2-Limited Broadcast Domination in Grid Graphs

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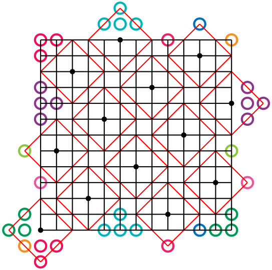
(with G. MacGillivray, W. Myrvold, & F. Ruskey)



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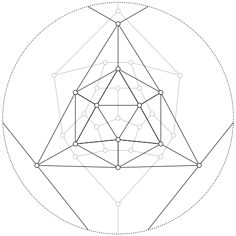
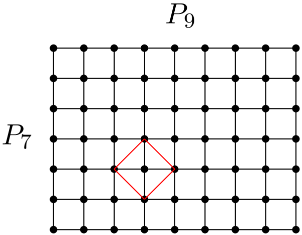


Warm Up

Let $G = (V, E)$ be a graph.



Suppose there is a **transmitter** located at each vertex in G and each transmitter can **broadcast** at strength 0 or 1.

Warm Up

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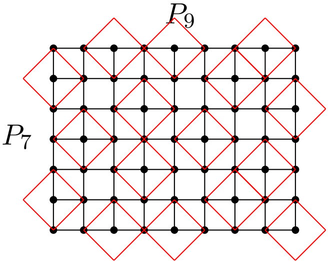


Suppose there is a **transmitter** located at each vertex in G and each transmitter can **broadcast** at strength 0 or 1.

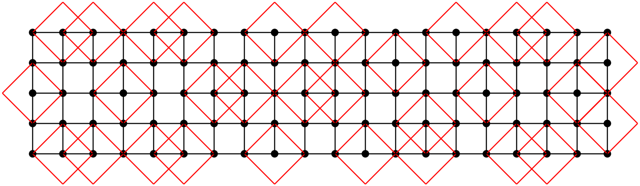
Goal: Assign strength to the transmitters such that every vertex not transmitting is adjacent to one that is.

Result: A dominating 1**-limited broadcast** on G.

Warm Up

1-Limited Broadcast Domination

The **cost** of a 1-limited broadcast is the sum of the strengths assigned to the transmitters.

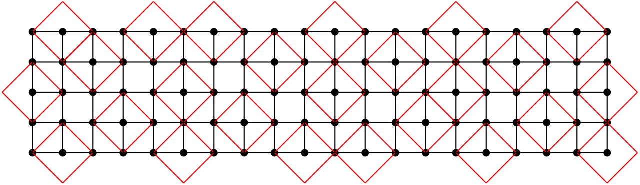


1-limited broadcast on $P_5 \square P_{20}$ of cost: 30

Warm Up

1-Limited Broadcast Domination Number

The **cost** of a 1-limited broadcast is the sum of the strengths assigned to the transmitters.



1-limited broadcast on $P_5 \square P_{20}$ of cost: 25

$\gamma_{b,1}(G)$ is defined as the **least** cost 1-limited broadcast on G.

Observe: $\gamma_{b,1} = \gamma$.

Definition

1-Limited Broadcast Domination Number: $\gamma_{b,1}$

Let $G = (V, E)$ be a graph. For each vertex $i \in V$, let

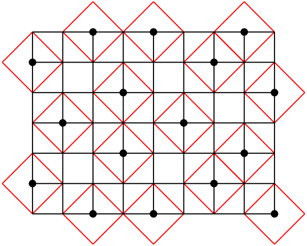
$$x_i = \begin{cases} 1 & \text{if vertex } i \text{ is broadcasting at strength } 1, \\ 0 & \text{otherwise} \end{cases}$$

Formulation of $\gamma_{b,1}(G)$ as an ILP:

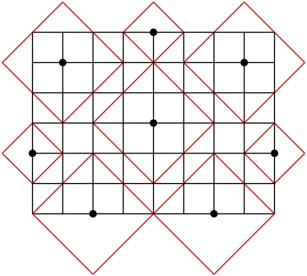
Minimize: $\sum_{i \in V} x_i$

Subject to: $\sum_{d(i,j) \le 1} x_i \ge 1$, for each vertex $j \in V$

2-Limited Broadcast Domination Number: $\gamma_{b,2}$



Optimal 1-limited broadcast on $P_7 \square P_9$
$\gamma_{b,1}(P_7 \square P_9) = 16$



Optimal 2-limited broadcast on $P_7 \square P_9$
$\gamma_{b,2}(P_7 \square P_9) = 13$

Definition

2-Limited Broadcast Domination Number: $\gamma_{b,2}$

Let $G = (V, E)$ be a graph. For each vertex $i \in V$, let

$$x_{i,k} = \begin{cases} 1 & \text{if vertex } i \text{ is broadcasting at strength } k, \\ 0 & \text{otherwise} \end{cases}$$

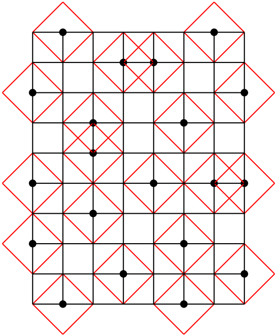
Formulation of $\gamma_{b,2}(G)$ as an ILP:

Minimize: $\sum_{k=1}^{2} \sum_{i \in V} k \cdot x_{i,k}$

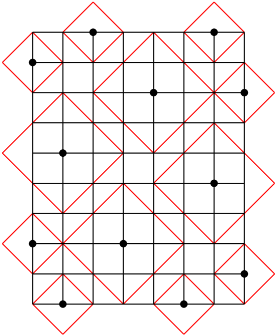
Subject to: $\sum_{d(i,j) \leq k} x_{i,k} \geq 1$, for each vertex $j \in V$

Example

1- vs. 2-Limited Broadcast Domination



Optimal 1-limited broadcast on $P_{10}\Box P_8$.
$\gamma_{b,1}(P_{10}\Box P_8) = 20$.



Optimal 2-limited broadcast on $P_{10}\Box P_8$,
$\gamma_{b,2}(P_{10}\Box P_8) = 16$.

Observe: $\gamma_{b,2} \leq \gamma_{b,1}$.

Previous Results

Broadcast Domination:

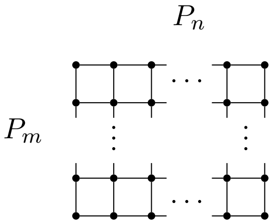
1. Introduced by Erwin in 2001 [Erw01],
2. Known on grids and toroidal grids [BS09],
3. Optimal broadcast domination can be computed in polynomial time: $O(n^6)$ [HL06].

Limited Broadcast Domination:

1. Tight bounds on trees ($\lceil \frac{4n}{9} \rceil$) [CHM$^+$18],
2. Limited broadcast domination is NP-complete [CHM$^+$18],
3. There exist algorithms for 2-limited broadcasts for some graph classes (e.g. trees, interval, strongly chordal) [Yan19].

$\gamma_{b,2}(P_m \square P_n)$

Given $P_m \square P_n$,



Goal: Determine $\gamma_{b,2}(P_m \square P_n)$ for all m and n. (Hard).

Realistic Goal: Determine "good" bounds for $\gamma_{b,2}(P_m \square P_n)$ for all m and n. (Less Hard).

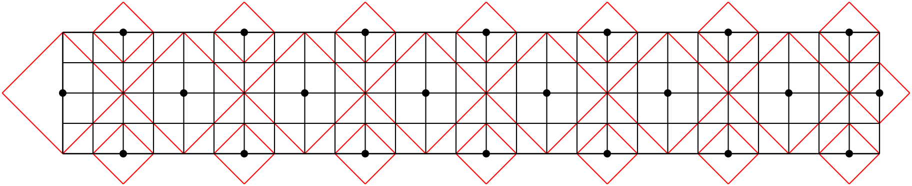
Methodology:

1. Fix m and use an ILP solver to determine $\gamma_{b,2}(P_m \square P_n)$ for small values of n (≤ 50),
2. Look for "patterns" in the broadcast structure,
3. Create general constructions based on these "patterns."

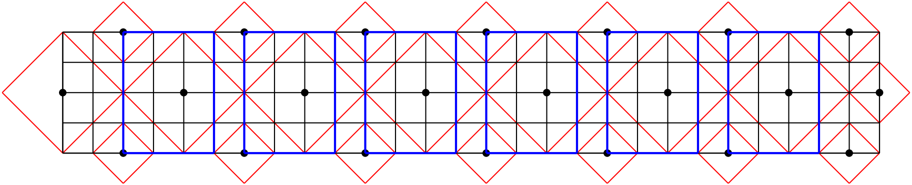
Example: $P_5 \Box P_n$

Optimal 2-Limited Broadcast on $P_5 \Box P_{28}$, cost $= 29$

Step 1:

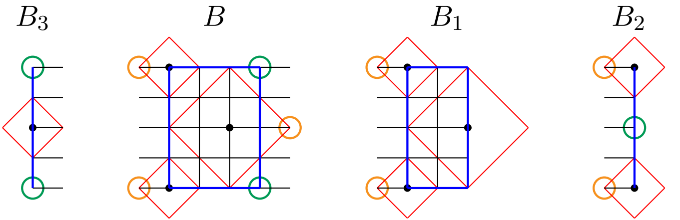


Step 2:



Example: $P_5 \square P_n$

Step 3:



$n \equiv 0 \pmod 4$: $B_3 + \underbrace{B + \cdots + B}_{\frac{n-4}{4}} + B_1$,

$n \equiv 1 \pmod 4$: $\underbrace{B + \cdots + B}_{\frac{n-1}{4}} + B_2$,

$n \equiv 2 \pmod 4$: $B_3 + \underbrace{B + \cdots + B}_{\frac{n-2}{4}} + B_2$,

$n \equiv 3 \pmod 4$: $\underbrace{B + \cdots + B}_{\frac{n-3}{4}} + B_1$.

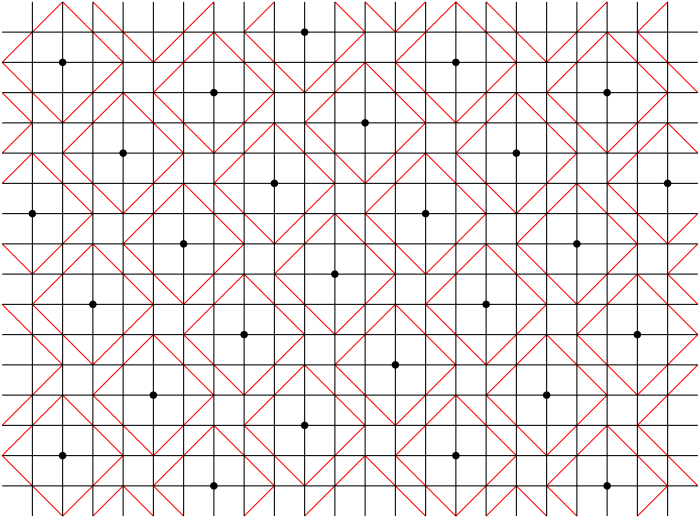
$$\gamma_{b,2}(P_5 \square P_n) \leq n+1$$

2-Limited Broadcasts on $P_m \square P_n$

2-Limited Broadcast #	Upper Bound
$\gamma_{b,2}(P_2 \square P_n)$	$\leq \lceil \frac{n+1}{2} \rceil$
$\gamma_{b,2}(P_3 \square P_n)$	$\leq \lceil \frac{2n}{3} \rceil$
$\gamma_{b,2}(P_4 \square P_n)$	$\leq 8 \left\lfloor \frac{n}{10} \right\rfloor + c_4(n)_{\leq 8}$
$\gamma_{b,2}(P_5 \square P_n)$	$\leq n + 1$
$\gamma_{b,2}(P_6 \square P_n)$	$\leq 18 \lfloor \frac{n}{16} \rfloor + c_6(n)_{\leq 18}$
$\gamma_{b,2}(P_7 \square P_n)$	$\leq 18 \lfloor \frac{n}{14} \rfloor + c_7(n)_{\leq 18}$
$\gamma_{b,2}(P_8 \square P_n)$	$\leq 32 \lfloor \frac{n}{22} \rfloor + c_8(n)_{\leq 32}$
$\gamma_{b,2}(P_9 \square P_n)$	$\leq 16 \lfloor \frac{n}{10} \rfloor + c_9(n)_{\leq 16}$
$\gamma_{b,2}(P_{10} \square P_n)$	$\leq 32 \lfloor \frac{n}{18} \rfloor + c_{10}(n)_{\leq 32}$
$\gamma_{b,2}(P_{11} \square P_n)$	$\leq 50 \lfloor \frac{n}{26} \rfloor + c_{11}(n)_{\leq 50}$
$\gamma_{b,2}(P_{12} \square P_n)$	$\leq 50 \lfloor \frac{n}{24} \rfloor + c_{12}(n)_{\leq 50}$
$\gamma_{b,2}(P_{m \geq 13} \square P_{n \geq 13})$	$\leq ?$

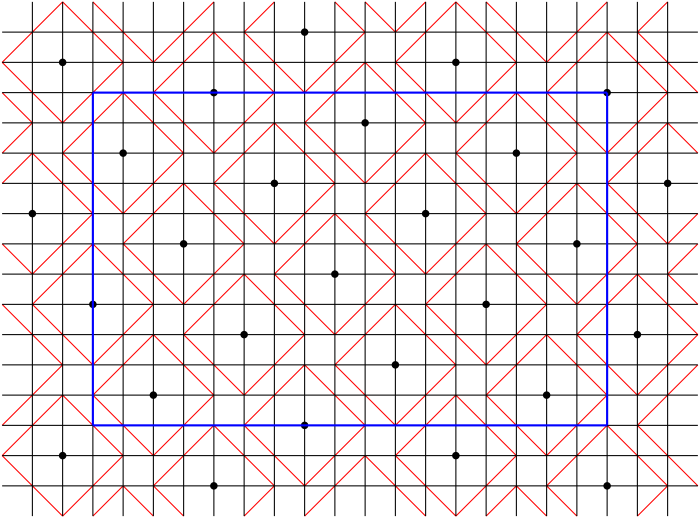
Infinite Plane

Quest for $\gamma_{b,2}\left(P_{m\geq 13}\square P_{n\geq 13}\right)$



Infinite Plane

Quest for $\gamma_{b,2}\left(P_{m\geq 13}\square P_{n\geq 13}\right)$



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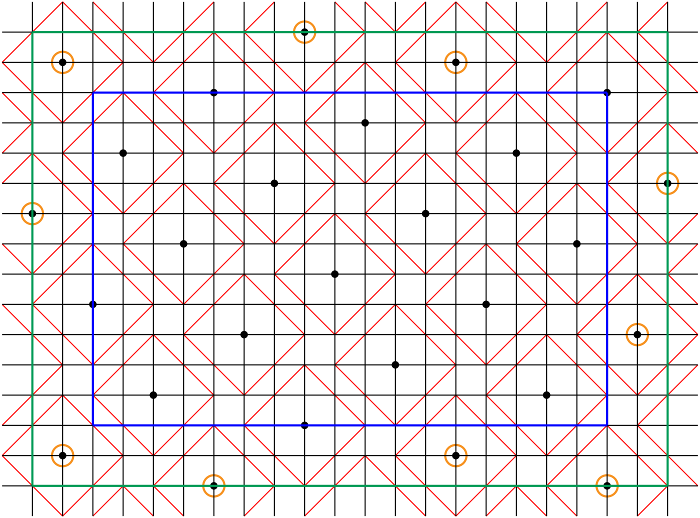

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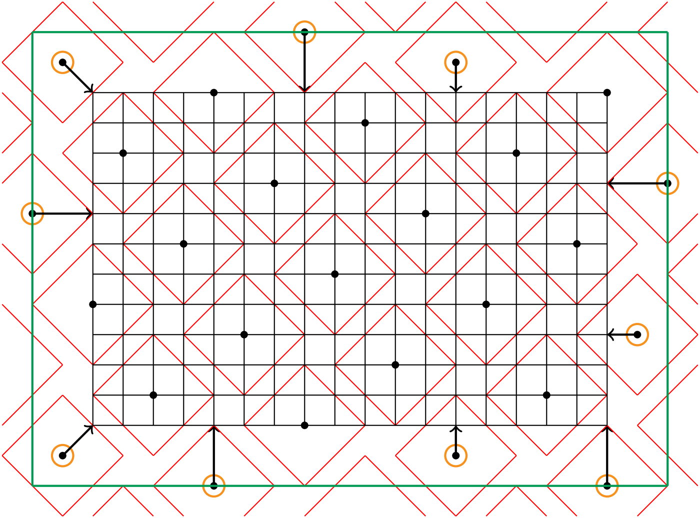
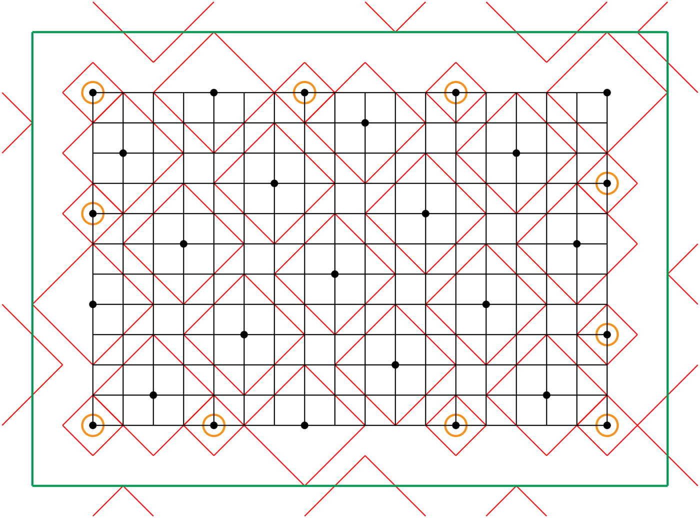


Infinite Plane

Quest for $\gamma_{b,2}\left(P_{m\geq 13}\square P_{n\geq 13}\right)$



2-Limited Broadcasts on $P_m \square P_n$

2-Limited Broadcast #	Upper Bound
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$\gamma_{b,2}(P_4 \square P_n)$	$\leq 8 \left\lfloor \frac{n}{10} \right\rfloor + c_4(n)_{\leq 8}$
$\gamma_{b,2}(P_5 \square P_n)$	$\leq n + 1$
$\gamma_{b,2}(P_6 \square P_n)$	$\leq 18 \lfloor \frac{n}{16} \rfloor + c_6(n)_{\leq 18}$
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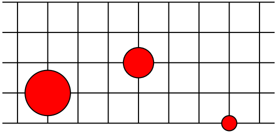
$\gamma_{b,2}(P_{11} \square P_n)$	$\leq 50 \lfloor \frac{n}{26} \rfloor + c_{11}(n)_{\leq 50}$
$\gamma_{b,2}(P_{12} \square P_n)$	$\leq 50 \lfloor \frac{n}{24} \rfloor + c_{12}(n)_{\leq 50}$
$\gamma_{b,2}(P_{m \geq 13} \square P_{n \geq 13})$	$\leq 2 \left(\frac{mn}{13} \right) + 4 \left(\frac{m+n}{13} \right) + c_{13}(n)_{\leq 2}$

The linear programming **dual** of 2-limited broadcast domination is 2-limited multipacking.

Fractional 2-Limited Multipacking

Lower Bounds

Let $G = (V, E)$ be a graph.



Suppose we can place a weight **between** 0 and 1 at each of vertex G.

Fractional 2-Limited Multipacking

Lower Bounds

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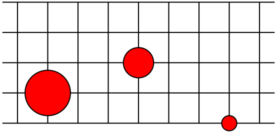


Suppose we can place a weight **between** 0 and 1 at each of vertex G.

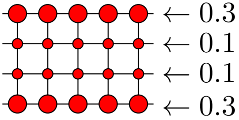
Goal: **Maximize** the sum of the weights such that, for each vertex,

1. the sum of the weights within distance 1 is ≤ 1, and
2. the sum of the weights within distance 2 is ≤ 2.

Result: A **fractional 2-limited multipacking** mp_2 on G.

Fractional 2-Limited Multipacking

Example: $P_4 \square C_5$



Optimal Fractional 2-Limited Multipacking on $P_4 \square C_5$: $mp_2(P_4 \square C_5) = 4$.

Sum of weights within 1 of a vertex on the bottom row:

$$3\,(0.3) + 0.1 = 1.$$

Sum of weights within 2 of a vertex on the bottom row:

$$5(0.3) + 4(0.1) = 1.9 \leq 2.$$

Definition

Fractional 2-Limited Multipacking

Let $G = (V, E)$ be a graph. For each vertex $i \in V$, let

$$y_i = \text{weight at vertex } i.$$

Formulation of $mp_2(G)$ as an LP:

Maximize: $\sum_{i \in V} y_i$

Subject to (1): $\sum_{d(i,j) \leq 1} y_i \leq 1$, for each vertex $j \in V$

(2): $\sum_{d(i,j) \leq 2} y_i \leq 2$, for each vertex $j \in V$

Fractional 2-Limited Multipackings on $P_m \square C_n$

Lower Bounds

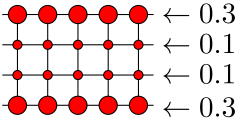
Methodology:

1. Fix m. Compute* a fractional 2-limited multipacking on $P_m \square C_5$ with where we require the weights assigned to each vertex within a row to be equal.
2. Use the weights in each row to construct general fractional 2-limited multipacking on $P_m \square C_n$ for all $n \geq 3$.
3. As $mp_2(P_m \square C_n) \leq \gamma_{b,2}(P_m \square C_n) \leq \gamma_{b,2}(P_m \square P_n)$, this achieves a lower bound.

$*$ using an exact LP solver [GSW15, GSW12] as a part of the SoPlex distribution [GEG+17].

Fractional 2-Limited Multipacking

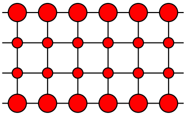
Example: $P_4 \square C_5$



Optimal Fractional 2-Limited Multipacking on $P_4 \square C_5$: $mp_2(P_4 \square C_5) = 4$

Fractional 2-Limited Multipacking

Example: $P_4 \Box C_6$



Optimal Fractional 2-Limited Multipacking on $P_4 \Box C_6$: $mp_2(P_4 \Box C_6) = 4.8$

Let $\mathbf{v} = \begin{bmatrix} 0.3 & 0.1 & 0.1 & 0.3 \end{bmatrix}$ and $\mathbf{J}_n$ be the all ones matrix of size $1 \times n$. The product $\mathbf{v}^T \mathbf{J}_n$ defines a fractional 2-limited multipacking on $P_4 \Box C_n$ for all $n \geq 3$.

The cost of such a multipacking is $(2 \cdot 0.3 + 2 \cdot 0.1)n = \frac{4n}{5}$.

Thus $\frac{4n}{5} \leq mp_2(P_4 \Box C_n) \leq \gamma_{b,2}(P_4 \Box P_n)$.

2-Limited Broadcasts on $P_m \Box P_n$

Lower Bound	2-LB #	Upper Bound
$\left\lceil \frac{n}{2} \right\rceil \leq$	$\gamma_{b,2}(P_2 \Box P_n)$	$\leq \left\lceil \frac{n+1}{2} \right\rceil$
$\left\lceil \frac{2n}{3} \right\rceil \leq$	$\gamma_{b,2}(P_3 \Box P_n)$	$\leq \left\lceil \frac{2n}{3} \right\rceil$
$\left\lceil \frac{4n}{5} \right\rceil \leq$	$\gamma_{b,2}(P_4 \Box P_n)$	$\leq 8 \left\lfloor \frac{n}{10} \right\rfloor + c_4(n)_{\leq 8}$
$\left\lceil \frac{26n}{27} \right\rceil \leq$	$\gamma_{b,2}(P_5 \Box P_n)$	$\leq n + 1$
$17.8 \left(\frac{n}{16} \right) \leq \left\lceil \frac{29n}{26} \right\rceil \leq$	$\gamma_{b,2}(P_6 \Box P_n)$	$\leq 18 \left\lfloor \frac{n}{16} \right\rfloor + c_6(n)_{\leq 18}$
$17.7 \left(\frac{n}{14} \right) \leq \left\lceil \frac{19n}{15} \right\rceil \leq$	$\gamma_{b,2}(P_7 \Box P_n)$	$\leq 18 \left\lfloor \frac{n}{14} \right\rfloor + c_7(n)_{\leq 18}$
$31.3 \left(\frac{n}{22} \right) \leq \left\lceil \frac{212n}{149} \right\rceil \leq$	$\gamma_{b,2}(P_8 \Box P_n)$	$\leq 32 \left\lfloor \frac{n}{22} \right\rfloor + c_8(n)_{\leq 32}$
$15.7 \left(\frac{n}{10} \right) \leq \left\lceil \frac{52n}{33} \right\rceil \leq$	$\gamma_{b,2}(P_9 \Box P_n)$	$\leq 16 \left\lfloor \frac{n}{10} \right\rfloor + c_9(n)_{\leq 16}$
$31.1 \left(\frac{n}{18} \right) \leq \left\lceil \frac{780n}{451} \right\rceil \leq$	$\gamma_{b,2}(P_{10} \Box P_n)$	$\leq 32 \left\lfloor \frac{n}{18} \right\rfloor + c_{10}(n)_{\leq 32}$
$48.9 \left(\frac{n}{26} \right) \leq \left\lceil \frac{81n}{43} \right\rceil \leq$	$\gamma_{b,2}(P_{11} \Box P_n)$	$\leq 50 \left\lfloor \frac{n}{26} \right\rfloor + c_{11}(n)_{\leq 50}$
$48.8 \left(\frac{n}{24} \right) \leq \left\lceil \frac{273n}{134} \right\rceil \leq$	$\gamma_{b,2}(P_{12} \Box P_n)$	$\leq 50 \left\lfloor \frac{n}{24} \right\rfloor + c_{12}(n)_{\leq 50}$
$\frac{2mn}{13} + \frac{2.4}{13}(m+n) \leq$	$\gamma_{b,2}(P_m \Box P_n)$	$\leq \frac{2mn}{13} + \frac{4}{13}(m+n) + c_{13}(n)_{\leq 2}$

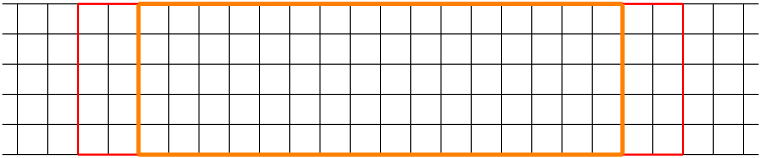
Data: $P_6 \square P_n$

n	Lower Bound	Upper Bound	Gap
$2^4 = 16$	18	20	2
$2^5 = 32$	36	38	2
$2^6 = 64$	72	74	2
$2^7 = 128$	143	146	3
$2^8 = 256$	286	290	4
$2^9 = 512$	572	578	6
$2^{10} = 1024$	1143	1154	11
$2^{11} = 2048$	2285	2306	21
$2^{12} = 4096$	4569	4610	41
$2^{13} = 8192$	9138	9218	80
$2^{14} = 16384$	18275	18434	159
$2^{15} = 32768$	36549	36866	317
$2^{16} = 65536$	73098	73730	632
$2^{17} = 131072$	146196	147458	1262
$2^{18} = 262144$	292392	294914	2522
$2^{19} = 524288$	584783	589826	5043
$2^{20} = 1048576$	1169566	1179650	10084

Future Work

Backtracking and linear programming **magic** to try and improve out lower bounds and yield periodically optimal values.

Given the upper bound: $\gamma_{b,2}(P_6 \Box P_n) \leq 18 \left\lfloor \frac{n}{16} \right\rfloor + O(1)$, if we can show that that in any 16 consecutive columns there must be weight 18, we yield a lower bound of $18 \left\lfloor \frac{n}{16} \right\rfloor$.



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