2-Limited Broadcast Domination in Grid Graphs

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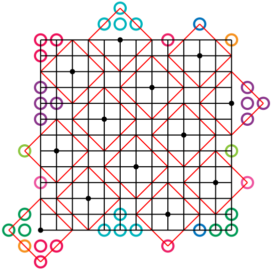
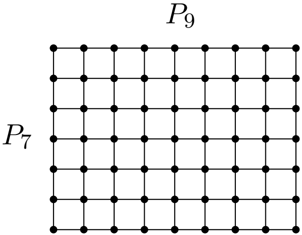


Society for Industrial and Applied Mathematics

Thank you SIAM for the SIAM Travel Award.

Warm Up

Let $G = (V, E)$ be a graph.



Warm Up

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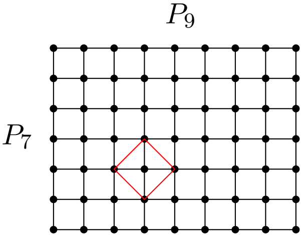
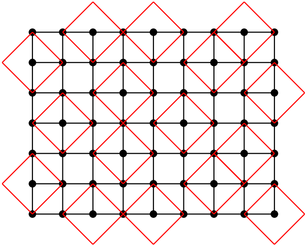


Suppose there is a **transmitter** located at each vertex in G and each transmitter can **broadcast** at strength 0 or 1.

Warm Up

Let $G = (V, E)$ be a graph.

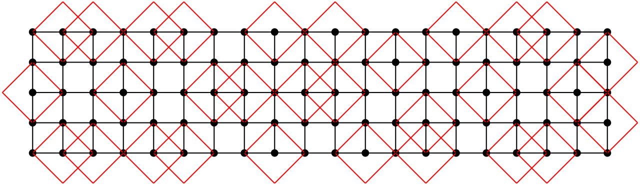


Suppose there is a **transmitter** located at each vertex in G and each transmitter can **broadcast** at strength 0 or 1.

Goal: Assign strength to the transmitters such that every vertex not transmitting is adjacent to one that is.

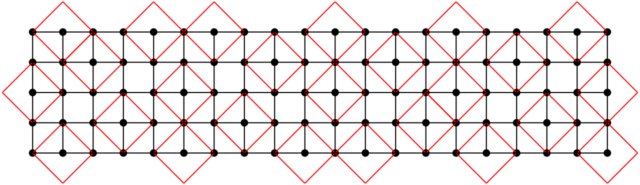
Result: A 1-**limited dominating broadcast** on G.

The **cost** of a 1-limited broadcast is the sum of the strengths assigned to the transmitters.



1-limited dominating broadcast on $P_5 \square P_{20}$ of cost 30

The **cost** of a 1-limited broadcast is the sum of the strengths assigned to the transmitters.



1-limited broadcast on $P_5 \square P_{20}$ of cost 25

$\gamma_{b,1}(G)$ is defined as the **least** cost 1-limited dominating broadcast on the graph G.

Observe: $\gamma_{b,1} = \gamma$.

Definition

1-Limited Broadcast Domination Number: $\gamma_{b,1}$

Let $G = (V, E)$ be a graph. For each vertex $i \in V$, let

$$x_i = \begin{cases} 1 & \text{if vertex } i \text{ is broadcasting at strength 1,} \\ 0 & \text{otherwise} \end{cases}$$

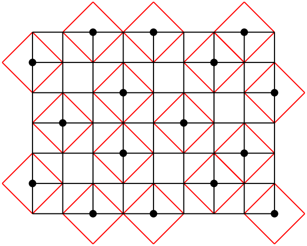
Formulation of $\gamma_{b,1}(G)$ as an ILP:

Minimize: $\displaystyle\sum_{i \in V} x_i$

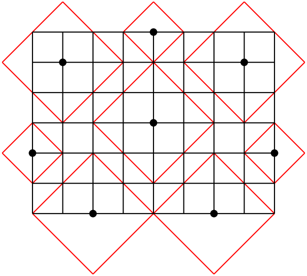
Subject to: $\displaystyle\sum_{d(i,j) \leq 1} x_i \geq 1$, for each vertex $j \in V$

Intuition

1- vs. 2-Limited Broadcast Domination



Optimal 1-limited dominating broadcast on $P_7 \square P_9$.
$\gamma_{b,1}(P_7 \square P_9) = 16$.



Optimal 2-limited dominating broadcast on $P_7 \square P_9$.
$\gamma_{b,2}(P_7 \square P_9) = 13$.

Definition

2-Limited Broadcast Domination Number: $\gamma_{b,2}$

Let $G = (V, E)$ be a graph. For each vertex $i \in V$, let

$$x_{i,k} = \begin{cases} 1 & \text{if vertex } i \text{ is broadcasting at strength } k, \\ 0 & \text{otherwise} \end{cases}$$

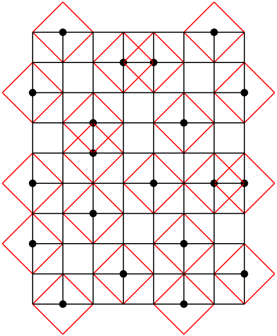
Formulation of $\gamma_{b,2}(G)$ as an ILP:

Minimize: $\displaystyle\sum_{k=1}^{2} \sum_{i \in V} k \cdot x_{i,k}$

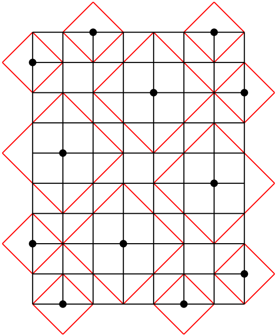
Subject to: $\displaystyle\sum_{d(i,j) \leq k} x_{i,k} \geq 1$, for each vertex $j \in V$

Example

1- vs. 2-Limited Broadcast Domination



Optimal 1-limited dominating broadcast on $P_{10} \square P_8$. $\gamma_{b,1}(P_{10} \square P_8) = 20$.



Optimal 2-limited dominating broadcast on $P_{10} \square P_8$. $\gamma_{b,2}(P_{10} \square P_8) = 16$.

Observe: $\gamma_{b,2} \leq \gamma_{b,1}$.

Previous Results

Broadcast Domination:

1. Introduced by Erwin in 2001 [Erw01],
2. Known on grids and toroidal grids [BS09],
3. Optimal broadcast domination can be computed in polynomial time: $O(n^6)$ [HL06].

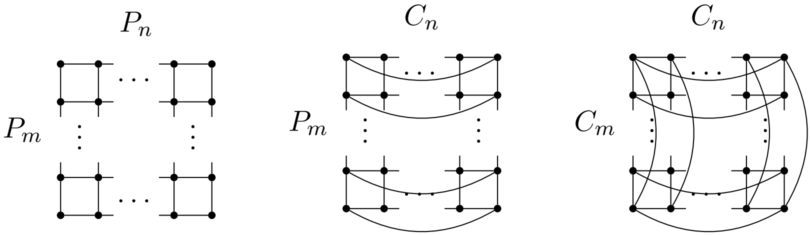
Limited Broadcast Domination:

1. Tight bounds on trees ($\lceil \frac{4n}{9} \rceil$) [CHM$^+$18],
2. Limited broadcast domination is NP-complete [CHM$^+$18],
3. There exist algorithms for 2-limited broadcasts for some graph classes (e.g. trees, interval, strongly chordal) [Yan19].

Grid Graphs

2-Limited Broadcast Domination

Given $P_m \square P_n$, $P_m \square C_n$, and $C_m \square C_n$



Goal: Determine $\gamma_{b,2}(P_m \square P_n)$, $\gamma_{b,2}(P_m \square C_n)$, and $\gamma_{b,2}(C_m \square C_n)$ for all m and n (Hard).

Realistic Goal: Determine "good" bounds for $\gamma_{b,2}$ on said graphs (Less Hard).

Finding Bounds

Sweeping many months of research under the rug in one slide...

Upper Bounds For all m and n, we provide 2-limited dominating broadcasts on $P_m \Box P_n$, $P_m \Box C_n$, and $C_m \Box C_n$.

Lower Bounds: Use the LP dual of the LP relaxation of 2-limited broadcast domination (*fractional 2-limited multipacking*) to establish generalized constructions.

Theorem (S+, 2021)

Lower Bound	2-Limited Broadcast #	Upper Bound
	$\gamma_{b,2}(P_2 \square C_n)$	$= \lceil \frac{n}{2} \rceil$
	$\gamma_{b,2}(P_3 \square C_n)$	$= \lceil \frac{2n}{3} \rceil$
$\lceil 8 \lfloor \frac{n}{10} \rfloor \rceil \leq$	$\gamma_{b,2}(P_4 \square C_n)$	$\leq 8 \left\lfloor \frac{n}{10} \right\rfloor + c_4(n)_{\leq 8}$
$\lceil 7.703 \lfloor \frac{n}{8} \rfloor \rceil \leq$	$\gamma_{b,2}(P_5 \square C_n)$	$\leq n + c_5(n)_{\leq 1}$
$\lceil 17.846 \lfloor \frac{n}{16} \rfloor \rceil \leq$	$\gamma_{b,2}(P_6 \square C_n)$	$\leq 18 \lfloor \frac{n}{16} \rfloor + c_6(n)_{\leq 18}$
$\lceil 16.466 \lfloor \frac{n}{14} \rfloor \rceil \leq$	$\gamma_{b,2}(P_7 \square C_n)$	$\leq 18 \lfloor \frac{n}{14} \rfloor + c_7(n)_{\leq 18}$
$\lceil 31.302 \lfloor \frac{n}{22} \rfloor \rceil \leq$	$\gamma_{b,2}(P_8 \square C_n)$	$\leq 32 \lfloor \frac{n}{22} \rfloor + c_8(n)_{\leq 32}$
$\lceil 15.757 \lfloor \frac{n}{10} \rfloor \rceil \leq$	$\gamma_{b,2}(P_9 \square C_n)$	$\leq 16 \lfloor \frac{n}{10} \rfloor + c_9(n)_{\leq 16}$
$\lceil 31.130 \lfloor \frac{n}{18} \rfloor \rceil \leq$	$\gamma_{b,2}(P_{10} \square C_n)$	$\leq 32 \lfloor \frac{n}{18} \rfloor + c_{10}(n)_{\leq 32}$
$\lceil 48.976 \lfloor \frac{n}{26} \rfloor \rceil \leq$	$\gamma_{b,2}(P_{11} \square C_n)$	$\leq 50 \lfloor \frac{n}{26} \rfloor + c_{11}(n)_{\leq 50}$
$\lceil 48.895 \lfloor \frac{n}{24} \rfloor \rceil \leq$	$\gamma_{b,2}(P_{12} \square C_n)$	$\leq 50 \lfloor \frac{n}{24} \rfloor + c_{12}(n)_{\leq 50}$

$P_5 \Box C_n$

Known Bounds: $\left\lceil 7.703 \left\lfloor \frac{n}{8} \right\rfloor \right\rceil \leq \gamma_{b,2}(P_5 \Box C_n) \leq n + c_5(n)_{\leq 1}$.

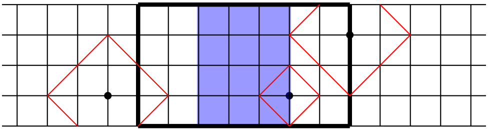
Goal: Prove n as a lower bound.

Methodology:

1. Let n be the least n such that there exists a 2-limited dominating broadcast f on $P_5 \Box C_n$ of cost $\leq n - 1$. Let C be the least cost eight consecutive columns in f.
2. C can have cost at most 7.
3. For each integer $x \leq 7$, consider all possible broadcasts of cost x within C and try to yield a contradiction.

$P_5 \square C_n$

Observations

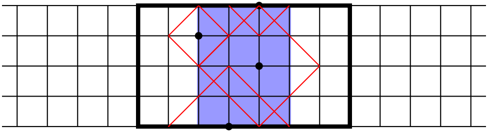


1. Vertices in C and within distance one of the left and right most border of C may be dominated by vertices outside or inside of C.
2. The inner $P_5 \square P_4$ formed by the other vertices of C must be dominated by vertices inside of C.
3. As $\gamma_{b,2}(P_5 \square P_4) = 5$, we need only considers broadcasts of costs $5 \leq x \leq 7$ in C.

$P_5 \Box C_n$

Consider when $\text{cost}(C) = 6$.

By Pólya's Theorem, there are 1,925,104 possible broadcasts (up to isomorphism) of cost 6 in $P_5 \Box P_8$. Of those, 314 dominate the inner columns.



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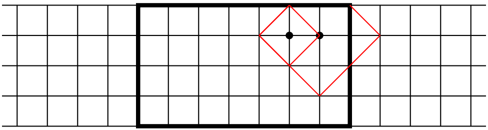
Note: There are trivially bad broadcasts.



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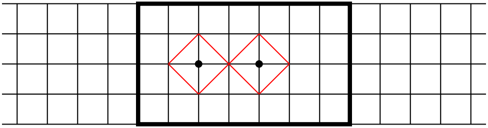

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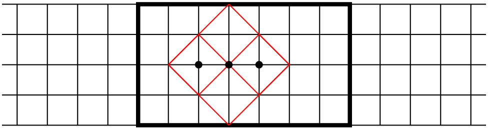

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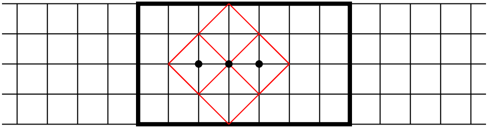

Of those, 226 are not trivially bad.

$P_5 \Box C_n$

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$P_5 \square C_n$
Contradiction Type: 1

The broadcast in C is not optimal.

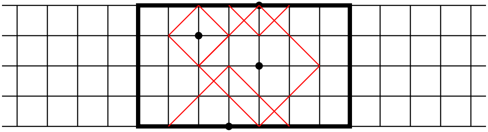

Figure: Broadcast in C of cost 6.

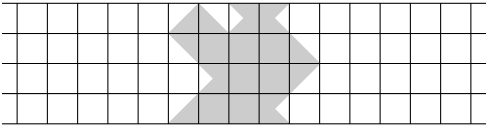

Figure: Corresponding dominated region.

$P_5 \square C_n$

Contradiction Type: 1

The broadcast in C is not optimal.

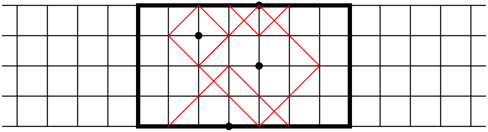

Figure: Broadcast in C of cost 6.

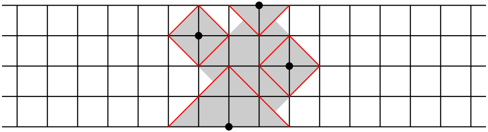

Figure: Broadcast of cost 5 which dominates the same region.

$P_5 \square C_n$

Contradiction Type: 2

Broadcast in C is optimal and forces a unique domination. This unique domination is not optimal.

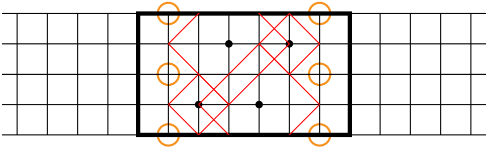

Figure: Broadcast in C of cost 6.

$P_5 \square C_n$

Contradiction Type: 2

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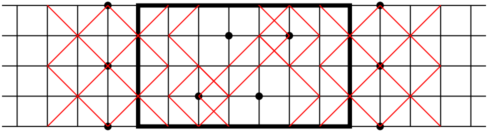

Figure: Broadcast in C and necessary external vertices of cost 18.

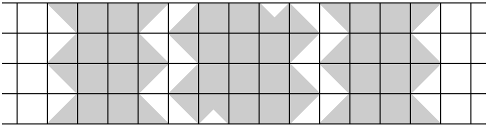

Figure: Corresponding dominated region.

$P_5 \square C_n$

Contradiction Type: 2

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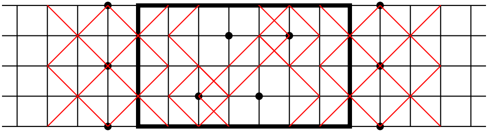


Figure: Broadcast in C and necessary external vertices of cost 18.

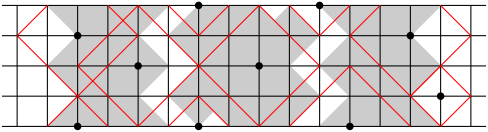


Figure: Broadcast of cost 15 which dominates the same region.

$P_5 \square C_n$

Contradiction Type: 3

Broadcast in C is optimal and does not force a unique domination. Consider all possible subcases to dominate remaining vertices. Subcase is not optimal.

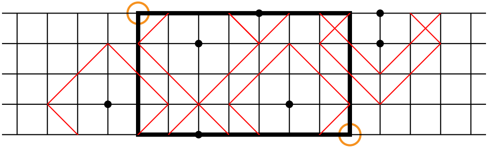


Figure: Broadcast in C and necessary external broadcasters of cost 12.

$P_5 \square C_n$
Contradiction Type: 3

Broadcast in C is optimal and does not force a unique domination. Consider all possible subcases to dominate remaining vertices. Subcase is not optimal.

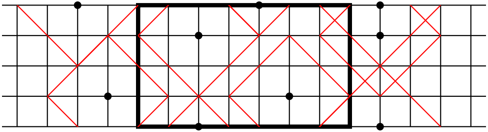


Figure: Broadcast of cost 16.

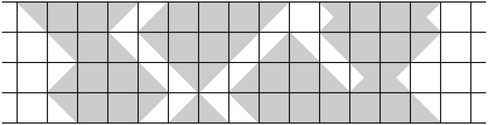


Figure: Corresponding dominated region.

$P_5 \square C_n$

Contradiction Type: 3

Broadcast in C is optimal and does not force a unique domination. Consider all possible subcases to dominate remaining vertices. Subcase is not optimal.

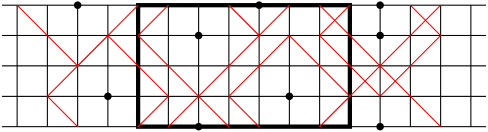


Figure: Broadcast of cost 16.

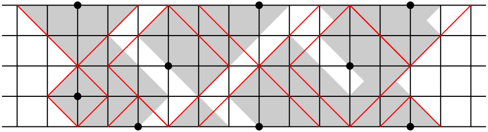


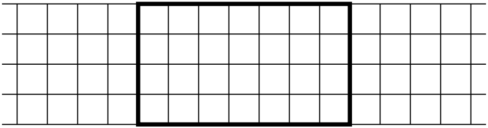
Figure: Broadcast of cost 15 which dominates the same region.

$P_5 \square C_n$

Contradiction Type: 4

Assumed global cost of broadcast is $\leq n-1$. Cost of local broadcast is x. If the region left undominated after deleting y columns can be dominated with cost $c \leq x-y$, there exists a broadcast on $P_5 \square C_{n-y}$ of cost at most

$$(n-1) - x + c \leq n - y - 1 < n - y.$$



$P_5 \square C_n$

Contradiction Type: 4

Assumed global cost of broadcast is $\leq n - 1$. Region left undominated after deleting 6 columns can be dominated with cost 6. There therefore exists a broadcast on $P_5 \square C_{n-6}$ of cost $n - 7$. Contradiction.

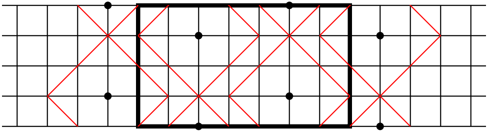


Figure: Broadcast of cost 12.

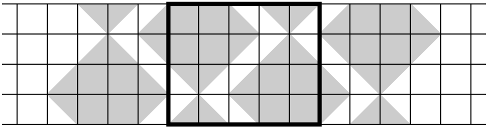


Figure: Corresponding dominated region.

$P_5 \square C_n$

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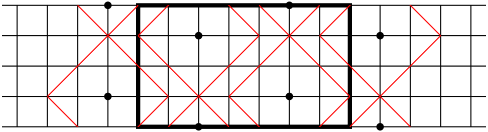

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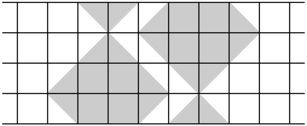

Figure: Corresponding region to be dominated after deleting 6 columns.

$P_5 \square C_n$

Contradiction Type: 4

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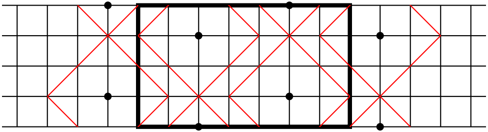

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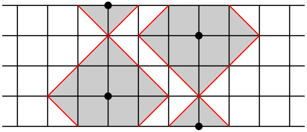

Figure: Broadcast of cost 6 which dominates the remaining region.

	Non-isomorphic	Middle dominated	Not foolish	Total	Time
Cost 5:	264,148	2	2	2	< 2 sec
Cost 6:	1,925,104	314	145	773	< 2 min
Cost 7:	12,162,548	11,632	2,699	59,546	< 40 min

Implemented in Python 3.8.6, made faster with Cython, and run on a 2012 4GB MacBook Pro with a 2.5 GHz Intel Dual-Core i5 processor.

2-Limited Broadcasts on $P_m \Box C_n$

Lower Bound	2-Limited Broadcast #	Upper Bound
	$\gamma_{b,2}(P_2 \Box C_n)$	$= \lceil \frac{n}{2} \rceil$
	$\gamma_{b,2}(P_3 \Box C_n)$	$= \lceil \frac{2n}{3} \rceil$
$\lceil 8 \lfloor \frac{n}{10} \rfloor \rceil \leq$	$\gamma_{b,2}(P_4 \Box C_n)$	$\leq 8 \lfloor \frac{n}{10} \rfloor + c_4(n)_{\leq 8}$
$\lceil 7.703 \lfloor \frac{n}{8} \rfloor \rceil \leq$	$\gamma_{b,2}(P_5 \Box C_n)$	$\leq n + c_5(n)_{\leq 1}$
$\lceil 17.846 \lfloor \frac{n}{16} \rfloor \rceil \leq$	$\gamma_{b,2}(P_6 \Box C_n)$	$\leq 18 \lfloor \frac{n}{16} \rfloor + c_6(n)_{\leq 18}$
$\lceil 16.466 \lfloor \frac{n}{14} \rfloor \rceil \leq$	$\gamma_{b,2}(P_7 \Box C_n)$	$\leq 18 \lfloor \frac{n}{14} \rfloor + c_7(n)_{\leq 18}$
$\lceil 31.302 \lfloor \frac{n}{22} \rfloor \rceil \leq$	$\gamma_{b,2}(P_8 \Box C_n)$	$\leq 32 \lfloor \frac{n}{22} \rfloor + c_8(n)_{\leq 32}$
$\lceil 15.757 \lfloor \frac{n}{10} \rfloor \rceil \leq$	$\gamma_{b,2}(P_9 \Box C_n)$	$\leq 16 \lfloor \frac{n}{10} \rfloor + c_9(n)_{\leq 16}$
$\lceil 31.130 \lfloor \frac{n}{18} \rfloor \rceil \leq$	$\gamma_{b,2}(P_{10} \Box C_n)$	$\leq 32 \lfloor \frac{n}{18} \rfloor + c_{10}(n)_{\leq 32}$
$\lceil 48.976 \lfloor \frac{n}{26} \rfloor \rceil \leq$	$\gamma_{b,2}(P_{11} \Box C_n)$	$\leq 50 \lfloor \frac{n}{26} \rfloor + c_{11}(n)_{\leq 50}$
$\lceil 48.895 \lfloor \frac{n}{24} \rfloor \rceil \leq$	$\gamma_{b,2}(P_{12} \Box C_n)$	$\leq 50 \lfloor \frac{n}{24} \rfloor + c_{12}(n)_{\leq 50}$

2-Limited Broadcasts on $P_m \square C_n$

Lower Bound	2-Limited Broadcast #	Upper Bound
	$\gamma_{b,2}(P_2 \square C_n)$	$= \lceil \frac{n}{2} \rceil$
	$\gamma_{b,2}(P_3 \square C_n)$	$= \lceil \frac{2n}{3} \rceil$
$\lceil 8 \lfloor \frac{n}{10} \rfloor \rceil \leq$	$\gamma_{b,2}(P_4 \square C_n)$	$\leq 8 \lfloor \frac{n}{10} \rfloor + c_4(n)_{\leq 8}$
$n \leq$	$\gamma_{b,2}(P_5 \square C_n)$	$\leq n + c_5(n)_{\leq 1}$
$\lceil 17.846 \lfloor \frac{n}{16} \rfloor \rceil \leq$	$\gamma_{b,2}(P_6 \square C_n)$	$\leq 18 \lfloor \frac{n}{16} \rfloor + c_6(n)_{\leq 18}$
$\lceil 16.466 \lfloor \frac{n}{14} \rfloor \rceil \leq$	$\gamma_{b,2}(P_7 \square C_n)$	$\leq 18 \lfloor \frac{n}{14} \rfloor + c_7(n)_{\leq 18}$
$\lceil 31.302 \lfloor \frac{n}{22} \rfloor \rceil \leq$	$\gamma_{b,2}(P_8 \square C_n)$	$\leq 32 \lfloor \frac{n}{22} \rfloor + c_8(n)_{\leq 32}$
$\lceil 15.757 \lfloor \frac{n}{10} \rfloor \rceil \leq$	$\gamma_{b,2}(P_9 \square C_n)$	$\leq 16 \lfloor \frac{n}{10} \rfloor + c_9(n)_{\leq 16}$
$\lceil 31.130 \lfloor \frac{n}{18} \rfloor \rceil \leq$	$\gamma_{b,2}(P_{10} \square C_n)$	$\leq 32 \lfloor \frac{n}{18} \rfloor + c_{10}(n)_{\leq 32}$
$\lceil 48.976 \lfloor \frac{n}{26} \rfloor \rceil \leq$	$\gamma_{b,2}(P_{11} \square C_n)$	$\leq 50 \lfloor \frac{n}{26} \rfloor + c_{11}(n)_{\leq 50}$
$\lceil 48.895 \lfloor \frac{n}{24} \rfloor \rceil \leq$	$\gamma_{b,2}(P_{12} \square C_n)$	$\leq 50 \lfloor \frac{n}{24} \rfloor + c_{12}(n)_{\leq 50}$

Improving Lower Bounds

Computational Methods

Theorem (S+, 2021)

For $n \geq 3$, $\gamma_{b,2}\left(C_3 \square C_n\right) = \left\lceil \frac{2n}{3} \right\rceil$.

Theorem (S+, 2021)

For $n \geq 5$, $\gamma_{b,2}\left(C_5 \square C_n\right) = n$.

These results imply bounds for $\gamma_{b,2}$ on $C_4 \square C_n$, $P_m \square C_3$, $P_m \square C_4$, $C_6 \square C_n$, $P_5 \square C_n$, $P_m \square C_5$, $P_m \square C_6$, and $P_5 \square P_n$.

Future Work

1. Use this method to prove more bounds on $P_m \square P_n$, $P_m \square C_n$, and $C_m \square C_n$.
2. Improve my methods to make the algorithm faster; there is some redundancy.

References I

- Bostjan Bresar and Simon Spacapan, *Broadcast domination of products of graphs*, Ars Combinatoria **92** (2009).
- Jose Caceres, M. Carmen Hernando, Merce Mora, Ignacio Pelayo, and Maria Puertas, *Dominating 2-broadcast in graphs: Complexity, bounds and extremal graphs*, Applicable Analysis and Discrete Mathematics **12** (2018), 205–223.
- David John Erwin, *Cost domination in graphs*, Ph.D. Thesis, Western Michigan University (2001).
- Pinar Heggernes and Daniel Lokshtanov, *Optimal broadcast domination in polynomial time*, Discrete Mathematics **306** (2006), no. 24, 3267 – 3280.

References II

F. Yang, *Limited broadcast domination*, PhD's Thesis, Department of Mathematics and Statistics, University of Victoria, Canada (2019).

$P_5 \square C_n$

Contradiction Type: 3

Broadcast in C is optimal and does not force a unique domination. Consider least cost domination and max possible region that could be dominated. Dominating the max region with less cost yields a contradiction.

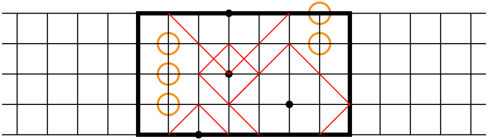

Figure: Broadcast in C of cost 6.

Broadcast in C is optimal and does not force a unique domination. Consider least cost domination and max possible region that could be dominated. Dominating the max region with less cost yields a contradiction.

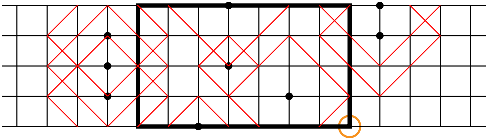

Figure: Broadcast in C and necessary external broadcasters of cost 16.

$P_5 \square C_n$

Contradiction Type: 3

Broadcast in C is optimal and does not force a unique domination. Consider least cost domination and max possible region that could be dominated. Dominating the max region with less cost yields a contradiction.

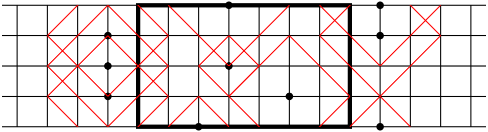

Figure: Least cost broadcast to dominate undominated vertex of cost 17.

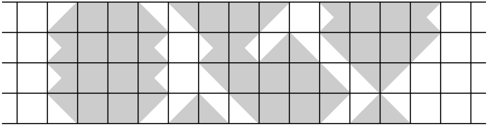

Figure: Corresponding least cost region.

$P_5 \Box C_n$

Contradiction Type: 3

Broadcast in C is optimal and does not force a unique domination. Consider least cost domination and max possible region that could be dominated. Dominating the max region with less cost yields a contradiction.

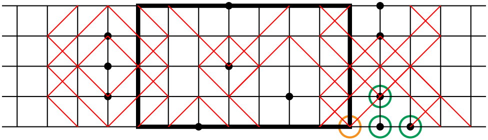

Figure: Max region which could be dominated.

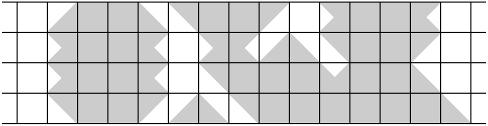

Figure: Corresponding max region.

$P_5 \square C_n$

Contradiction Type: 3

Broadcast in C is optimal and does not force a unique domination. Consider least cost domination and max possible region that could be dominated. Dominating the max region with less cost yields a contradiction.

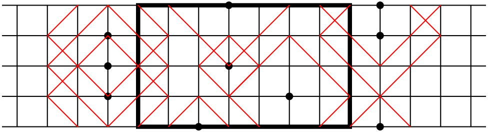

Figure: Least cost broadcast of cost 17.

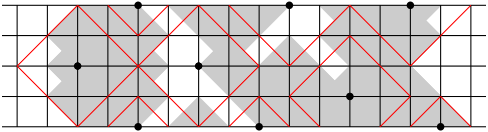

Figure: Broadcast of cost 14 which dominates the max region.

2-Limited Broadcasts on $P_m \square C_n$

Asymptotic Bounds

As $m, n \to \infty$,

$$\left\lceil 2\left(\frac{mn}{13}\right) + (0.608)\,4\left(\frac{m}{13}\right)\right\rceil \leq \gamma_{b,2}\left(P_m \square C_n\right)$$

$$\gamma_{b,2}\left(P_m \square C_n\right) \leq 2\left(\frac{mn}{13}\right) + 4\left(\frac{m}{13}\right) + c(n)\left(\frac{n}{13}\right) + O(1)$$

where $0 \leq c(n) \leq 4$, dependant upon the least residue of $n \pmod{13}$.