The 2-adic Solenoid

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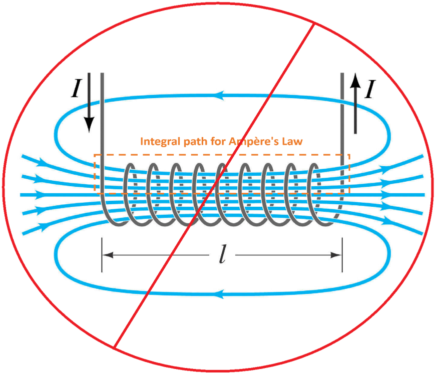
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Integral path for Ampère's Law
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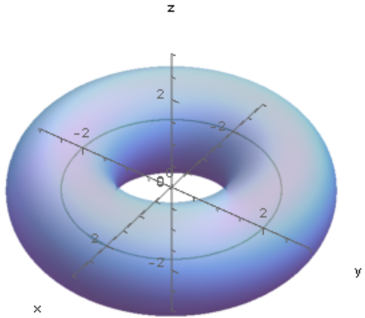
History

- The solenoid was first known to topologists and it was first introduced by Vietoris in
  L. Vietoris, ber den hheren Zusammenhang kompakter Rume und eine Klasse von zusammenhangstreuen Abbildungen, Math. Ann. 97 (1927), pp. 45447.
- Was introduced as an example in dynamical systems by Stephen Smale (1967).
- Then picked up by R.F. Williams who developed the theory of one-dimensional expanding attractors. (1967,1974).

Consider the solid torus

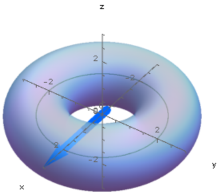
$$S^1 \times \overline{\mathbb{D}} = \{(\theta, r, s) \mid \theta \in S^1, r^2 + s^2 \leq 1\} = M.$$

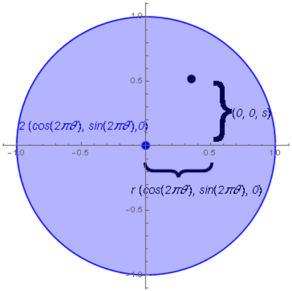
S^1 is thought of as $\mathbb{R}/\mathbb{Z}$ i.e we say that $r = s$ iff $r - s \in \mathbb{Z}$. $\overline{\mathbb{D}}$ is the solid unit disk.



We may visualize this in $\mathbb{R}^3$ by the embedding map:

$$
\begin{aligned}
(\theta, r, s) &\mapsto \\
2\big(\cos(2\pi\theta), \sin(2\pi\theta), 0\big) \\
+r(\cos(2\pi\theta), \sin(2\pi\theta), 0) \\
+(0, 0, s)
\end{aligned}
$$

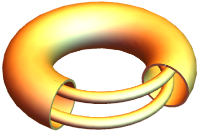




Defining a map f on the torus

The following map stretches the solid torus and then wraps the stretched torus inside the original:

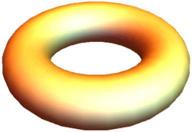
$$f(\theta, r, s) = (2\theta, 5^{-1}r + 2^{-1}\cos(2\pi\theta), 5^{-1}s + 2^{-1}\sin(2\pi\theta))$$



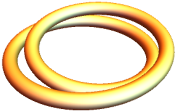
Notice that f stretches in one direction and contracts in another. Also, notice that $f(M) \subseteq M$ and so f is well defined.

Stretch and Twist

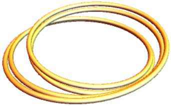
M



$f(M)$



$f^2(M)$



$f^3(M)$

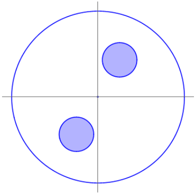
Let $D(\theta_0) = \{\theta_0\} \times \overline{\mathbb{D}}$.

Proposition

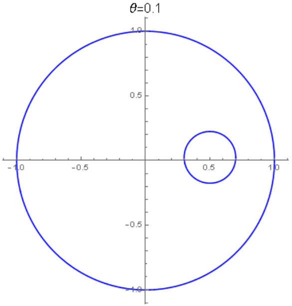
The set $f^n(M) \cap D(\theta_0)$ is the union of 2^n closed disks of radius $(1/5)^n$

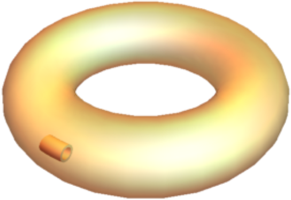
Proof.

Proof by induction! $\square$



The Hyperloop experience-traveling inside the $f(M)$ tube



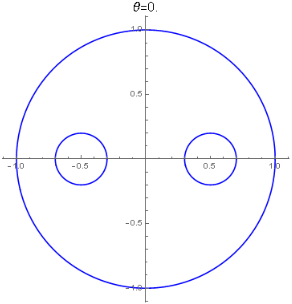


Look Anna! I fixed it!
The map f, contracts by a factor of $1/5$ on each slice. The radius of each smaller dark blue disk (the tube) is $1/5$.

A slice of M and $f(M)$ for various θ

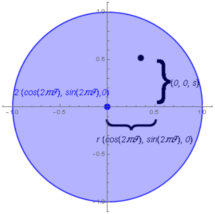
At each fixed θ, the centre of each of the disks is given by:

$$2^{-1}(\cos(2\pi\theta), \sin(2\pi\theta))$$
$$\text{and } 2^{-1}(\cos(2\pi(\theta + 1/2)), \sin(2\pi(\theta + 1/2)).$$

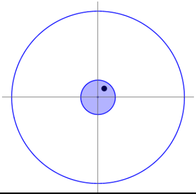


What happens to a point (θ, r, s) under this map?

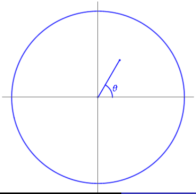
$$f(\theta, r, s) = (2\theta, 5^{-1}r + 2^{-1}\cos(2\pi\theta), 5^{-1}s + 2^{-1}\sin(2\pi\theta))$$



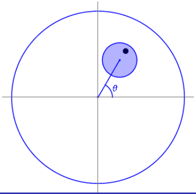
$5^{-1}(r, s)$



$2^{-1}(\cos(2\pi\theta), \sin(2\pi\theta))$



shift



We may form a nested chain of compact sets.

$$M \supseteq f(M) \supseteq f^2(M) \ldots$$

So, Nat'ralists observe, a Flea
Hath smaller Fleas that on him prey:
And these hath smaller Fleas to bite 'em;
And so proceed ad infinitum.

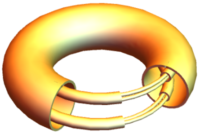
Jonathan Swift, *On Poetry*

The intersection of a non-empty, nested sequence of compact sets is non-empty and compact.

$$\mathcal{S} = \bigcap_{n \geq 0} f^n(M)$$

We call this beauty the solenoid!

$$(\mathcal{S}, f|_{\mathcal{S}})$$



The solenoid as a dynamical system

Proposition

The map $f : \mathcal{S} \to \mathcal{S}$ is a homeomorphism.

Proof.

We will show that f is injective, surjective and its inverse is continuous. $\square$

Properties of $(\mathcal{S}, f)$

- Connected?
- Locally connected?
- Strange attractor.
- Hyperbolic.

Connectedness

Definition

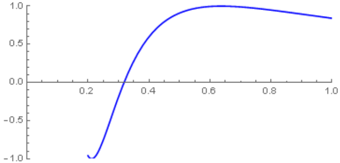
Let $(X, \mathcal{T})$ be a topological space. We say that X is disconnected if there is a subset A, other than X, and the empty set which is both open and closed. If X is not disconnected, then it is connected.

The Topologist's sine curve

Examples:

- The set $X = \{x \in \mathbb{R} \mid |x| < 1 \ \text{ or} |x-3| < 1\}$ is not connected.
- The set of rationals with the Euclidean metric is not connected.
- The Topologist's sine curve is connected.

  $T = \{(x, \sin(1/x) : x \in (0, 1]\} \cup \{(0, y) \mid -1 \leq y \leq 1\}.$



Proposition

The solenoid is connected.

Proof.

Take the intersection of a nested sequence of non-empty, compact, connected sets of a metric space. $\square$

Definition

A topological space is locally connected at the point x if every neighborhood of x contains a connected open neighborhood. It is called locally connected if it is locally connected at every point.

Examples:

- The set $X = \{x \in \mathbb{R} \mid |x| < 1 \ \text{or} |x-3| < 1\}$ is locally connected.
- The set of rationals with the Euclidean metric is not locally connected.
- A discrete space is locally connected.
- The Topologist's sine curve is not locally connected.

Another non-example is the solenoid

Proposition

The solenoid is not locally connected.

Proof.

$\square$

Now we move on to the next property on our list. Does anyone remember what that was? Here is a hint:

or

Strange Attractor

Definition

A set X is an attractor for a map f if there exists a neighborhood U of X such that $\overline{f(U)} \subseteq U$ and $X = \bigcap_{n \geq 0} f^n(U)$. The set U is called an attracting set.

The definition for “strange” will be explained later on.

The term "strange attractor" first appeared in
D. Ruelle, F. Takens, On the Nature of Turbulence. 1971.

"I asked Floris Takens if he had created this remarkably successful expression. Here is his answer: 'Did you ever ask God whether he created this damned universe?...I don't remember anything...I often create without remembering it...' The creation of strange attractors thus seems to be surrounded by clouds and thunder. Anyway, the name is beautiful, and well suited to these astonishing objects, of which we understand so little." - Ruelle

Hyperbolicity

Definition

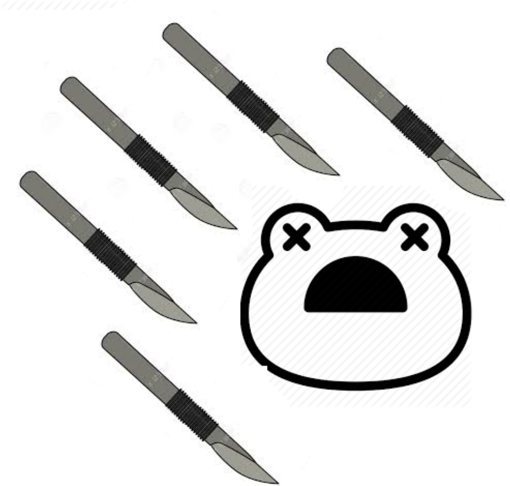
The set Λ is called a hyperbolic set for f if there exists a continuous splitting of $TM|_\Lambda$, the restriction of the tangent bundle TM to Λ, which is invariant under the action of the derivative map Df;

$$TM_\Lambda = E^s \oplus E^u, Df(E^s) = E^s,\ Df(E^u) = E^u,$$

for which there are constants $c > 0$ and $0 < \lambda < 1$, such that for $t \in \mathbb{Z}$,

$$\|Df^t|_{E^s}\| < c\lambda^t,\ t \geq 0$$
$$\|Df^{-t}|_{E^u}\| < c\lambda^t,\ t \geq 0.$$

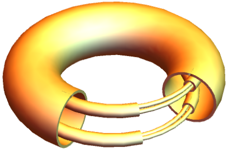
What happens if you act like a biologist and start dissecting the solenoid?



What does $\mathcal{S}$ look like locally

Our method here is much more animal friendly :).

Suppose we fix $\theta = \theta_0$ what would we see?



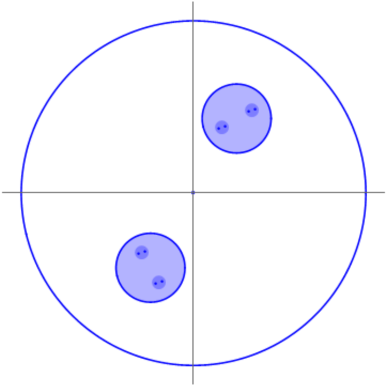
We will look for the stable sets and unstable sets of $(\mathcal{S}, f)$. One proprety to note about stable sets is that if x and y are stably equivalent, then

$$d(f^n(x), f^n(y)) \to 0 \text{ as } n \to \infty$$

Similarly, x and y are unstably equivalent if,

$$d(f^{-n}(x), f^{-n}(y)) \to 0 \text{ as } n \to \infty$$

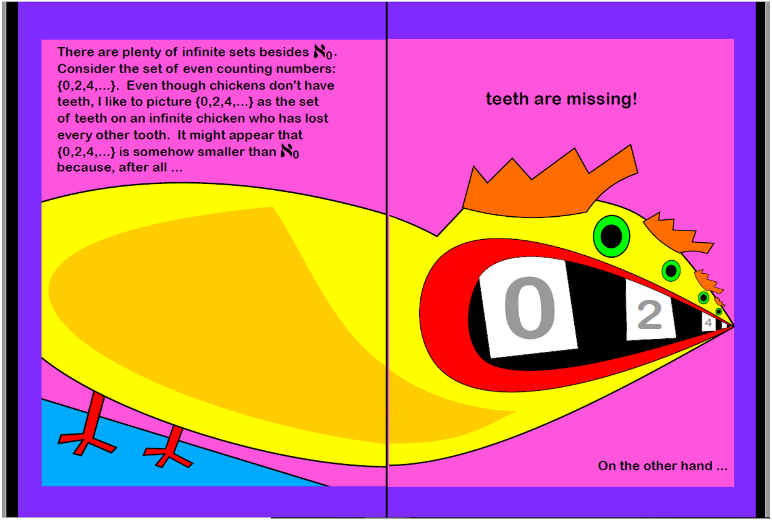
Let $D(\theta_0) = \{\theta_0\} \times \overline{\mathbb{D}}$. The following is a stable set of the solenoid.



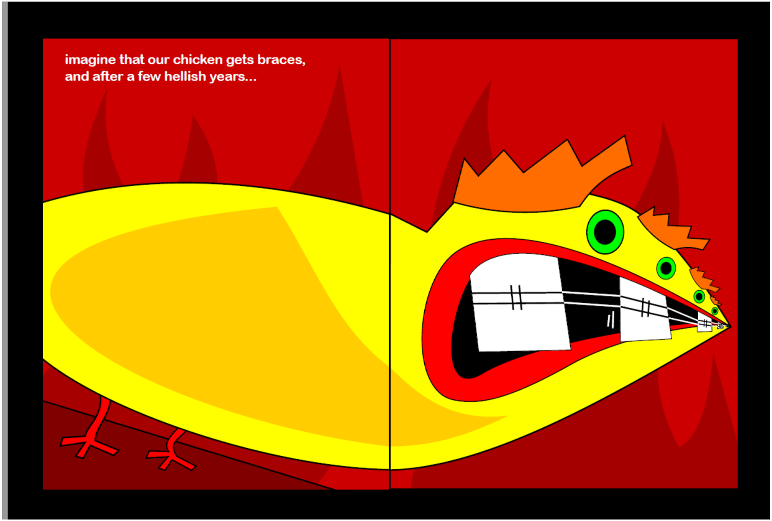
$D(\theta_0) \cap \mathcal{S}$

A point on the solenoid is determined by a $\theta_0 \in S^1$ and choosing a path along the following diagram:

$$\begin{tikzcd}
 & & f^2(D(\theta/2^2)) \arrow[r] & {} \\
 & f(D(\theta/2)) \arrow[ur] \arrow[dr] & & \\
 & & f^2(D(\theta/2^2+1/2)) \arrow[r] & {} \\
D(\theta_0) \arrow[uur] \arrow[ddr] & & & \\
 & & f^2(D(\theta/2^2+1/4)) \arrow[r] & {} \\
 & f(D(\theta_0/2+1/2)) \arrow[ur] \arrow[dr] & & \\
 & & f^2(D(\theta/2^2+3/4)) \arrow[r] & {}
\end{tikzcd}$$



source: Richard Schwartz's book called "Gallery of the Infinite"



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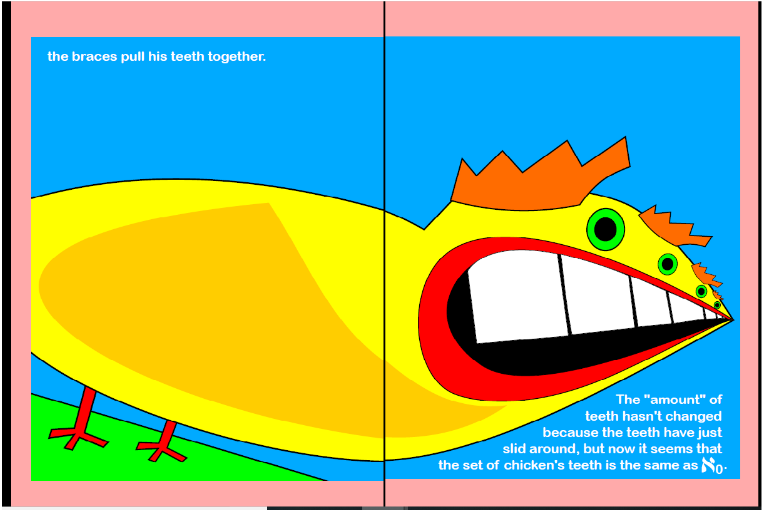

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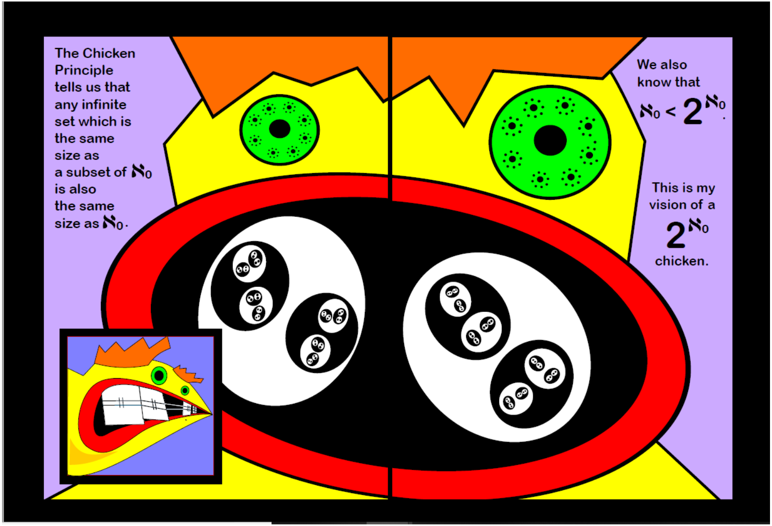


source: Richard Schwartz's book called "Gallery of the Infinite"

Proposition

The set $D(\theta_0) \cap \mathcal{S}$ is a Cantor set.

Proof.

We show this set is compact, totally disconnected and perfect. $\square$

What if we fix (r, s)?

...

We get an immersed line!

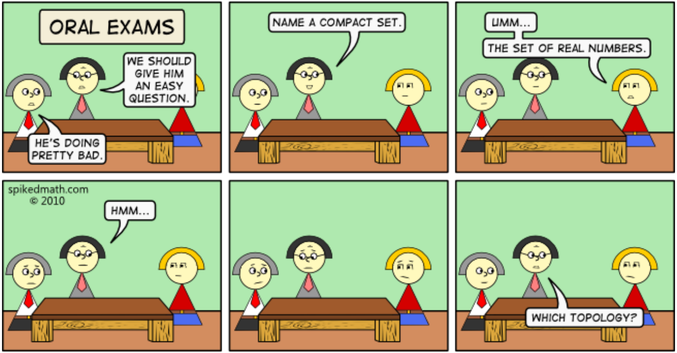
Locally, the solenoid is a (Cantor set $\times$ $\mathbb{R}$).

Theorem (R.F. Williams)

Each point of an n-solenoid has a neighborhood of the form (Cantor set $\times$ n-disk).

We will (sort of) show this in the next section.
Stay tuned.

Now we move on to the topological description of the solenoid.



Spiked Math "Oral Exams"

Definition

A inverse (projective) system of groups consists of the following data

- a directed set $(I, \leq)$
- a family $(G_i)_{i \in I}$ of groups and
- a family of group homomorphisms

$$\pi_i^j : G_j \to G_i, \quad \text{if } i \leq j,$$

such that the following axioms are met:

$$\pi_i^i = ID_{G_i} \;\text{ and }\; \pi_i^j \circ \pi_j^k = \pi_i^k, \;\text{ if } i \leq j \leq k.$$

Definition

Let (G_i, π_i^j) be a inverse (projective) system of groups. The inverse (projective) limit of the system is the set

$$G = \lim_{\longleftarrow} G_i$$

of all $a \in \prod_{i \in I} G_i$ such that $a_i = \pi_i^j(a_j)$ holds for every pair $i \leq j$ in I.

Universal property

Inverse limit construction

Let

$$\{X_i, \sigma_i\} = S^1 \xleftarrow{\times 2} S^1 \xleftarrow{\times 2} S^1 \dots$$

be an inverse sequences of compact metric spaces S^1 with the doubling map. We define the inverse limit as the subpsace of the product space,

$$(X, \sigma) = \varprojlim \{X_k, \sigma_k\}_{k=0}^{\infty} = \{(x_0, x_1, \dots) \mid x_k \in \mathbb{R}/\mathbb{Z}, 2x_{k+1} = x_k, k \geq 0\}.$$

Where σ on X is defined as,

$$\sigma(x_0, x_1, \dots) = (2x_0, 2x_1, \dots) = (2x_0, x_0, x_1, \dots)$$

Extension!

Theorem

X is a topological group that is compact using the metric,

$$d((x_0, x_1 \ldots), (y_0, y_1, \ldots)) = \sum_{k \geq 0} 2^{-k} \inf\{x_k - y_k + l \mid l \in \mathbb{Z}\}$$

Proof.

The group operation is componentwise addition mod 1. The product topology on $(S^1)^{\mathbb{N}_0}$ induces the subspace topology on X. The map $(x, y) \mapsto x - y$ is continuous. □

Proposition

The map $\sigma : X \to X$ is a homeomorphism.

Proof.

We have that σ is continuous since multiplacation by 2 is continuous on each component. The inverse of σ is a well-defined map given by

$$\sigma^{-1}(x_0, x_1, x_2 \ldots) = (x_1, x_2 \ldots).$$

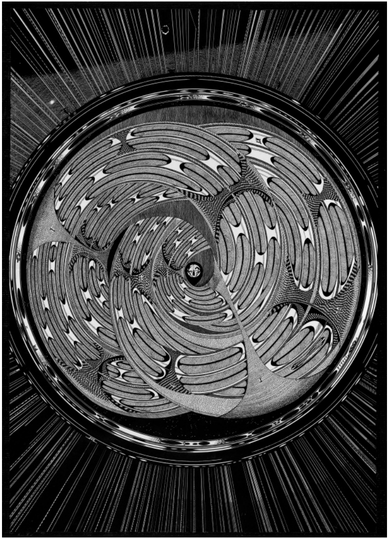
It is clear that $\sigma \circ \sigma^{-1}(x) = x$ and $\sigma^{-1} \circ \sigma(x) = x$. The space, X, is compact and so σ^{-1} is continuous. $\square$

Definition

Let $f : X \to X$ and $g : Y \to Y$ be two functions. We say that f and g are *topologically conjugate* if there exists a homeomorphism $h : X \to Y$ such that $fh = hg$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"'] & X \arrow[d, "h"] \\ Y \arrow[r, "g"] & Y \end{tikzcd}$$

Artwork depicting the solenoid by Anatoly Fomenko.



These two dynamical systems are actually the "same"

Theorem

There is a topological conjugacy, h, between the solenoid $(\mathcal{S}, f)$ and the inverse limit (X, σ).

Proof.

Take $\pi : S^1 \times \overline{\mathbb{D}} \to S^1$ such that $\pi(\theta, r, s) = \theta$ and then $h : X \to \mathcal{S}$ to be $h(p)_n = \pi f^{-n}(p)$. $\square$

More properties of the solenoid

- Back to hyperbolicity
- Chaotic dynamical system

Proposition

The unstable set of 0 consists precisely those sequence of the form

$$(x, \frac{x}{2}, \frac{x}{2^2}, \frac{x}{2^3}, \ldots)$$

for any x in $\mathbb{R}$.

Proof.

$\square$

What is chaos?

Jurassic Park chaos clip

Devaney's definition of Chaos (1986)

Definition

Let V be a set. $f : V \to V$ is said to be chaotic on V if:

1. f has sensitive dependence on initial conditions.
2. f is topologically transitive.
3. Periodic points are dense in V.

The first is a measure of unpredictability, the second of indecomposibility, and the third an element "in the midst of this random behaviour, we nevertheless" have regularity.

Theorem

If $f : X \to X$ is a chaotic dynamical system on an infinite metric space and $g : Y \to Y$ is topologically conjugate to f then g is also a chaotic dynamical system.

In 1992, Banks, Brooks, Cairns, Davis and Stacey proved the following

Theorem

If $f : X \to X$ is a transitive continuous map on an infinite metric space and has a dense set of periodic points then f has sensitive dependence on initial conditions.

Assaf IV and Gadbois also proved that for general maps, this is the only redundancy.

"Ed Lorenz is like an owl . He can turn his head 180 degrees. No one else can. With the one position of the head, you look at the universe in which the butterfly did not flap its wings ; with the other position of the neck, you look at that other universe in which it did. Both universes diverge exponentially from each other. "
Rossler (Chaos Avant-Garde)

Transitive.

Definition

The dynamical system (X, f) is topologically transitive if for every pair of non-empty open sets U and V in X, there is a non-negative integer n such that $f^n(U) \cap V \neq \emptyset$.

Theorem

A dynamical system is topologically transitive iff there is a point x_0 in X with a dense orbit in X.

Proposition

The dynamical system (X, σ) is transitive.

Proof.

$\square$

Periodic points are dense.

Proposition

The periodic points of (X, σ) are dense in X.

Proof.

$\square$

Answer Chris's question about writing an element down.

Theorem

If $f : X \to X$ is a chaotic dynamical system on an infinite metric space and $g : Y \to Y$ is topologically conjugate to f then g is also a chaotic dynamical system.

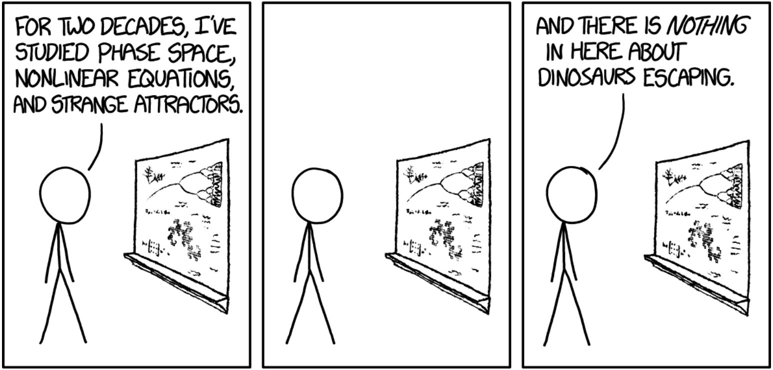
Proof.

$\square$

Before we saw that the solenoid is an attractor now that we know it is a **chaotic** dynamical system, we can call it a **strange attractor**.

Strange Attractor on Band Camp

xkcd "Chaos"


FOR TWO DECADES, I'VE STUDIED PHASE SPACE, NONLINEAR EQUATIONS, AND STRANGE ATTRACTORS.
AND THERE IS NOTHING IN HERE ABOUT DINOSAURS ESCAPING.

p-adic numbers

First we assume that for any rational number a we can factor out as many possible 2's from a and then write $a = 2^k \frac{r}{s}$. This is essentially the idea for a type of absolute value function on $\mathbb{Q}$.

$$\begin{cases} |0|_2 = 0 \\ |2^k \frac{r}{s}|_2 = 2^{-k} \quad k \in \mathbb{Z}, r, s \text{ non-zero integers and relatively prime to .2} \end{cases}$$

Note: $|\cdot|_2$ can only take a "discrete" set of values,

$$\{p^n, n \in \mathbb{Z}\} \cup \{0\}$$

From here, we define the metric $|a - b|_2$ on $\mathbb{Q}$.

Surprise!

2080 and 2 are closer together than 2 and 3.

$$|2080|_2 = |2^5 * 65|_2 = 2^{-5} < |3-2|_2 = |1|_2 = 2^0 = 1$$

Multiplication by 2 is a contraction.
Suppose $x = 2^k \frac{r}{s}$, then $2x = 2^{k+1}\frac{r}{s}$ and so,

$$|2x|_2 = 2^{-(k+1)} = 2^{-1}2^{-k} = 2^{-1}|x|_2$$

Division by 2 is an expansion.

The metric defined before, is actually an *ultrametric* i.e it satisfies the strong triangle inequality,

$$|a - b| \leq \max\{|a - c|, |c - b|\}$$

In an ultrametric space, all triangles are isosceles, with at most one short side.

$\mathbb{Q}_2$-The bonafide approach to the *p*-adics

Definition

We define $\mathbb{Q}_2$ as equivalence classes of Cauchy sequences of $\mathbb{Q}$ with respect to $|\cdot|_2$. Where two Cauchy sequences are equivalent if their difference converges to 0. Lastly, for any a in $\mathbb{Q}_2$,

$$|a|_2 = \lim_{n \to \infty} |a_n|_2$$

"One of my students once asked me what the p-adic norm measures. I told him it measures the p-ness of a rational number."
- Paul Sally

Algebraic Structure

Theorem

The p-adic numbers (for p prime) are a field.

Proof.

□

More strangeness

(Now we just think of elements in $\mathbb{Q}_2$ as values and supress the fact that we thought of them as sequences)

The following sequence is convergent in $\mathbb{Q}_2$.

$$(1, 2, 4, 8, 16, 32, 64, 128, \ldots)$$

What is

$$1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 \ldots = ?$$

p-adic digit expansion

What do elements in $\mathbb{Q}_2$ look like?

Theorem

Every 2-adic number $a \in \mathbb{Q}_2$ has a unique 2-adic expansion

$$\sum_{i=k}^{\infty} a_i 2^i,$$

Where $a_i \in \{0, 1\}$ and k is some integer such that $a_k \neq 0$.

Proof.

$\square$

$\mathbb{Q}_p$ can be thought of as the collection of formal laurent series in p.

"The topology of the p-adic field, $\mathbb{Q}_p$, is weird."

- William Stein 2004 Quantum field theory III and gauge theory.

What is the unit disk in $\mathbb{Q}_2$?

$$\overline{B_1(0)} = \{a \in \mathbb{Q}_2 \mid |a|_2 \leq 1\}$$

This means that a must have a non-negative power of 2 as a factor.

$$\overline{B_1(0)} = \left\{ \sum_{k \geq 0} a_k 2^k \mid a_k \in \{0, 1\} \right\}$$

What is the open ball of radius p?

$\mathbb{Z}_p$

The set of 2-adic integers is denoted by $\mathbb{Z}_2$, and given by

$$\mathbb{Z}_2 = \left\{ \sum_{k\geq 0} a_k 2^k \mid a_k \in \{0,1\} \right\} = \overline{B_1(0)} = B_p(0)$$

This means that the set of 2-adic integers is both closed and open in $\mathbb{Q}_2$.
Fun fact: It is also a subring of $\mathbb{Q}_2$.

Middle thirds Cantor set

Proposition

The space $\mathbb{Z}_2$ with $|\cdot|_2$ is homeomorphic to the middle thirds Cantor set with $|\cdot|$.

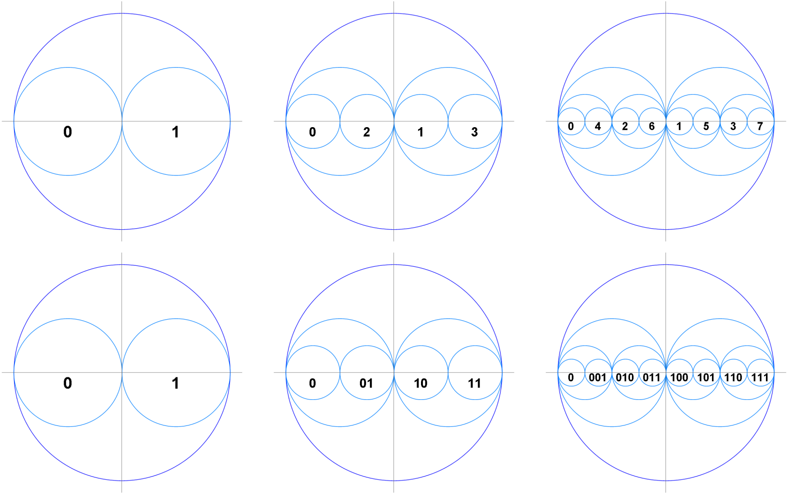
Proof.

Let $\phi : \mathbb{Z}_2 \to \mathcal{C}$.

$$\phi : \sum_{i=0}^{\infty} a_i 2^i \mapsto \sum_{i=1}^{\infty} \frac{2a_i}{3^{i+1}}$$

$\square$

Another visualization



$(\mathbb{R} \times \mathbb{Z}_2, \phi)$

The metric is defined as

$$d((r,z),(r',z')) = \max\{|r - r'|, |z - z'|_2\}$$

Now, we define a map on this space which gives us a dynamical system.

$$\phi(r,a) = (2r, 2a)$$

Notice that this map expands in the first coordinate and contracts in the second!

The solenoid-in disguise

Theorem

The dynamical system $((\mathbb{R} \times \mathbb{Z}_2)/\{(a, -a) \mid a \in \mathbb{Z}\}, \phi)$ *is topologically conjugate to* (X, σ).

Proof.

We define a map, $\hat{\psi} : \mathbb{R} \times \mathbb{Z}_2 \to X$.

$$(r, a) \mapsto (r, 2^{-1}r + 2^{-1}a_0, 2^{-2}r + 2^{-2}\sum_{k=0}^{1} a_k 2^k \ldots)$$

Kernel is given by

$$\Gamma = \{(a, -a) \mid a \in \mathbb{Z}\}$$

□

Some nice short exact sequences

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \times \mathbb{Z}_2 \longrightarrow (\mathbb{R} \times \mathbb{Z}_2)/\{(a, -a) \mid a \in \mathbb{Z}\} \longrightarrow 0.$$

Another fun short exact sequence is,

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow (\mathbb{R} \times \mathbb{Z}_2)/\{(a, -a) \mid a \in \mathbb{Z}\} \longrightarrow \mathbb{R}/\mathbb{Z} \longrightarrow 0.$$

The solenoid-in disguise (again)

Theorem

The dynamical system $((\mathbb{R} \times \mathbb{Q}_2)/\{(a, -a) \mid a \in \mathbb{Z}[1/2]\}\ , \phi)$ *is topologically conjugate to* (X, σ)*.*

Proof.

We define a map, $\hat{\psi} : \mathbb{R} \times \mathbb{Q}_2 \to X$.

$$(r, \sum_{i=-n}^{\infty} a_k 2^k) \mapsto (r + \sum_{i=-n}^{0} a_k 2^k, 2^{-1} r + 2^{-1} \sum_{i=-n}^{1} a_k 2^k, \dots)$$

Kernel is given by

$$\Gamma_2 = \{(2^{-i} j, -2^{-i} j) \mid i \in \mathbb{N},\ j \in \mathbb{Z}\}$$

Thank you for listening!