Betti Table Stabilization of Homogeneous Monomial Ideals

Aaron Slobodin (Research Advisor: Dr. Sarah Mayes-Tang)

Quest University Canada

Introduction

Let $R = \Bbbk[x_1, ..., x_n]$, where $\Bbbk$ is an algebraically closed field of characteristic zero. Let $I \subseteq R$ be a homogeneous monomial ideal. We investigate the asymptotic behaviour of the Betti tables shapes of I^d as we vary d. We build off Whieldon's definition of the stabilization index of I, $\text{Stab}(I)$, to define the stabilization sequence of I, $\text{StabSeq}(I)$.

There is little known about the relationship between the Betti tables $\beta(I^d)$ of I^d as we vary d. Elena Guardo and Adam Van Tuyl compute the Betti numbers of homogeneous complete intersection ideals, in [GT05]. In [Whi14], Whieldon proved that the shapes of the Betti tables of equigenerated ideals will stabilize:

Theorem 1. (Theorem 4.1 in [Whi14]) Let $I = (f_0, f_1, ..., f_k) \subseteq R$ be an equigenerated ideal of degree r. Then there exists a D such that for all $d \geq D$, we have

$$\beta_{i,j+rd}(I^d) \neq 0 \iff \beta_{i,j+rD}(I^D) \neq 0.$$

Betti Table Shape

We say that the Betti tables of I^x and I^y share the same *shape* if there exits an integer r such that, for all i and j,

$$\beta_{i,j+rx}(I^x) \neq 0 \iff \beta_{i,j+ry}(I^y) \neq 0.$$

Example 1. Let $I = (x_1x_2x_3x_4, x_2^4, x_1x_4^3) \subseteq R$. If we consider the Betti tables of the first few powers of I,

I^1

-	0	1	2
total:	3	3	1
4:	3	·	·
5:	·	1	·
6:	·	1	·
7:	·	1	1

I^2

-	0	1	2
total:	6	9	4
8:	6	·	·
9:	·	3	·
10:	·	4	2
11:	·	2	2

I^3

-	0	1	2
total:	10	18	9
12:	10	·	·
13:	·	6	·
14:	·	9	6
15:	·	3	3

we see two distinct Betti table shapes, first expressed in the Betti tables of I^1 and I^2.

Stabilization Index

Definition 1. (Definition 5.1 in [Whi14]) Let $I \subseteq R$ be an equigenerated ideal of degree r. Let the *stabilization index Stab*(I) of I be the smallest D such that for all $d \geq D$,

$$\beta_{i,j+rd}(I^d) \neq 0 \iff \beta_{i,j+rD}(I^D) \neq 0.$$

Example 2. One can test higher powers of I, from Example 1, and see that it appear that $\beta_{i,j+4(2)}(I^2) \neq 0 \iff \beta_{i,j+4(d)}(I^d) \neq 0$, for all i and j, and $d \geq 2$. Given Theorem 1, we predict that $\text{Stab}(I) = 2$.

Stabilization Sequence

Definition 2. Let $I \subseteq R$. Let the *stabilization sequence StabSeq*(I) of I be the sequence of powers for which we see new shapes of the Betti tables of I.

$$\text{StabSeq}(I) = \left\{ d : \begin{array}{c} I^d \text{ does not share the same Betti table} \\ \text{shape as } I^{d-1}, d \in \mathbb{Z}^+ \end{array} \right\}$$

Example 3. In Example 1 we saw two distinct Betti table shapes, first expressed at I^1 and I^2. Therefore $1, 2 \in \text{StabSeq}(I)$ and $3 \notin \text{StabSeq}(I)$. We predict that $\text{StabSeq}(I) = \{1, 2\}$.

Linearly Connected Family of Ideals

Recall we may write a monomial in R as $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$.

Definition 3. A collection of monomial ideals $\{I_j\} \subseteq R$ is a *linearly connected family* if there exists linear functions $\alpha_k^i(j)$ such that

$$I_j = (x^{\alpha^1(j)}, x^{\alpha^2(j)}, ..., x^{\alpha^l(j)})$$

where $\alpha^i(j) = (\alpha_1^i(j), \alpha_2^i(j), ..., \alpha_n^i(j))$ for $1 \leq i \leq l$ and $j \in \mathbb{Z}^+$.

Example 4. The ideal $\{I_b\} = (x_1^b x_2^{3b}, x_2^{4b}, x_1^{2b} x_3^{2b}) \subseteq R$ is a *linearly connected family* as the powers of the variables of the generators of I are all linear functions dependent on b. The $\alpha^i(b)$ are listed below

$$\alpha^1(b) = (b, 3b, 0), \qquad \alpha^2(b) = (0, 4b, 0), \qquad \alpha^3(b) = (2b, 0, 2b).$$

Main Results

We will now discuss the linearly connected family of homogeneous ideals $\{I_n\}$, where

$$I_n = (a^{2n}b^{2n}c^{2n}, b^{4n}c^{2n}, a^{3n}c^{3n}, a^{6n-1}b) \subseteq \Bbbk[a, b, c],$$

as we vary n and the powers of I_n. Below we have listed the stabilization sequences of I_n for $1 \leq n \leq 6$, determined by testing the shapes of the Betti tables of I_n^d for $d \leq 100$.

$$\text{StabSeq}(I_1) = \{1, 2, 6\}$$

$$\text{StabSeq}(I_2) = \{1, 2, 3, 5, 6, 11\}$$

$$\text{StabSeq}(I_3) = \{1, 2, 3, 5, 6, 11, 17, 23\}$$

$$\text{StabSeq}(I_4) = \{1, 2, 3, 5, 6, 11, 17, 23, 29, 35\}$$

$$\text{StabSeq}(I_5) = \{1, 2, 3, 5, 6, 11, 17, 23, 29, 35, 41, 47\}$$

$$\text{StabSeq}(I_6) = \{1, 2, 3, 5, 6, 11, 17, 23, 29, 35, 41, 47, 53, 59\}$$

We had no reason to expect that the stabilization indexes or sequences of a linearly connected family of ideals should have any relation. Yet we see that the stabilization index of I_n, for $n \geq 2$, appears to be given by the linear function

$$\text{Stab}(I_n) = 12n - 13$$

and the stabilization sequence of I_n appears to given by

$$\text{StabSeq}(I_n) = \text{StabSeq}(I_{n-1}) \cup \{12n - 13, 12n - 19\}.$$

Future Research

The unexpected structure of the stabilization indexes and sequences found in the linearly related family of homogeneous ideals above presents a wealth of questions for future research. Below we list some questions which we believe to be fruitful avenues for future research.

Question 1. Do all linearly related family of homogeneous ideals exhibit structure in their stabilization indexes and sequences? If so, in what ways can we characterize this structure?

Question 2. What sorts of functions are the stabilization indexes and sequences of other linearly related family of homogeneous ideals dependent upon? Are these functions always linear?

Question 3. Is there a formula for $\text{Stab}(I)$ and $\text{StabSeq}(I)$ for certain classes of ideals? If so what characteristics of the ideals does it depend upon?

References

[GS] Daniel R. Grayson and Michael E. Stillman, *Macaulay2, a software system for research in algebraic geometry*, Available at http://www.math.uiuc.edu/Macaulay2/.

[GT05] Elena Guardo and Adam Van Tuyl, *Powers of Complete Intersections: Graded Betti numbers and Applications*, Illinois Journal of Mathematics **49** (2005), no. 1, 265–279, http://arxiv.org/abs/math/0409090v2.

[Pee11] Irena Peeva, *Graded Free Resolutions*, pp. 1–158, Springer London, London, 2011, http://dx.doi.org/10.1007/978-0-85729-177-6_1.

[Whi14] Gwyneth Whieldon, *Stabilization of Betti tables*, Journal of Commutative Algebra **6** (2014), no. 1, 113–126, https://arxiv.org/abs/1106.2355.

Acknowledgements

I would like to thank the Quest Summer Fellows Committee for providing me the funding and opportunity for this research and Dr. Sarah Mayes-Tang for her support as my faculty advisor. Calculations were performed using the computer software Macaulay2 [GS].

Contact Author: aaron.slobodin@questu.ca