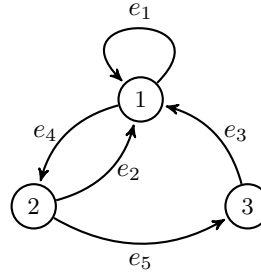


Set Possibilities for Labelings of Finite Graphs.

We start with a graph which we will call the Tribonacci graph, with the corresponding adjacency matrix sitting right beside it. The directed graph has edge set $E = \{e_1, e_2, e_3, e_4, e_5\}$ and vertex set $V = \{1, 2, 3\}$, along with range and source maps $r : E \rightarrow V$ and $s : E \rightarrow V$ that identify the range and source of an edge, respectively.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$



Adjacency Matrix

Each entry e_{ij} in the adjacency matrix corresponds to the number of edges going from vertex j to i . (Sorry it is all transposed). This matrix is pretty special because it is what is called a *hyperbolic* matrix.

For a general matrix, A in $GL_d(\mathbb{R})$ we define, (s for stable and u for unstable)

$$E^s = \{x \in \mathbb{R}^d \mid \|\hat{A}^n x\| \rightarrow 0, n \rightarrow +\infty\}$$

$$E^u = \{x \in \mathbb{R}^d \mid \|\hat{A}^n x\| \rightarrow 0, n \rightarrow -\infty\}$$

Definition: We say a matrix A is **hyperbolic** if A is in $GL_d(\mathbb{R})$ and,

$$\mathbb{R}^d = E^s \oplus E^u.$$

From now on, we will denote the fixed adjacency matrix above, A . For this case we have three eigenvalues, one of which is real with modulus greater than 1 and the other two which are complex with modulus less than 1. The expanding eigenvalue corresponds to an eigenspace E^u on which the matrix A is expanding, while the two complex eigenvalues corresponding to E^s on which the matrix is contracting. Since we have that

$$\mathbb{R}^3 = E^s \oplus E^u$$

Then every element, x of \mathbb{R}^3 will have a stable part x^s in E^s and an unstable part x^u in E^u .

Path space

We will consider the space of all bi-infinite paths along the edges of the finite directed graph. This space, Σ , along with the shift map, σ is a dynamical system called a Shift of Finite Type (Σ, σ) .

$$\Sigma = \{(\dots, x_{-1}, x_0, x_1, x_2, \dots) \mid r(x_n) = s(x_{n+1})\}$$

$$\sigma(x)_n = x_{n-1}$$

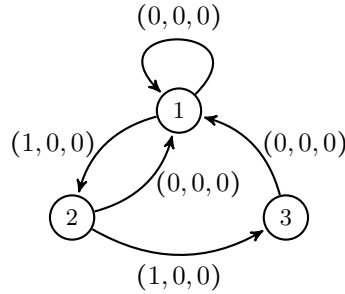
Relabel

We may relabel the edges of our finite directed graph with tuples (a, b, c) from \mathbb{Z}^3 (possibly even \mathbb{R}^3 , but we can keep things simple for now). Let the following function ν be a relabeling of the edges.

$$\nu : E \rightarrow (\mathbb{Z}^3)^5$$

For, example we can define

$$\nu(e_1) = (0, 0, 0), \nu(e_2) = (0, 0, 0), \nu(e_3) = (0, 0, 0), \nu(e_4) = (1, 0, 0), \nu(e_5) = (1, 0, 0)$$



We can extend this labeling function to the entire path space by applying ν to each edge within a path.

Projection Functions

Next, we define our projection functions. These will take a path in Σ and use the labeling with the Tribonacci matrix to project onto the expanding line and the contracting plane.

$$\pi^s : \Sigma \rightarrow E^s \quad \text{and} \quad \pi^u : \Sigma \rightarrow E^u$$

$$\pi^s(x) = \sum_{i \leq 0} A^{-i} \nu(x_i)^s \quad \text{and} \quad \pi^u(x) = \sum_{i \geq 1} A^{-i} \nu(x_i)^u$$

From here we can consider the set $\pi^s(\Sigma)$ and $\pi^u(\Sigma)$. In fact, we can break this set down in terms of the vertices. Recall that the vertex set V is given by $V = \{1, 2, 4\}$

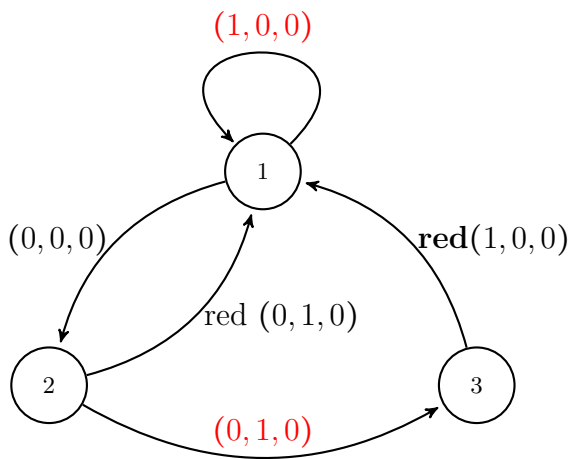
$$\mathcal{R}^s = \pi^s(\Sigma)$$

$$\mathcal{R}_1^s = \pi^s\{x \mid r(x_0) = 1\}$$

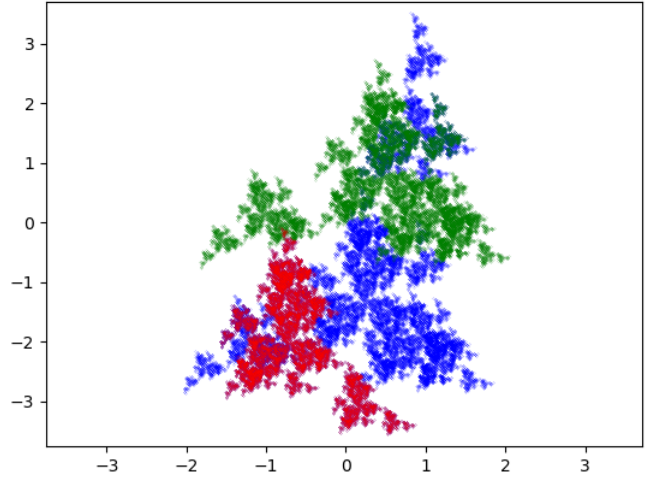
$$\mathcal{R}_2^s = \pi^s\{x \mid r(x_0) = 2\}$$

$$\mathcal{R}_3^s = \pi^s\{x \mid r(x_0) = 3\}$$

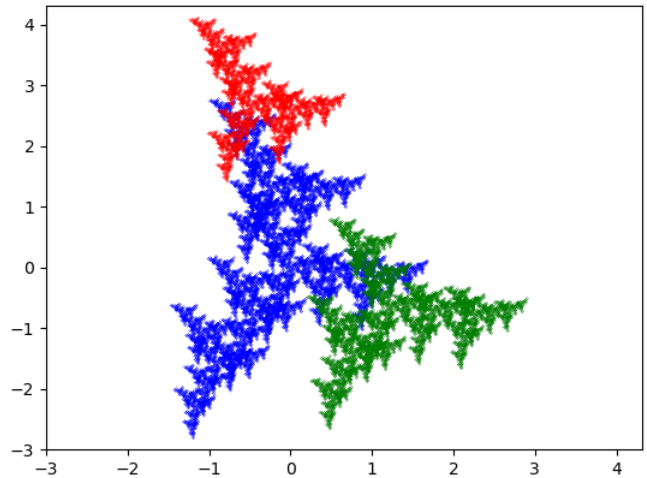
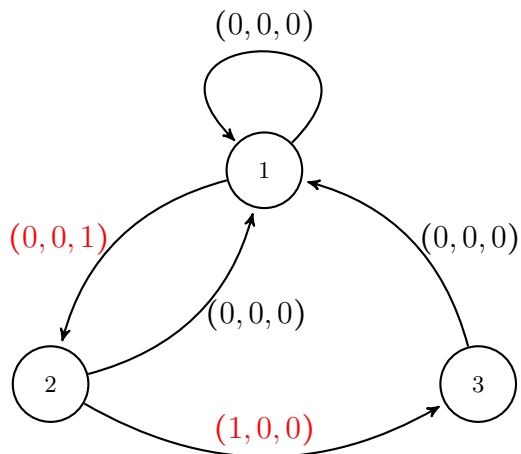
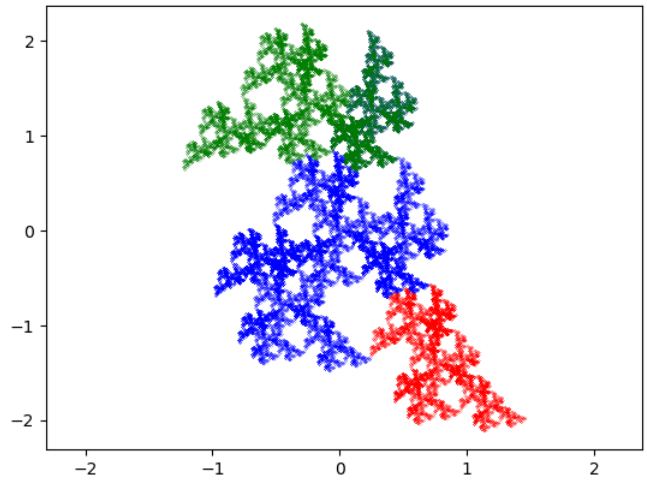
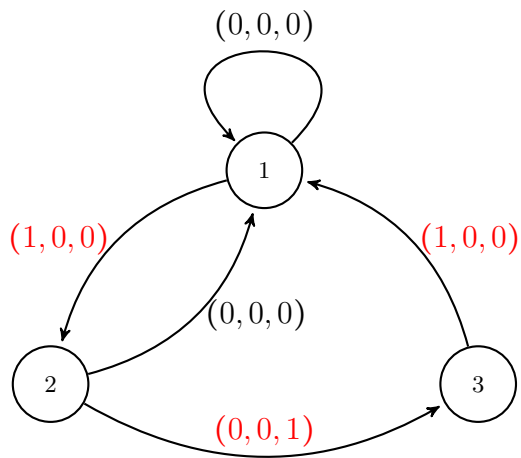
We can do the same on the unstable side. And it turns out that if we add the stable and unstable part we get \mathcal{R} which will be a Markov partition for the hyperbolic toral automorphism determined by A for special labels on the edges. We won't go into that here. We actually have all that we need to define these beautiful sets. Using Python to code this up, we can input a random labeling of the edges and see what sort of sets is returned.

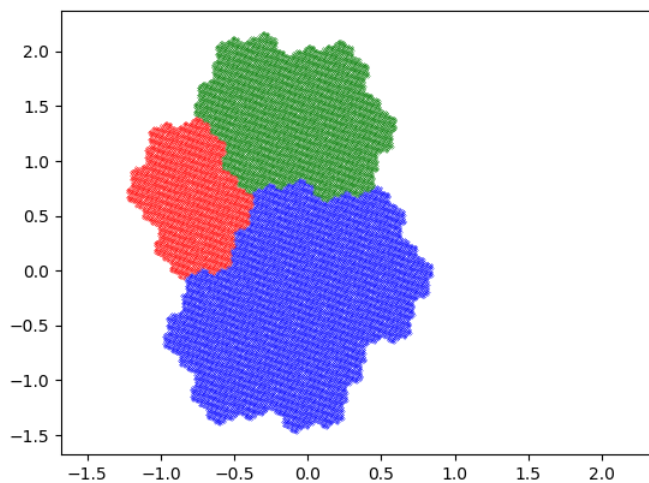
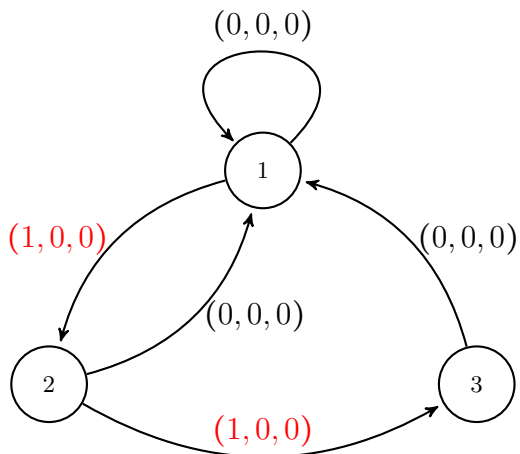
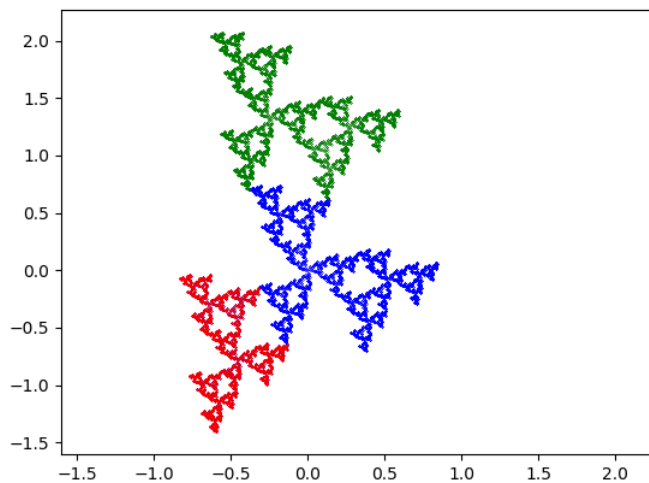
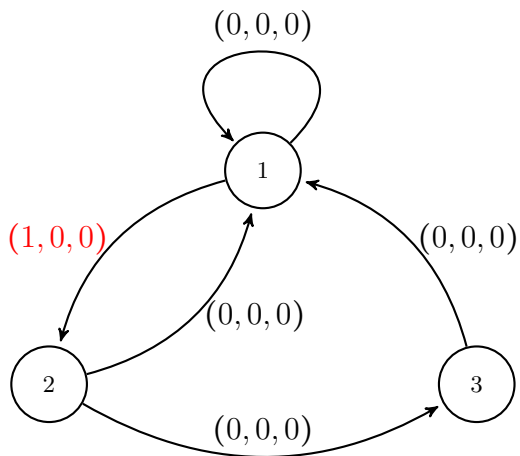
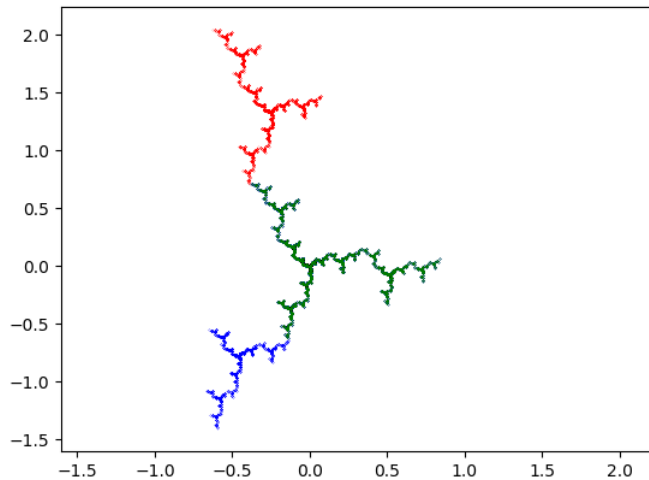
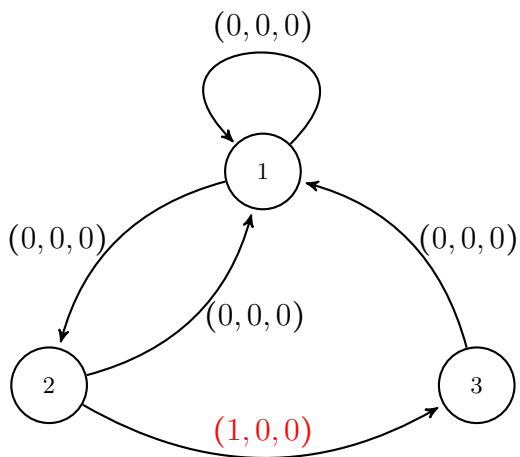


I've colored the non-zero labels in red.



Each of the sets $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$ are colored differently.





Surprisingly, we may add the coordinates the two first images, defined by the same path to get the the Rauzy fractal.