy properties. We will use in this part certain properties of measure-theoretic entropy. The relevant material is contained (in the one-dimensional case) in [8]. The proofs are the same in higher dimensions.

To this end, let $S = \bigcup_{a \in \Lambda(2n-1)} \sigma^{-a}([\tilde{B}] \cup [\tilde{C}])$. We then define two partitions of $X_0$ as follows: $P_1 = \{[\tilde{B}] \subset S'; i \in \mathcal{A}\} \cup \{S\}$ and $P_2 = \{[[\tilde{B}],[\tilde{C}],[X \setminus (\{[\tilde{B}] \cup [\tilde{C}]\})]\}$. Then we see that $P = P_1 \vee P_2$ is a generating partition as follows: If we know which element of $P$ each of $\sigma_a(x)$ lies in for $a$ running over $\Lambda(2N - 1)$, then we know $x_0$. To see this, note that if $x_0$ lies inside the core of a $\tilde{C}$ or $\tilde{B}$ occurrence, then knowing the element of $P_2$ partition which the centralized copy of $x$ lies in would give us a point of $X_0 \setminus \tilde{C}$.
in tells us whether it is in a $\tilde{B}$ or a $\tilde{C}$ and this allows us to determine the value of $x_0$. If $x_0$ does not lie inside the core of a $\tilde{C}$ or $\tilde{B}$ occurrence, then knowing the element of the $P_1$ partition in which $x$ lies tells us the value of $x_0$. It follows that the partition $P$ generates the Borel $\sigma$-algebra of $X_0$. Putting this together, we have

$$h(X) = h(X_0) = h_\mu(X_0) = h_\mu(X_0, P_1 \lor P_2)$$
$$\leq h_\mu(X_0, P_1) + H_\mu(P_2|P_1)$$
$$\leq h_\mu(X_0, P_1) + H_\mu(P_2).$$

This yields $h_\mu(X_0, P_1) \geq h(X) - H_\mu(P_2)$. Write $\theta$ for the map sending $X_0$ to $X_0 \setminus \tilde{C}$ by replacing each occurrence of $\tilde{C}$ with a $\tilde{B}$ (simultaneously). Then we see that $\theta$ is a conjugacy between the systems $(X_0, \bigvee_{n \in \mathbb{Z}^2} \sigma_\mathbf{a} P_1, \mu)$ and $(X_0 \setminus \tilde{C}, B, \mu \circ \theta^{-1})$ where $B$ is the Borel $\sigma$-algebra of $X_0 \setminus \tilde{C}$. It follows that

$$h(X_0 \setminus \tilde{C}) \geq h_{\mu \circ \theta^{-1}}(X_0 \setminus \tilde{C}) = h_\mu(X_0, P_1)$$
$$\geq h(X) - H_\mu(P_2).$$

Now let $\epsilon > 0$ be given. Since $H_\mu(P_2) = -2\mu[\tilde{C}] \log(\mu[\tilde{C}]) - (1 - 2\mu[\tilde{C}]) \log(1 - 2\mu[\tilde{C}])$, we may ensure that $H_\mu(P_2) < \epsilon$ if we can ensure that $\tilde{C}$ has arbitrarily small measure. But $\tilde{C}$ had the property that any two occurrences must be at least $N + n + 1$ apart, so it follows that $\mu[\tilde{C}] \leq (N + n + 1)^{-2}$. In particular by ensuring that $N$ is sufficiently large, $H_\mu(P_2)$ may be made less than $\epsilon$. We now have $h(X) - \epsilon < h(X \setminus \tilde{C}) < h(X)$. There exists a decreasing sequence of subshifts $Y_n$ of $X$ which are shifts of finite type and satisfy $X_0 \setminus \tilde{C} = \bigcap_{n=1}^{\infty} Y_n$. It is then known (see [5], prop. 4.4.6) that $h(Y_n) \to h(X_0 \setminus \tilde{C})$. In particular, there exists an $n$ such that $h(X_0 \setminus \tilde{C}) \leq h(Y_n) < h(X)$. This completes the proof. \qed

Remark With $\tilde{C}$ and $\tilde{B}$ defined as above, changing a $\tilde{C}$ to a $\tilde{B}$ in a configuration $x$ neither creates nor destroys any existing $\tilde{C}$s. Formally if $\tilde{C}$ occurs in $x$ with left corner at $\mathbf{a}$ and if $y$ is the corresponding point with $\tilde{C}$ replaced by $\tilde{B}$, then for each $\mathbf{b} \in \mathbb{Z}^2 \setminus \{\mathbf{a}\}$, $x$ has a $\tilde{C}$ at $\mathbf{b}$ if and only if $y$ has a $\tilde{C}$ at $\mathbf{b}$.

Corollary 2.13.

If $X$ is a shift of finite type and $h(X) > 0$, then $X$ has a subshift $Y$ which is not of finite type.

Proof. Let $\tilde{C}$ and $\tilde{B}$ be as in Theorem 2.9. Then let $Y$ be the subset of $X$ consisting of those points of $X$ in which $\tilde{C}$ occurs at most once. Then $Y$ is shift-invariant, non-empty (it contains $X \setminus \tilde{C}$) and is closed (since the limit of points containing no more than one $\tilde{C}$ contains no more than one $\tilde{C}$). To see that $Y$ is not of finite type, observe that for each $N$, there is a point $x$ of $X$ containing exactly two $\tilde{C}$s, separated by at least $N$ (this follows from the above remark). Then all $N$-blocks in $x$ are allowed $N$-blocks in $Y$, but $x$ does not belong to $Y$. \qed
3. Factor Maps and Diamonds

In this section we discuss continuous, shift-commuting maps between shift spaces, and generalize some known results for one-dimensional shifts (see [4], Chapters 8-10).

Suppose that $X$ and $Y$ are $d$-dimensional shift spaces. Let $k$ be a positive integer, and $g: \mathcal{B}_{2k-1}(X) \to \mathcal{B}_1(Y)$ a map on finite blocks. A sliding block code (or simply a code) is a map $f: X \to Y$ defined by $f(x)_a = g(x_{a+\lambda(2k-1)})$ for $x \in X$. If $k = 0$, we say that $f$ is a one-block code.

By an easy generalization of the Curtis-Hedlund-Lyndon Theorem, any continuous map $f: X \to Y$ such that $f \sigma_a = \sigma_a f$, for all $a \in \mathbb{Z}^d$, is a sliding block code for some $k$ (see [5], Theorem 6.2.9 for a proof in dimension one). Any sliding block code can be recoded to a one-block map; that is, there is a shift space $\hat{X}$, a one-block code $\hat{f}: \hat{X} \to Y$ and a conjugacy $\alpha: X \to \hat{X}$ such that $\hat{f}\alpha = f$. See [5], Prop. 1.5.12 for a proof in dimension one. A surjective code is also known in the literature as a factor map.

If $f: X \to Y$ is a code between shift spaces, a diamond for $f$ is a pair of points $x, y \in X$ such that there is a finite subset $S \subseteq \mathbb{Z}^d$, with $f(x) = f(y)$ and $x_S \neq y_S$, $x_a = y_a$ for $a \in \mathbb{Z}^d \setminus S$. If no such pair exists, we say that $f$ has no diamonds. If $X$ is a shift of finite type and $f$ is a one-block code, this definition is equivalent to saying that there is a pair of allowed blocks $B \neq C$, with $\partial(B) = \partial(C)$, such that $f(B) = f(C)$.

**Proposition 3.1.** (see [5], Theorem 8.1.16). Let $f: X \to Y$ be a code, where $X$ is an irreducible shift of finite type. If $f$ is countable-to-one, then it has no diamonds.

**Proof.** We may assume that $f$ is a one-block code. Suppose $f$ has a diamond, so that there is a pair of allowed $k$-blocks $B, C$, with $\partial(B) = \partial(C)$, such that $f(B) = f(C)$. Since $X$ is irreducible, there exists a point in which $B$ occurs infinitely often, in non-overlapping coordinates. Since the block $B$ can be replaced by $C$ wherever it occurs, and $f(B) = f(C)$, we obtain an uncountable collection in $f^{-1}(y)$. $\square$

For one-dimensional shift spaces (not necessarily shifts of finite type), the converse is true, and in fact $f$ must be uniformly finite-to-one ([5], Theorem 8.1.16). But for higher dimensional shifts, the converse is false. For example, if $X$ is the three-dot system (see [3] for a definition), and $f$ collapses all points in $X$ to a single fixed point, then $f$ is uncountable-to-one, since $X$ is uncountable. However, $f$ has no diamonds, since no two distinct $N$-blocks for $X$ can have the same border.

**Theorem 3.2.** (see [5], Theorem 8.1.16). Let $X$ be a shift of finite type, $Y$ be a shift space and $f: X \to Y$ be a surjective code. If $f$ has no diamonds, then $h(X) = h(Y)$. If $X$ is entropy minimal, the converse holds.

**Proof.** We may assume that $f$ is a one-block code. Suppose that $f$ has no diamonds. Let $\mathcal{A}_X$ denote the alphabet of $X$. Then for any $B \in \mathcal{B}_k(Y)$, we have $|f^{-1}(B)| \leq |\mathcal{A}_X|^{2^d k^{d-1}}$, since $|\partial(A(k))| \leq 2^d k^{d-1}$ and no two preimages of $B$ can share the same border. It follows that

$$|\mathcal{B}_k(X)| \leq |\mathcal{A}_X|^{2^d k^{d-1}} |\mathcal{B}_k(Y)|.$$ 

Therefore

$$\frac{1}{kd} \log |\mathcal{B}_k(X)| \leq \frac{1}{kd} \log |\mathcal{A}_X|^{2^d k^{d-1}} |\mathcal{B}_k(Y)| = \frac{2^d}{k} \log |\mathcal{A}_X| + \frac{1}{kd} \log |\mathcal{B}_k(Y)|.$$
Since $2^d \log |A_X|$ is constant, the first term in the sum on the right tends to 0 as $k \to \infty$, and so $h(X) = h(Y)$.

Now assume that $X$ is an entropy minimal shift of finite type. Suppose $f$ has a diamond $B, C$, where $B$ and $C$ are $N$-blocks. Then $f$ restricts to a map $X \setminus C \to Y$, and we claim that the restriction is still onto. To see this, observe that if $f(x) = y$, and we change any occurrence of $C$ in $x$ to $B$, the resulting point still maps to $y$. Let $y \in Y$. Since $f$ is surjective, there exists $x \in f^{-1}(y)$. Now, following the proof of Lemma 2.2, for any positive integer $N$ we may replace all occurrences of $C$ in $x_{N-1}^{2N-1}$ with $B$, so that the resulting point $x^N \in f^{-1}(y)$, and $x^N_{N-1}$ contains no occurrences of $C$. Now, by compactness, choose a limit point $x'$ of the sequence $x^N$. Then $x' \in X \setminus C$ and $f(x') = y$. Therefore $f|_{X \setminus C}$ is surjective.

It follows that $h(X \setminus C) \geq h(Y)$. Since $X$ is entropy minimal, we have $h(X) > h(X \setminus C) \geq h(Y)$. Therefore, $h(Y^N) \leq h(X^N \setminus C) < h(X^N)$, and so $h(Y) < h(X)$. □

If $f : X \to Y$ is a surjective, entropy preserving code (i.e. $h(X) = h(Y)$), and $\nu$ is a measure of maximal entropy for $Y$, then there exists an invariant measure $\mu$ for $X$ such that $\hat{f} \mu = \nu$, where $\hat{f}$ is the induced map on measures (see [2], Theorem 1.1). It is easy to see that $\mu$ is a measure of maximal entropy for $X$. Consequently, an entropy preserving code is measure preserving for some measures of maximal entropy for $X$ and $Y$. If $X$ and $Y$ are one-dimensional irreducible shifts of finite type, they have unique measures of maximal entropy, which are preserved by $f$ ([7]and [2]). But in higher dimensions there may be more than one measure of maximal entropy ([1]).

**Lemma 3.3.** Let $X$ be a shift of finite type, $Y$ be a shift space and $f : X \to Y$ be a surjective code. Suppose that $Y$ is entropy minimal, $f$ has no diamonds and $h(X) = h(Y)$. Then $f$ is onto.

**Proof.** Assume that $f$ is a one-block code. We have $h(X) = h(f(X))$, by Theorem 3.2. Therefore $h(f(X)) = h(Y)$. Since $Y$ is entropy minimal, we must have $f(X) = Y$. □

The following is a generalization of [5], Corollary 8.1.20], in which $f$ having no diamonds replaces $f$ being finite-to-one.

**Theorem 3.4.** Let $f : X \to Y$ be a code between entropy minimal shifts of finite type. Then any two of the following statements implies the third.

1. $f$ has no diamonds.
2. $f$ is surjective.
3. $h(X) = h(Y)$.

**Proof.** The fact that (1) and (2) implies (3), and that (2) and (3) imply (1), follows from Theorem 3.2. That (1) and (3) imply (2) follows from Lemma 3.3. □

Sliding block codes are important in the classification of shifts of finite type up to finite equivalence (see [5], Section 8.3). Two shift spaces are finitely equivalent if there is a shift of finite type which is a common finite-to-one extension of both of them ([5], Def. 8.3.1). It is known that one-dimensional shifts of finite type are finitely equivalent if and only if they have the same entropy ([5], Theorem 8.3.7). In higher dimensions this is false: for example, no infinite zero-entropy shift of finite type, such as the three-dot system, can be finitely equivalent to a shift consisting...
of a finite periodic orbit. So we ask, under what conditions are two shift spaces finitely equivalent? We can define a weaker equivalence relation by saying that two shift spaces $X_1$ and $X_2$ are equal-entropy equivalent if there is a shift of finite type $W$ and surjective codes $\phi_1 : W \to X_1$ and $\phi_2 : W \to X_2$ which preserve entropy. We conclude by asking the following question: if $X_1$ and $X_2$ are shifts of finite type, with $h(X_1) = h(X_2)$, are they equal-entropy equivalent?

References


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