Convexity and the Structure of Designs

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2009

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Preface

In this manuscript, various known and original inequalities concerning the structure of combinatorial designs are established using 'convexity arguments' and 'inclusion matrices'. The emphasis is on a unification, through this method, of various structural results in design theory.

This work begins by giving definitions and elementary facts concerning t-designs. The inclusion matrix W_t of t-subsets versus k-subsets of a finite set is defined. The opening chapter also discusses relevant facts in convex geometry. The purpose of Chapter 2 is to study the symmetries of the cone generated by columns of W. The two subsequent chapters, 3 and 4, derive inequalities on block density and intersection patterns in t-designs. Chapter 5 outlines various possible generalizations of W_t, with a discussion of applications. Finally, chapter 6 further generalizes the setting to include other association schemes and structures therein.

Although the motivation for this work rests in design theory, I have attempted a small glance at a few related questions in linear algebra, linear programming, algebraic geometry, and coding theory. I am disappointed to not yet fully understand these connections. However, it is my hope that the knowledgeable reader finds some use in what follows, perhaps as a stepping stone to broader use of convexity arguments in combinatorics.

It should be noted that certain of the methods and results in this manuscript have been recently published by me and R.M. Wilson, my doctoral supervisor, in [11] and [32].

Professor R.M. Wilson deserves my sincere thanks for a variety of reasons. I appreciate his suggestion of this research topic and the many helpful ideas arising from our discussions. In this respect alone, I could not have asked for

a better mentor. But this is not even to mention his patience, kindness and humor, all of which made this work very enjoyable.

This work was done mostly during my doctoral studies at the Department of Mathematics at the California Institute of Technology. Fellow students, faculty, and administrators all made for an excellent experience during my stay there. Additional thanks extend to the members of my thesis defense committee.

Financial assistance from the department and from my NSERC postgraduate and postdoctoral fellowships was indispensable and also much appreciated.

Finally, I would like to thank my parents. They lovingly and selflessly honed my curiosity for learning.

Chapter 1

Preliminaries

Suppose 10 triangles on six points cover each of the 15 pairs of points exactly twice. There cannot be two disjoint triangles T_1 and T_2 in this collection, for the following reason. If there were, the other 8 triangles would have to cover $3^2 = 9$ pairs crossing between T_1 and T_2, twice each. But each triangle affords at most two crossing pairs, and this is a contradiction.

On one hand, this is merely simple counting. However, viewed in greater generality, we made a structural conclusion about a certain combinatorial block design. The technique – almost hidden in the background – hinged on the convexity of pair coverage by triangles.

1.1 Combinatorial block designs

Let $\mathbb{N}$ denote the set of positive integers, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Suppose $t, \lambda \in \mathbb{N}_0$. A *$t$-wise balanced block design* (tBD) of *index* λ is a triple $(V, \mathcal{B}, \iota)$, where V and $\mathcal{B}$ are (disjoint) sets of *points* and *blocks*, respectively, and $\iota \subseteq V \times \mathcal{B}$ is a set of *flags* with the property that for any t-subset T of V, there are precisely λ blocks B of $\mathcal{B}$ satisfying $(x, B) \in \iota$ for all $x \in T$.

The *supplement* of such a tBD is $(V, \mathcal{B}, \bar{\iota})$, where $\bar{\iota} = (V \times \mathcal{B}) \setminus \iota$. When $(x, B) \in \iota$, it is said that x and B are *incident*. For a block $B \in \mathcal{B}$, notation is usually abused by writing B also for the set of points in V which are incident with B. With this in mind, it makes sense to drop the flags from this notation:

by a point x lying 'in' a block B, we simply mean $(x, B) \in \iota$. Therefore, $\mathcal{B}$ can be regarded as a multiset, or family, of subsets of V. If this family of blocks is itself a set, or in other words when there are no repeated blocks, the tBD is called *simple*.

Although $\lambda = 0$ is permitted in the definition, we assume unless otherwise mentioned that $\lambda > 0$. Otherwise, $\mathcal{B}$ is empty and the tBD is trivial.

Observe also that a tBD is uninteresting for $t = 0$. There is no condition and λ is just the number of blocks. For $t = 1$, each point must be incident with same number of blocks. If also $\lambda = 1$, then $\mathcal{B}$ is just a partition of V. These systems are trivial to construct, though specialized results for $t = 1$ are sometimes of interest. In any case, it is generally assumed that $t \geq 2$. The well known Fano plane is an example of a 2BD with index unity and all blocks of size 3.

Until Chapter 5, all blocks are be assumed to have a common size k with $t \leq k \leq v = |V|$. The relevant structure is then often referenced by its parameters as a t-(v, k, λ) *design*, or simply a t-*design*.

The situations $k = t$ and $k = v$ lead again to designs which are 'uninteresting'. We assume throughout that $k > t$ and – with minor exceptions – that $v > k$

The reader is cautioned that the word 'design' has many meanings in discrete mathematics. Not only have we already introduced 't-designs' and the more general 'tBDs', but as we will see in Chapter 6 the term is used in other completely different contexts. However, the common ground in all uses of the terminology is that a design is in some sense an 'approximation'. For our initial investigations of t-(v, k, λ) designs, we can loosely think of these set systems as a 'level t' approximation of the set $\binom{V}{k}$ of all $\binom{v}{k}$ subsets of size k. Loosely speaking, a t-design balances all t-subsets by k-subsets.

A *configuration* $\mathcal{C}$ is a collection of subsets from some – usually small – generic set U. To say that a design $(V, \mathcal{B})$ *contains* a configuration means that there exists an injection $U \hookrightarrow V$ so that (the image of) $\mathcal{C}$ is a subcollection of $\mathcal{B}$. A very large amount of research has gone into the construction and enumeration of designs containing or avoiding various configurations. The focus of this manuscript is on structural constraints – which can be seen as nonexistence results – for designs containing a given configuration. In fact, this configuration can be as simple as a single block, which of course is present

in any design with $\lambda > 0$.

The first constraint for design parameters stems from a well known 'counting argument'.

Proposition 1.1. *For $0 \le i \le t$ and $I \subset V$ with $|I| = i$, the number of blocks containing I in a t-(v, k, λ) design is a constant $\lambda\binom{v-i}{t-i}/\binom{k-i}{t-i}$. In particular, there are $\lambda\binom{v}{t}/\binom{k}{t}$ blocks in a t-(v, k, λ) design.*

Proof. Count in two ways the number of ordered pairs (T, B), where $|T| = t$ and B is a block with $I \subseteq T \subseteq B$. □

It follows that every t-design is also an i-design for $i \le t$. As the count of blocks above is obviously an integer, this result implies the necessary conditions

$$\binom{k-i}{t-i} \Big| \lambda \binom{v-i}{t-i}, \quad 0 \le i \le t. \tag{1.1}$$

Parameters t, k, v, λ which satisfy all these divisibility requirements are said to be *admissible*. A detailed treatment of this and other first principles on t-designs can be found in [26], chapter 19.

A somewhat more subtle family of constraints exists on the parameters of a t-design. Let H be any subset of V with $|H| = h$. Suppose there are z_j blocks which intersect H in j points for $j = 0, 1, \ldots, k$. Count the ordered pairs (I, B), where B is a block and $I \subseteq B \cap H$ with $|I| = i$ in two ways. Starting with a choice of either B (and using the z_j) or I (and using Proposition 1.1) yields the system

$$\sum_{j=0}^{k} \binom{j}{i} z_j = \lambda \binom{h}{i} \binom{v-i}{t-i} \binom{k-i}{t-i}^{-1}, \quad i = 0, 1, \ldots, t. \tag{1.2}$$

These are the *moment equations*. The existence of nonnegative integral z_j solving (1.2) has been frequently exploited to obtain inequalities or other nonexistence results on designs. This technique is often called the 'method of moments'. Dropping one of either the integrality or nonnegativity condition on the z_j makes the solubility issue for (1.2) more tractable. The work that follows here pursues the nonnegativity condition, relaxing integrality, but for a more general system. Working from the integrality condition results in *signed designs*; see [31], for example.

Let $t, k, v \in \mathbb{N}$ with $t \leq k \leq v$. From now on, the v-set V is assumed to have some arbitrary ordering, say $\{1, \ldots, v\}$. By a *t-vector* on V, we mean a vector in $\mathbb{R}^{\binom{v}{t}}$ indexed by the t-subsets of V, which for this purpose also have some fixed ordering. If $X \subseteq V$, let $\mathbf{1}_X$ be the *characteristic* t-vector of X, defined by

$$\mathbf{1}_X(T) = \begin{cases} 1 & \text{if } T \subseteq X, \\ 0 & \text{otherwise.} \end{cases}$$

The $\binom{v}{t} \times \binom{v}{k}$ *inclusion matrix* W^v_{tk} (or simply W_t) has rows and columns indexed by all t-subsets and k-subsets of V, respectively, with

$$W^v_{tk}(T, K) = \begin{cases} 1 & \text{if } T \subseteq K, \\ 0 & \text{otherwise.} \end{cases}$$

At times, one may wish to restrict the possible choices of k-subsets. For a family $\mathcal{K} \subseteq \binom{V}{k}$, define the modified matrix $W^v_{tk}|\mathcal{K}$ to be the restriction of W^v_{tk} to those columns indexed over $\mathcal{K}$.

Let $\mathbf{1}$ denote the vector (whose dimension is understood from context) with all entries equal to 1. Using matrix multiplication, there exists a t-(v, k, λ) design if and only if the system

$$W^v_{tk}\boldsymbol{\phi} = \lambda\mathbf{1} \tag{1.3}$$

has a nonnegative integral solution $\boldsymbol{\phi}$. The k-vector $\boldsymbol{\phi}$ simply encodes the number of occurrences of each possible k-subset as a block. Indeed, we shall not distinguish between $\boldsymbol{\phi}$ and the corresponding design.

Note that since $W^v_{tk}\mathbf{1} = \binom{v-t}{k-t}\mathbf{1}$, it is always true that (1.3) has a nonnegative *rational* solution. This at first seems unfortunate for the prospects of structural results through relaxing integrality. However, in the work which follows, a simple modification of (1.3) leads to useful results.

Suppose a design $(V, \mathcal{B})$ contains a certain configuration $\mathcal{C}$, with respective k-vectors $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$. Consider the blocks not in $\boldsymbol{\psi}$. They yield a nonnegative k-vector $\mathbf{x}$ with $\boldsymbol{\phi} = \mathbf{x} + \boldsymbol{\psi}$.

Theorem 1.2. *The existence of a t-(v, k, λ) design containing a configuration $\boldsymbol{\psi}$ is equivalent to a nonnegative integral solution $\mathbf{x}$ of*

$$W_t\mathbf{x} = \lambda\mathbf{1} - W_t\boldsymbol{\psi}. \tag{1.4}$$

As we shall see, relaxing to nonnegative real (thus rational) $\mathbf{x}$ becomes nontrivial in general.

For instance, we often consider the case $\boldsymbol{\psi} = (1, 0, \ldots, 0)$, corresponding to $\mathcal{C}$ being a single block. It is shown in Section 2.4 that for this configuration (1.4) leads to a variant of equations (1.2). In Section 3.1, from the same configuration, we recover the famous inequality

$$b \geq \binom{v}{s}$$

of Ray-Chaudhuri and Wilson [24] on the number of blocks b in a $2s$-design on $v \geq k + s$ points. And in Section 6.1, we actually prove that (1.4) with $\boldsymbol{\psi} = (1, 0, \ldots, 0)$ is nearly equivalent to Delsarte's inequalities, [7]. This is not even to mention the possibility of taking different configurations $\boldsymbol{\psi}$, which in Chapter 4 leads to a unification of various old and new inequalities on block intersection.

1.2 Convexity and polyhedral cones

For more on the definitions and proofs omitted in this section, see the book [27]. A *cone* κ in a finite-dimensional real vector space U is a subset of U for which $c_1u_1 + c_2u_2 \in \kappa$ whenever $u_1, u_2 \in \kappa$ and $c_1, c_2 \geq 0$. Therefore, κ is a convex subset of U. The cone *generated by* $\{u_1, \ldots, u_n\} \subset U$ is the set $\kappa = \{c_1u_1 + \cdots + c_nu_n : c_i \geq 0\}$. Should these u_i be linearly independent, the cone is said to be of *dimension* n. A cone $\kappa \subseteq U$ is *full* if its dimension agrees with that of U, and is *pointed* if $u, -u \in \kappa$ implies $u = 0$. Here, all cones are assumed to be *polyhedral*; that is, they are full, pointed, and generated by a finite set. A *face* of κ is a cone $\eta \subseteq \kappa$ such that for all $u \in \eta$, if $u = u_1 + u_2$ with $u_1, u_2 \in \kappa$, then $u_1, u_2 \in \eta$. A face of dimension 1 is called an *extremal ray* of κ, while a face of codimension 1 is called a *facet* of κ.

The following is a simplified 'cone version' of the Krein-Milman Theorem, which states that every compact, convex set in a finite dimensional space is the convex hull of its extreme points.

Proposition 1.3. *Let $\kappa \subset U$ be a polyhedral cone and suppose $\{u_1\}, \ldots, \{u_n\}$ generate all the extremal rays of κ. Then $\{u_1, \ldots, u_n\}$ generates κ.*

Let U' be the dual space of U and let κ be a cone in U. Then $\kappa' = \{y \in U' : \langle y, u\rangle \geq 0\}$ is a cone called the *dual* of κ. The space U'' can be identified with U so that $\kappa'' = \kappa$. The following correspondence is of particular interest:

The dual of a facet of κ is an extremal ray of κ'.

For $y \in U'$, $y \neq 0$, the dual of the cone generated by $\{y\}$ is a *half-space* of U, and y is a *supporting vector* for any cone contained in this half-space. If y is a supporting vector for κ and $\eta = \kappa \cap y^{\perp}$ is a face of κ, then y is said to *support* κ *along* η. A result of fundamental importance is that a cone κ is the intersection of all half-spaces described by supporting vectors of κ. Theorem 1.4 below states this in the concrete setting which shall be used herein.

The discussion from now on focuses on polyhedral cones in real Euclidean space. The usual inner product $\langle \cdot, \cdot \rangle$ is used, and we must agree on a convention for vectors. The vector $\mathbf{x}$ is used in what follows to represent a column vector, while $\mathbf{y}$ normally represents a row vector, say in the dual space. Otherwise, we tend to avoid the use of transpose, leaving the shape of a vector to be deduced from context.

Given an $m \times n$ matrix A, the set $\texttt{cone}(A) = \{A\mathbf{x} : \mathbf{x} \in \mathbb{R}^n, \mathbf{x} \geq \mathbf{0}\}$ is a closed and polyhedral cone in $\mathbb{R}^m$. The dimension of $\texttt{cone}(A)$ is equal to the rank of A. The following well known result provides necessary and sufficient conditions for a point to belong to $\texttt{cone}(A)$.

Theorem 1.4 (Farkas Lemma)**.** *Let A be an $m \times n$ matrix, and $\mathbf{b} \in \mathbb{R}^m$. The equation $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x} \geq \mathbf{0}$ (that is, $\mathbf{b} \in \texttt{cone}(A)$) if and only if $\langle \mathbf{y}, \mathbf{b}\rangle \geq 0$ for all $\mathbf{y} \in \mathbb{R}^m$ such that $\mathbf{y}A \geq \mathbf{0}$.*

Remarks: One direction of this result is immediate. Suppose $A\mathbf{x} = \mathbf{b}$ has a nonnegative solution $\mathbf{x} \in \mathbb{R}^n$, and let $\mathbf{y}$ be such that $\mathbf{y}A \geq \mathbf{0}$. Then

$$\langle \mathbf{y}, \mathbf{b}\rangle = \langle \mathbf{y}, A\mathbf{x}\rangle = \langle \mathbf{y}A, \mathbf{x}\rangle \geq 0.$$

The converse is deeper, relying on the existence of a *separating hyperplane* between $\texttt{cone}(A)$ and a point not in this cone.

When $\texttt{cone}(A)$ is full and pointed, it is enough by Proposition 1.3 to check the condition in Theorem 1.4 for $\mathbf{y}$ corresponding to facets of $\texttt{cone}(A)$.

Roughly speaking, facets of $\texttt{cone}(A)$ provide the family of strongest tests for $\mathbf{b} \in \texttt{cone}(A)$. Since there are a finite number of facets of $\texttt{cone}(A)$, it is a finite problem to determine whether $A\mathbf{x} = \mathbf{b}$ has a nonnegative solution $\mathbf{x}$. However this problem is seldom easy in practice.

A variant of the *simplex algorithm* can be used to find facets of $\texttt{cone}(A)$. This is given below for completeness. In the second step, $\mathrm{col}(A)$ is used to denoting the set of columns of A.

1. Start with a random $\mathbf{y} \in \mathbb{R}^m$ such that $\mathbf{y}A \geq \mathbf{0}$.
2. If $\dim(\mathrm{span}\{\mathbf{a} \in \mathrm{col}(A) : \langle \mathbf{y}, \mathbf{a} \rangle = 0\}) = m - 1$, then $\mathbf{y}$ already supports $\texttt{cone}(A)$ on a facet. Otherwise, choose a random $\mathbf{z} \in \mathbb{R}^m$ such that $\mathbf{z}A$ has a positive coordinate but vanishes on at least the same coordinates as $\mathbf{y}A$.
3. Let
$$\epsilon = \min \frac{(\mathbf{y}A)_i}{(\mathbf{z}A)_i},$$
where the minimum is taken over all i for which the quantity is defined and positive.
4. Set $\mathbf{y} := \mathbf{y} - \epsilon\mathbf{z}$ and return to step 2.

It should be noted that the columns of W^v_{tk} (defined in Section 1.1) are linearly independent and all lie in the nonnegative orthant of the space of t-vectors. So step 1 in the above algorithm is trivial. The rank of W^v_{tk} (and of any $W^v_{tk}|\mathcal{K}$ we consider) is $\binom{v}{t}$; see [33]. Therefore, the cones we consider are indeed full. Unfortunately, when the parameters – particularly t – are large, the simplex algorithm is too slow to be of much use, though infinite families of facets of may be guessed by observing the output from this algorithm.

Our main contribution in this manuscript is an analysis of the Farkas Lemma applied to (1.4).

Theorem 1.5 (Cone condition)**.** *In a t-(v, k, λ) design containing a configuration $\boldsymbol{\psi}$, we have*

$$\lambda\langle \mathbf{y}, \mathbf{1} \rangle \geq \langle \mathbf{y}W_t, \boldsymbol{\psi} \rangle \tag{1.5}$$

for every supporting t-vector $\mathbf{y}$ of $\texttt{cone}(W_t)$. Equality occurs in (1.5) only if $\mathbf{y}W_t$ vanishes at every block not in $\boldsymbol{\psi}$. If (1.5) holds for all supporting t-vectors $\mathbf{y}$ of $\texttt{cone}(W_t)$, then $\lambda\mathbf{1} - W_t\boldsymbol{\psi} \in \texttt{cone}(W_t)$.

In certain of our proofs, it is often the case that other supporting vectors $\mathbf{y}$ of $\texttt{cone}(W_t)$ – those which do not necessarily support along a facet – are easier to use with Theorem 1.5. However, facets are of some combinatorial interest on their own; see Section 2.2 for more.

In any case, it remains to investigate supporting t-vectors $\mathbf{y}$ for $\texttt{cone}(W_t)$. This begins in Section 2.1.

We should close this section by pointing out an important connection to linear programming (LP). Given an $m \times n$ matrix A, constant vectors $\mathbf{b}$ and $\mathbf{c}$, the LP *primal* problem is

$$\begin{array}{rl} \text{maximize} & \langle \mathbf{c}, \mathbf{x} \rangle \\ \text{subject to} & A\mathbf{x} \leq \mathbf{b}, \quad \mathbf{x} \geq \mathbf{0}. \end{array} \tag{1.6}$$

It's *dual* problem is

$$\begin{array}{rl} \text{minimize} & \langle \mathbf{y}, \mathbf{b} \rangle \\ \text{subject to} & \mathbf{y}A \geq \mathbf{c}, \quad \mathbf{y} \geq \mathbf{0}. \end{array} \tag{1.7}$$

It is easy to see that

$$\langle \mathbf{y}, \mathbf{b} \rangle \geq \mathbf{y}A\mathbf{x} \geq \langle \mathbf{c}, \mathbf{x} \rangle.$$

On the other hand, it is also true that, when both problems are feasible, the maximum in (1.6) and the minimum in (1.7) agree. See [27] for further details.

A small modification – obtained by replacing A with $\begin{bmatrix} A \\ -A \end{bmatrix}$ – allows for replacing '$\leq$' with '$=$' in (1.6) and dropping '$\mathbf{y} \geq \mathbf{0}$' in (1.7). Put $\mathbf{c} = \mathbf{0}$ in this modified system. We obtain

$$\begin{array}{rl} \text{maximize} & 0 \\ \text{subject to} & A\mathbf{x} = \mathbf{b}, \quad \mathbf{x} \geq \mathbf{0}, \end{array} \tag{1.8}$$

and

$$\begin{array}{rl} \text{minimize} & \langle \mathbf{y}, \mathbf{b} \rangle \\ \text{subject to} & \mathbf{y}A \geq \mathbf{0}. \end{array} \tag{1.9}$$

Then, we see through this alternative approach that (1.8) is feasible if and only if the conditions of the Farkas Lemma hold. So we may occasionally view the cone condition as an LP minimization problem (1.9) over $\mathbf{y}$. Should we obtain a minimum $\langle \mathbf{y}, \mathbf{b} \rangle < 0$, it follows that $A\mathbf{x} = \mathbf{b}$ has no nonnegative solution.

1.3 Some binomial identities

For use in later chapters, some facts and identities involving binomial coefficients are presented here. The simple relation

$$\binom{\alpha}{\beta}\binom{\beta}{\gamma} = \binom{\alpha}{\gamma}\binom{\alpha-\gamma}{\beta-\gamma} \tag{1.10}$$

is used frequently.

As is quite standard, the top argument of a binomial coefficient may take on non-integer values. The meaning to be understood is that for $t \in \mathbb{N}_0$ and $x \in \mathbb{R}$,

$$\binom{x}{t} \equiv \frac{1}{t!}\, x(x-1)\cdots(x-t+1) \equiv \frac{(x)_t}{t!},$$

a polynomial of degree t in x. We observe without proof that

$$\left\{\binom{x}{0}, \binom{x}{1}, \ldots, \binom{x}{t}\right\}$$

forms a basis for the vector space of polynomials in x of degree $\leq t$. Similar statements are true for several variables.

For binomial identities involving summations, it is convenient at times to use the hypergeometric notation

$${}_pF_q\left[\begin{array}{cccc} \alpha_1, & \alpha_2, & \ldots, & \alpha_p \\ & \beta_1, & \ldots, & \beta_q \end{array} ;\xi\right] \equiv \sum_{j=0}^{\infty} \frac{(\alpha_1)_j(\alpha_2)_j\ldots(\alpha_p)_j}{j!(\beta_1)_j\ldots(\beta_q)_j}\xi^j,$$

where $(\alpha)_j = \alpha(\alpha+1)\ldots(\alpha+j-1)$. The transformation from a (finite) sum of products of binomial coefficients into this notation is routine and omitted in what follows. References and proofs for many hypergeometric identities can be found in [1]. A vintage formula of Gauss is now given as a starting point.

Proposition 1.6. *If $a, b, c \in \mathbb{R}$ with $c > a + b$, then*

$${}_2F_1\left[\begin{array}{cc} a, & b \\ & c \end{array} ;1\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

Recall

$$\Gamma(z) \equiv \int_0^\infty u^{z-1}e^{-u}\, du,$$

which reduces to $(z-1)!$ if $z \in \mathbb{N}$. So when meaningful, the right side in Proposition 1.6 can be written in terms of binomial coefficients. Some easy consequences are the 'convolution identities'

$$\sum_{j=0}^{t} \binom{x}{j}\binom{y}{t-j} = \binom{x+y}{t}, \tag{1.11}$$

$$\sum_{j=0}^{t} (-1)^{t-j}\binom{x}{j}\binom{y-j}{t-j} = \binom{x-y+t-1}{t}, \tag{1.12}$$

$$\begin{aligned}\sum_{j=0}^{t} &\binom{j}{i}\binom{t-j}{r-i}\binom{x}{j}\binom{y}{t-j} \\ &= \binom{x+y-r}{t-r}\binom{x}{i}\binom{y}{r-i}, \quad (t \geq r),\end{aligned} \tag{1.13}$$

and

$$\begin{aligned}&\sum_{j=0}^{t-r} (-1)^j \binom{t}{j}^{-1}\binom{x}{j}\binom{y}{t-j} \\ &= \frac{t+1}{x+y-t}\left[\binom{y}{t+1} + (-1)^{t-r}\binom{x}{t-r+1}\binom{y}{r}\binom{t+1}{r}^{-1}\right], \quad (0 \leq r \leq t).\end{aligned} \tag{1.14}$$

Proof of (1.13). Apply equation (1.10) to the summand, shift the index of summation, and use (1.11). □

Proof of (1.14). The given (terminating) sum is

$$\left(\binom{y}{t} - (-1)^{t-r+1}\binom{x}{t-r+1}\binom{y}{r-1}\binom{t}{r-1}^{-1}\right) {}_2F_1\left[\begin{matrix} 1, & -x \\ & 1-t+y \end{matrix}; 1\right],$$

which simplifies to the right side by Proposition 1.6 and the special case $\frac{\beta}{\alpha}\binom{\alpha}{\beta} = \binom{\alpha-1}{\beta-1}$ of equation (1.10). □

Now, a classical identity of Saalschütz is presented. A reference such as [1] can be consulted for details.

Proposition 1.7. *Suppose* $1+w+x-n = y+z$ *with* $n \in \mathbb{N}_0$. *Then*

$${}_3F_2\left[\begin{matrix} -n, & w, & x \\ & y, & z \end{matrix}; 1\right] = \frac{(y-w)_n (y-x)_n}{(y)_n (y-w-x)_n}.$$

The identity of Saalschütz is essentially used, in Chapter 3, for an important summation that arises.

Lemma 1.8. *For $v \geq k+s$,*

$$\sum_{j=0}^{s}(-1)^j\binom{v-i-j}{k-i-j}\binom{v-s}{j}\binom{k-j}{s-j}=\binom{v-s}{k-s}.$$

Proof. Let $f(k,i)$ denote the given sum. The familiar $\binom{x-1}{i-1}+\binom{x-1}{i}=\binom{x}{i}$ gives rise to $f(k,i)=f(k,i+1)+f_{\text{Saal}}(k+1,i+1)$, where

$$\begin{aligned} f_{\text{Saal}}(k,i) &= \sum_{j=0}^{s}(-1)^j\binom{v-i-j}{k-i-j}\binom{v-s}{j}\binom{k-1-j}{s-j} \\ &= \binom{v-i}{k-i}\binom{k-1}{s}{}_3F_2\left[\begin{matrix} -s, & s-v, & i-k & ;1 \\ & 1-k, & i-v & \end{matrix}\right]. \end{aligned}$$

By Proposition 1.7, $f_{\text{Saal}}(k,i)=0$ unless $i=0$. So

$$\begin{aligned} f(k,i)=f(k,0) &= \sum_{j=0}^{s}(-1)^j\binom{v-j}{k-j}\binom{v-s}{j}\binom{k-j}{s-j} \\ &= \binom{v-s}{k-s}\sum_{j=0}^{s}(-1)^j\binom{v-j}{s-j}\binom{v-s}{j} \\ &= \binom{v-s}{k-s}, \end{aligned}$$

where equations (1.10) and (1.12) have been used. □

We close with an important family of polynomials. For $0<s\leq k, w\leq v$, define

$$g_{s,k}^{w}(x)=\sum_{i=0}^{s}(-1)^{s-i}\frac{\binom{v-s}{i}\binom{w-1-i}{s-i}\binom{k-i}{s-i}}{\binom{s}{i}}\binom{x}{i}.$$

This is a multiple of a (terminating) hypergeometric series of type ${}_3F_2$ with unit argument. Alternate presentations of it arise from hypergeometric identities or facts related to orthogonal polynomials, as is mentioned in [29]. For instance, one has the relations

$$g_{s,k}^{w}(x)=g_{s,w-1}^{k+1}(x)=(-1)^s g_{s,k}^{v-w+1}(k-x) \tag{1.15}$$

and

$$g^w_{s,k}(w) = \binom{v}{s}^{-1}\binom{k}{s}\binom{v-w}{s}\sum_{i=0}^{s}\left[\binom{v}{i}-\binom{v}{i-1}\right]\frac{\binom{w}{i}\binom{v-k}{i}}{\binom{k}{i}\binom{v-w}{i}}. \tag{1.16}$$

In the special case $w = k$, the polynomial $g_s(x) \equiv g^k_{s,k}(x)$ – also known as a *Hahn polynomial* – frequently appears in the design theory literature. We will revisit these polynomials in Chapters 3 and 4.

Chapter 2

Symmetries of the Inclusion Matrix

In general, it is difficult to determine if an arbitrary vector $\mathbf{b} \in \mathbb{R}^{\binom{v}{t}}$ is contained in $\texttt{cone}(W^v_{tk})$. Dimensions alone often render this question impractical. However, most design-theoretic applications enjoy abundant symmetry, which is usually prudent to exploit.

2.1 Invariant partitions and polynomials

In this section, $\mathbf{b}$ is some fixed t-vector as defined in Section 1.1. Consider the action of the symmetric group $\mathcal{S}_V$ on V. For $\sigma \in \mathcal{S}_V$ and $\mathbf{y}$ a t-vector, define $\mathbf{y}^\sigma$ by $\mathbf{y}^\sigma(T) = \mathbf{y}(\sigma^{-1}(T))$. This vector is obtained from $\mathbf{y}$ simply by permuting its coordinates according to the inherited action on t-subsets of V. It is clear that

$$\langle \mathbf{y}^\sigma, \mathbf{b}^\sigma \rangle = \sum_{|T|=t} \mathbf{y}(\sigma^{-1}(T))\mathbf{b}(\sigma^{-1}(T)) = \sum_{|T|=t} \mathbf{y}(T)\mathbf{b}(T) = \langle \mathbf{y}, \mathbf{b} \rangle. \tag{2.1}$$

If $\mathbf{b}^\sigma = \mathbf{b}$, then it is said that $\mathbf{b}$ is *invariant* under σ. The set of all such $\sigma \in \mathcal{S}_V$ is a group because $(\mathbf{b}^\sigma)^\tau = \mathbf{b}^{(\sigma\tau)}$ follows immediately from the definition. Define this group to be $\mathrm{stab}(\mathbf{b})$.

Let $\mathbb{H}^d(V)$ denote the set of partitions of V into d parts which are ordered according to the implicit ordering in V. For a set $S \subseteq V$ and $\Omega = (U_1, \ldots, U_d) \in \mathbb{H}^d(V)$, define

$$S \cap \Omega = (S \cap U_1, \ldots, S \cap U_d) \in \mathbb{H}^d(S).$$

For $s \in \mathbb{N}_0$, define the simplex of lattice points

$$\mathbb{H}^d(s) = \left\{ (n_1, \ldots, n_d) \in \mathbb{N}_0^d : \sum n_i = s \right\}.$$

Let $|\Omega|$ denote the integer partition $(|U_1|, \ldots, |U_d|) \in \mathbb{H}^d(v)$. The set of $\sigma \in \mathcal{S}_V$ which leave each U_i invariant is the subgroup $\mathrm{stab}(\Omega) = \mathcal{S}_{U_1} \times \cdots \times \mathcal{S}_{U_d}$ of $\mathcal{S}_V$. Consider the usual ordering $\preceq$[1] and the associated lattice structure on $\mathbb{H}^d(V)$.

Call $\Omega \in \mathbb{H}^d(V)$ an *invariant partition* for $\mathbf{b}$ if $\mathrm{stab}(\Omega) \subseteq \mathrm{stab}(\mathbf{b})$. Of primary interest are invariant partitions which are maximal in $\mathbb{H}^d(V)$, in the sense that any other such Ω' satisfies $\Omega' \preceq \Omega$. For the remainder of this section, assume $\Omega = (U_1, \ldots, U_d)$ is some invariant partition for $\mathbf{b}$, with $\omega = |\Omega|$.

Define

$$\overline{\mathbf{y}} = \frac{1}{|\mathrm{stab}(\Omega)|} \sum_{\sigma \in \mathrm{stab}(\Omega)} \mathbf{y}^\sigma.$$

An elementary consequence of the definitions and equation (2.1) is that $\langle \mathbf{y}, \mathbf{b} \rangle = \langle \overline{\mathbf{y}}, \mathbf{b} \rangle$. for any $\mathbf{y} \in \mathbb{R}^{\binom{v}{t}}$.

Two subsets $S, S' \subseteq V$, which satisfy $\tau(S) = S'$ for some $\tau \in \mathrm{stab}(\Omega)$, are called *equivalent* under Ω. Note that S, S' are equivalent under Ω if and only if $|S \cap \Omega| = |S' \cap \Omega| \in \mathbb{H}^d(s)$, where $s = |S| = |S'|$.

Lemma 2.1. *Let $T_1, T_2 \subset V$ be t-sets equivalent under Ω. Then $\overline{\mathbf{y}}(T_1) = \overline{\mathbf{y}}(T_2)$.*

Proof. Let $\tau \in \mathrm{stab}(\Omega)$ be such that $\tau(T_1) = T_2$. Then

$$\begin{aligned} |\mathrm{stab}(\Omega)|\ \overline{\mathbf{y}}(T_1) &= \sum_{\sigma \in \mathrm{stab}(\Omega)} \mathbf{y}^\sigma(T_1) = \sum_{\sigma \in \mathrm{stab}(\Omega)} \mathbf{y}(\sigma^{-1}(T_1)) \\ &= \sum_{\sigma \in \mathrm{stab}(\Omega)} \mathbf{y}(\sigma^{-1}\tau^{-1}(T_2)) = \sum_{\sigma \in \mathrm{stab}(\Omega)} \mathbf{y}^{\tau\sigma}(T_2) \end{aligned}$$

[1] If each part of Ω_1 belongs to a single part of Ω_2, then $\Omega_1 \preceq \Omega_2$.

$$= \sum_{\sigma' \in \mathrm{stab}(\Omega)} \mathbf{y}^{\sigma'}(T_2) = |\mathrm{stab}(\Omega)|\ \overline{\mathbf{y}}(T_2).$$

□

This allows for writing

$$\overline{\mathbf{y}} = \sum_{\varphi \in \mathbb{H}^d(t)} a_\varphi \sum_{|T \cap \Omega| = \varphi} \mathbf{1}_T \tag{2.2}$$

for some real coefficients a_φ. Of course, the contribution to the sum is 0 unless $\varphi \leq \omega$.

For $\mathbf{x} = (x_1, \ldots, x_d)$ and $\varphi \in \mathbb{N}_0^d$, define

$$\mathbf{x}^\varphi = \prod_{i=1}^d x_i^{\varphi_i} \quad \text{and} \quad \binom{\mathbf{x}}{\varphi} = \prod_{i=1}^d \binom{x_i}{\varphi_i}.$$

Let $k \in \mathbb{N}$. Consider the real algebra $\Lambda = \mathbb{R}[x_1, \ldots, x_d]/(-k + \sum_1^d x_i)$. Then Λ can be expressed as an increasing union $\Lambda = \cup_{t=0}^\infty \Lambda_t$, where

$$\Lambda_t = \mathrm{span}\{[\mathbf{x}^\varphi] : \varphi \in \mathbb{H}^d(t)\}.$$

It is easy to show that another basis for Λ_t is $\{[\binom{\mathbf{x}}{\varphi}] : \varphi \in \mathbb{H}^d(t)\}$. Indeed, the transition matrix expressing the binomial coefficients in terms of the monomials is upper triangular (after appropriate indexing) with diagonal entries $1/\varphi!$. Consider any $[f] \in \Lambda_t$. It follows that when $\sum_1^d x_i = k$,

$$f(\mathbf{x}) = \sum_{\varphi \in \mathbb{H}^d(t)} a_\varphi \binom{\mathbf{x}}{\varphi}$$

for some $a_\varphi \in \mathbb{R}$. For such an f expressed in this way, define the corresponding t-vector $\mathbf{y}_f$ as on the right side of equation (2.2).

Lemma 2.2. *Suppose K is a k-set with $|K \cap \Omega| = \psi \in \mathbb{H}^d(k)$. Then $\langle \mathbf{y}_f, \mathbf{1}_K \rangle = f(\psi)$*

Proof. The dot product on the left counts a_φ times the number of t-subsets T of K for which $|T \cap \Omega| = \varphi$, summed over all $\varphi \in \mathbb{H}^d(t)$. There are $\binom{\psi}{\varphi}$ such t-sets for a given φ, so this count agrees with the right hand side. □

We now arrive at our main result of this section, relating this polynomial correspondence $f \leftrightarrow \mathbf{y}_f$ to the cone condition.

Theorem 2.3. *Suppose $\Omega \in \mathbb{H}^d(V)$ is an invariant partition for $\mathbf{b}$, and let $\omega = |\Omega|$. Define $W = W^v_{tk}|\mathcal{K}$. Then $\mathbf{b} \in \mathtt{cone}(W)$ if and only if*

$$\langle \mathbf{y}_f, \mathbf{b} \rangle = \sum_{\varphi \in \mathbb{H}^d(t)} a_\varphi b_\varphi \binom{\omega}{\varphi} \geq 0$$

for all $f \in \mathbb{R}[x_1, \ldots, x_d]$ of degree $\leq t$ nonnegative on $\{|K \cap \Omega| : K \in \mathcal{K}\}$, where

$$f(\mathbf{x}) = \sum_{\varphi \in \mathbb{H}^d(t)} a_\varphi \binom{\mathbf{x}}{\varphi}$$

and

$$\mathbf{b} = \sum_{\varphi \in \mathbb{H}^d(t)} b_\varphi \sum_{|T \cap \Omega| = \varphi} \mathbf{1}_T.$$

Proof. By Lemma 2.2, the nonnegativity constraint on f is equivalent to $\langle \mathbf{y}_f, \mathbf{1}_K \rangle \geq 0$ for all k-sets K, or $\mathbf{y}_f W \geq \mathbf{0}$. Theorem 1.4 states that $\mathbf{b} \in \mathtt{cone}(W)$ if and only if $\langle \mathbf{y}, \mathbf{b} \rangle \geq 0$ whenever $\mathbf{y}W \geq \mathbf{0}$. Thus it is enough to prove this condition is equivalent to that when quantified over the Ω-invariant vectors $\mathbf{y}_f$. Suppose $\mathbf{y}W \geq \mathbf{0}$ implies $\langle \mathbf{y}, \mathbf{b} \rangle \geq 0$ for all $\mathbf{y} \in \mathbb{R}^{\binom{v}{t}}$. Then certainly $\mathbf{y}_f W \geq \mathbf{0}$ implies $\langle \mathbf{y}_f, \mathbf{b} \rangle \geq 0$ for all polynomials f of the given form. Conversely, suppose $\mathbf{y}_f W \geq \mathbf{0}$ implies $\langle \mathbf{y}_f, \mathbf{b} \rangle \geq 0$ for all f. Let $\mathbf{y} \in \mathbb{R}^{\binom{v}{t}}$ be arbitrary and assume $\mathbf{y}W_t \geq \mathbf{0}$. Observe for any $\sigma \in \mathcal{S}_V$ that the vector $\mathbf{y}^\sigma W_t$ is a rearrangement of $\mathbf{y}W_t$. So $\overline{\mathbf{y}}W_t \geq \mathbf{0}$. But $\overline{\mathbf{y}}$ is of the form $\mathbf{y}_f$ for some f. So $\langle \mathbf{y}, \mathbf{b} \rangle = \langle \overline{\mathbf{y}}, \mathbf{b} \rangle \geq 0$. $\square$

2.2 Facets and the extremal polynomials

Here, write $W = W^v_{tk}$. Define $W^\sigma(T, K) = W(\sigma^{-1}(T), K)$. This can be viewed as changing W by either a row or column permutation. For $\Omega \in \mathbb{H}^d(V)$, set

$$W_\Omega = \frac{1}{|\mathrm{stab}(\Omega)|} \sum_{\sigma \in \mathrm{stab}(\Omega)} W^\sigma.$$

Note that the cone $\texttt{cone}(W_\Omega)$ is full if and only if $d \geq 2$. It is a straightforward observation that $\Omega' \preceq \Omega$ implies $\texttt{cone}(W_\Omega) \subseteq \texttt{cone}(W_{\Omega'})$. This motivates the view of $\texttt{cone}(W)$ as a refinement of cones

$$\texttt{cone}(W) = \bigcup_{\Omega \in \mathbb{H}^d(V)} \texttt{cone}(W_\Omega)$$

indexed over the lattice of partitions of V.

Proposition 2.4. *Suppose $\Omega \in \mathbb{H}^d(V)$, $(d \geq 2)$ is an invariant partition for $\mathbf{y}$. Then $\mathbf{y}W_\Omega \geq \mathbf{0}$ implies $\mathbf{y}W \geq \mathbf{0}$. Moreover, if $\mathbf{y}$ supports W_Ω at a facet, then $\mathbf{y}$ supports W at a facet.*

Proof. Since $\mathbf{y}$ is Ω-invariant,

$$\mathbf{y}W = \frac{1}{|\mathrm{stab}(\Omega)|} \sum_{\sigma \in \mathrm{stab}(\Omega)} \mathbf{y}^{\sigma^{-1}} W = \frac{1}{|\mathrm{stab}(\Omega)|} \sum_{\sigma \in \mathrm{stab}(\Omega)} \mathbf{y}W^\sigma = \mathbf{y}W_\Omega.$$

This proves the first statement and, since each column of W_Ω is in the span of the columns of W,

$$\mathrm{span}\{\mathrm{col}(W_\Omega) : \mathbf{y}W_\Omega = 0\} \subseteq \mathrm{span}\{\mathrm{col}(W) : \mathbf{y}W = 0\}.$$

So if the subspace on the left has codimension 1, then so does the subspace on the right provided it is not full. But $\mathbf{y}$ supporting a facet of W_Ω implies $\langle \mathbf{y}, \sum_{\sigma \in \mathrm{stab}(\Omega)} \mathbf{1}_{\sigma K} \rangle > 0$ for some K. Again by invariance under Ω, it must be that $\langle \mathbf{y}, \mathbf{1}_K \rangle > 0$, and so $\mathbf{y}$ does in fact support a facet of $\texttt{cone}(W)$. □

Theorem 2.3 essentially describes the dual cone of $\texttt{cone}(W_\Omega)$. Rather than directly considering the supporting vectors, it is interesting to view this dual as the cone of d-variable polynomials of degree $\leq t$ which are nonnegative on the appropriate lattice points. By the remarks in Section 1.2, there is a correspondence between facets of $\texttt{cone}(W_t)$ and extremal rays of these cones of polynomials. It is not the aim of this work to thoroughly investigate such extremal rays. Indeed, this appears to be related to the subject of polynomial interpolation in several variables, for which relatively little is known in general; see [12]. However, the remainder of this section attempts to contribute some initial observations on these extreme polynomials.

For a set $S \subset \mathbb{R}^d$, let $P_t^d(S)$ denote the cone of d-variable polynomials of degree $\leq t$ which are nonnegative on S. It may be of interest to the reader that the (non-polyhedral) case when $d = 1$ and $S = [0, 1] \subset \mathbb{R}$ is investigated in [3]. Here though, it is assumed that S is a finite set.

Lemma 2.5. *Let $f, g \in P_t^d(S)$, $g \not\equiv 0$, and suppose $\{\mathbf{x} \in S : f(\mathbf{x}) = 0\}$ is a proper subset of $\{\mathbf{x} \in S : g(\mathbf{x}) = 0\}$. Then f does not generate an extremal ray of $P_t^d(S)$.*

Proof. Choose $\epsilon > 0$ such that $f - \epsilon g \in P_t^d(S)$. Neither this polynomial, nor ϵg are identically zero by the condition given. Furthermore, ϵg is not in the ray generated by f. Thus by the definition in Section 1.2, f cannot generate an extremal ray since $f = (f - \epsilon g) + \epsilon g$. □

While characterizing the nonzero 'maximally vanishing' polynomials in $P_t^d(S)$ is difficult in general, there is an easy solution in one variable. The following is a variant of Gale's evenness condition, which characterizes the facets of cyclic polytopes, [27].

Theorem 2.6. *Suppose $|S| \geq t + 1$ and let $f \in P_t^1(S)$ with $Z = \{x \in S : f(x) = 0\}$. Then f generates an extremal ray of $P_t^1(S)$ if and only if $|Z| = t$ and every two points of $S \setminus Z$ are separated by an even number of points of Z.*

Proof. By Lemma 2.5, any f generating an extremal ray must vanish maximally on S, so $|Z| = t$ and $f(x) = C \prod_{\zeta \in Z}(x - \zeta)$ for some $C \neq 0$. So in order for $f \geq 0$ on S, the evenness condition on Z must hold. Conversely, if $|Z| = t$ and $f(x) = C \prod_{\zeta \in Z}(x - \zeta) \in P_t^1(S)$ can be written as $f = g_1 + g_2$, where $g_1, g_2 \in P_t^1(S)$, then both g_1 and g_2 vanish on all of Z. As all degrees are $\leq t$, it follows that g_1 and g_2 are multiples of f. □

Vanishing subsets Z as in the theorem are here called *even sets*. It should be mentioned that a lower bound on the number of facets of W can be obtained by counting even subsets. In what follows, polynomials in the variables $x_1, \ldots, x_d$ which are nonnegative on $\mathbb{H}^d(k) \cap \{(x_1, \ldots, x_d) \leq \omega\}$ are often identified with polynomials in $d - 1$ variables, say $x_1, \ldots, x_{d-1}$, that are nonnegative on

$$\bigcup_{j \leq \omega_d} \mathbb{H}^{d-1}(k - j) \cap \{(x_1, \ldots, x_{d-1}) \leq (\omega_1, \ldots, \omega_{d-1})\}.$$

For instance, Theorems 2.3 and 2.6 applied to the interval of integers

$$S = \{\max(0, k - \omega_2), \ldots, \min(k, \omega_1)\}$$

give a characterization of facets for W_Ω when $\Omega \in \mathbb{H}^2(V)$ is a bipartition of V. A concrete description of these facets appears in Section 3.2.

It should be stressed that 'most' facets, even of $\texttt{cone}(W_2)$, remain uncharacterized. On a computer, we implemented the facet algorithm in Section 1.2 for $\texttt{cone}(W_{23}^v)$, $v \leq 10$. The algorithm helped us discover several facets in addition to those described by bipartitions. For instance,

$$\mathbf{y} = \begin{cases} -1 & \text{if } T = \{1,2\}, \\ 1 & \text{if } T = \{1,x\},\ x \neq 1,2, \\ 0 & \text{otherwise} \end{cases}$$

is a supporting 2-vector for the cone, as all triples receive weight 0 or 2. On each of the $v-2$ triples through $\{1,2\}$, and all $\binom{v-1}{3}$ triples not through 1, the weight vanishes. These span a space of t-vectors of codimension 1, thus intersecting the cone on a facet.

In fact, we found 19 different facets (up to permutation of points) for $\texttt{cone}(W_{23}^8)$. Some have associated 2-vectors $\mathbf{y}$ with up to 4 different values, and with extremely intricate symmetries. Typical facets, computed and displayed with *Mathematica*, are shown in Figure 2.1. Edges may be colored brighter shades of green (respectively, red) according to increasingly positive (respectively, negative) weights.

We conclude with a detailed look at one interesting example.

Example 2.1. One facet for $\texttt{cone}(W_{23}^8)$ is illustrated in Figure 2.2. The associated normal 2-vector $\mathbf{y}_f \in \mathbb{R}^{\binom{8}{2}}$ is formed from the edge weights in the diagram. A labeled edge between the circled sets represents all edges between the two sets receiving the indicated weight. Otherwise, missing edges correspond to a weight of zero.

Of the $\binom{8}{3} = 56$ possible 3-subsets of V, 36 have total inherited weight zero. Again, the fact that $\mathbf{y}_f$ supports a facet means the characteristic vectors of these triples span a codimension 1 subspace of $\mathbb{R}^{\binom{8}{2}}$. This vector $\mathbf{y}_f$ is invariant under $\mathrm{stab}(\Omega) \cong \mathcal{S}_2 \times \mathcal{S}_2 \times \mathcal{S}_4$. A (three variable) polynomial class $[f] \in \Lambda_2$ for $\mathbf{y}_f$ is given by

$$f(x_1,x_2,x_3) = 2\binom{x_1}{2} + 2\binom{x_2}{2} - x_1x_2 + x_2x_3.$$

Reducing modulo the ideal $(x_1+x_2+x_3-3)$ allows for the simplification $[f] = [(1-x_1)(2x_2-x_1)]$. The relevant values of $f^*(x_1,x_2) = (1-x_1)(2x_2-x_1)$ are given in the table below.

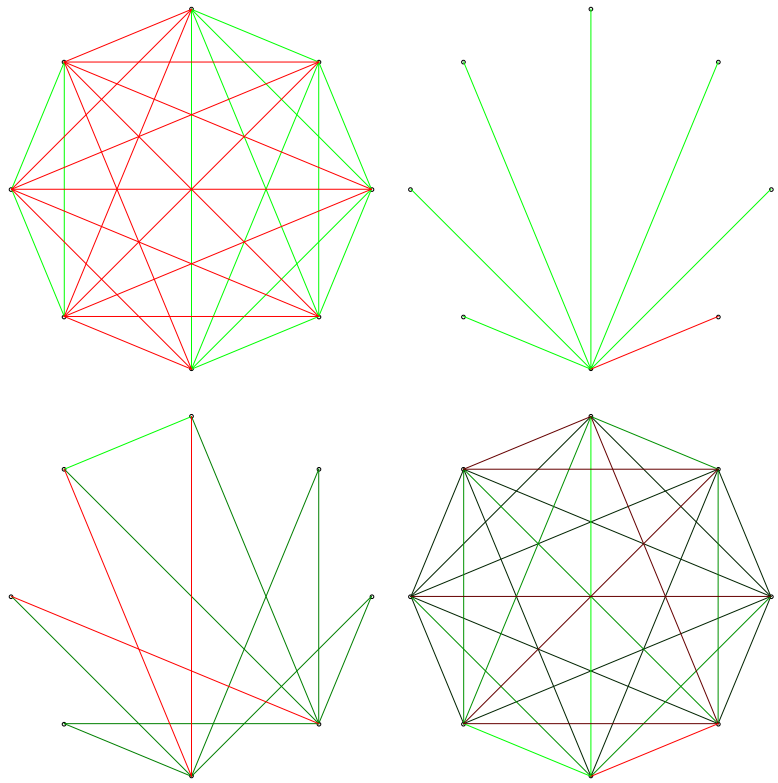

Figure 2.1: A color scheme for small facets of $\texttt{cone}(W_{23}^8)$.

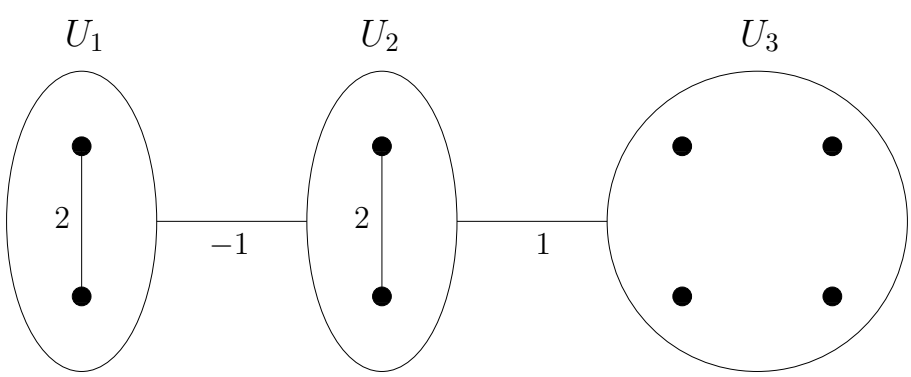

Figure 2.2: Edge weights for a facet of W_{23}^8.

f^*	0	1	2	x_2
0	0	2	4	
1	0	0	0	
2	2	0		
x_1				

2.3 Convex combinations

This brief section addresses a curious limitation of the cone condition, and has connections with the symmetry in W_t. Suppose Theorem 1.2 *fails* to rule out each of two configurations $\boldsymbol{\psi}_1$ and $\boldsymbol{\psi}_2$ in a hypothetical design. Then

$$\lambda\mathbf{1} - W_t\boldsymbol{\psi}_i \in \texttt{cone}(W_t), \qquad i = 1, 2.$$

By convexity, one also has

$$\lambda\mathbf{1} - \frac{1}{2}W_t(\boldsymbol{\psi}_1 + \boldsymbol{\psi}_2) \in \texttt{cone}(W_t) \tag{2.3}$$

What (2.3) is saying is that the 'average' of configurations $\boldsymbol{\psi}_1$ and $\boldsymbol{\psi}_2$ is also not ruled out.

Theorem 2.7. *Suppose the cone condition does not rule out a list of configurations $\{\boldsymbol{\psi}_i\}$. Then it does not rule out any convex combination of the configurations $\sum_i w_i\boldsymbol{\psi}_i$.*

For example, we present detailed results in Section 3.3 on repeated blocks in designs. Later, in Section 4.2, we rule out certain intersection numbers between pairs of blocks. Each of these methods invokes the cone condition.

Consider then a two-block configuration $\{B_1, B_2\}$ with characteristic vector $\boldsymbol{\psi}$. We have

$$W_t\boldsymbol{\psi} = \mathbf{1}_{B_1} + \mathbf{1}_{B_2} = \frac{1}{2}(2\mathbf{1}_{B_1} + 2\mathbf{1}_{B_2}).$$

Therefore, by Theorem 2.7, unless repeated blocks are disallowed by the cone condition, one should not expect any results which rule out two blocks intersecting in less than k points either. This remarkable fact is actually exploited to speed up certain computational work in Chapter 4.

Loosely speaking, high block density 'permits' lower block density in the sense of convexity.

2.4 The method of moments

Here, a generalization of the moment equations (1.2) is proposed. Suppose $\Omega \in \mathbb{H}^d(V)$ is an invariant partition for $\mathbf{b}$, and let $|\Omega| = \omega$. Define W^*_Ω to be the matrix having rows and columns indexed by $\mathbb{H}^d(t)$ and $\mathbb{H}^d(k)$, respectively, with

$$W^*_\Omega(\varphi, \psi) = \binom{\omega}{\psi}\binom{\psi}{\varphi}.$$

Then $W^*_\Omega = MD_k$, where $M = \left[\binom{\psi}{\varphi}\right]_{\varphi,\psi}$ has the same dimensions as W^*_Ω and $D_k = \operatorname{diag}\left(\binom{\omega}{\psi}\right)$ is a square diagonal matrix indexed over $\psi \in \mathbb{H}^d(k)$. Let $\mathbf{b}^*$ be the $|\mathbb{H}^d(t)| \times 1$ vector indexed over $\mathbb{H}^d(t)$ and defined by

$$\mathbf{b}^*(\varphi) = \binom{\omega}{\varphi} b_\varphi,$$

where, as in Theorem 2.3, $b_\varphi = \mathbf{b}(T)$ for any T with $|T \cap \Omega| = \varphi$, and $b_\varphi = 0$ if no such T exists. Invariance under Ω allows for 'averaging' Ω-equivalent entries of W and $\mathbf{b}$, as in the previous sections. It follows that $\mathbf{b} \in \texttt{cone}(W)$ if and only if $\mathbf{b}^* \in \texttt{cone}(W^*_\Omega) = \texttt{cone}(MD_k)$. And since D_k is diagonal with nonnegative entries, this latter condition is equivalent to $\mathbf{b}^* \in \texttt{cone}(M)$. A concrete restatement of this is now given.

Theorem 2.8. (Generalized Method of Moments) *With notation as above,* $\mathbf{b} \in \texttt{cone}(W_\Omega)$ *if and only if there exist nonnegative rational solutions* z_φ *to the equations*

$$\sum_{\psi \in \mathbb{H}^d(k), \psi \leq \omega} \binom{\psi}{\varphi} z_\varphi = \binom{\omega}{\varphi} b_\varphi, \quad \varphi \in \mathbb{H}^d(t).$$

The goal for the rest of this section is to show that the system of equations in Theorem 2.8 reduces to the moment equations (1.2) when Ω is a bipartition and $\mathbf{b} = \lambda\mathbf{1}$. This motivates the consideration of the cone condition as a generalization of the method of moments.

Consider the bipartition $\Omega = (H, V \setminus H)$ with $|H| = w$ and $t \leq w \leq v - t$. Suppose that some collection of k-subsets from V has the property that every t-set T is contained in precisely b_h members of this collection, where $h = |T \cap H|$. (From now on, indexing over the ordered bipartitions of, say $s \in \mathbb{N}_0$, will be changed to simply indicate the first coordinate, from 0 to s.) Let z_j be the

number of k-subsets in the collection that meet H in exactly j points. The following system of equations holds by the same double-counting proof as was mentioned before equations (1.2).

$$\sum_{j=0}^{k} \binom{j}{i} z_j = \binom{w}{i} \binom{k-i}{t-i}^{-1} \sum_{h=0}^{t} \binom{w-i}{h-i} \binom{v-w}{t-h} b_h, \quad i = 0, 1, \ldots, t. \tag{2.4}$$

Define the vector $\tilde{\mathbf{b}}$ indexed on $\{0, 1, \ldots, t\}$ by $\tilde{\mathbf{b}}(h) = b_h$. Note that when $\tilde{\mathbf{b}} = \lambda \mathbf{1}$, the equation above reduces to (1.2) via equation (1.11). Observe $\mathbf{b}^* = \operatorname{diag}\left(\binom{w}{i}\binom{v-w}{t-i}\right) \tilde{\mathbf{b}}$. Define

$$N = \left[\binom{j}{i}\right]_{\substack{i \in \{0,1,\ldots,t\} \\ j \in \{0,1,\ldots,k\}}},$$

$$Q = \left[\binom{w-i}{h-i}\binom{v-w}{t-h}\right]_{\substack{i \in \{0,1,\ldots,t\} \\ h \in \{0,1,\ldots,t\}}},$$

and $D = \operatorname{diag}\left(\binom{w}{i}\binom{k-i}{t-i}^{-1}\right)$. Then the existence of nonnegative rational solutions z_j to the equations (2.4) is equivalent to $DQ\tilde{\mathbf{b}} \in \mathtt{cone}(N)$.

For Ω a bipartition, the matrix M defined earlier is

$$M = \left[\binom{j}{i}\binom{k-j}{t-i}\right]_{\substack{i \in \{0,1,\ldots,t\} \\ j \in \{0,1,\ldots,k\}}}.$$

Write M_0 and N_0 for the square submatrices formed from the first $t+1$ columns (indexed by $\{0, 1, \ldots, t\} \subset \{0, 1, \ldots, k\}$) of M and N respectively. The inverse of M_0 is important for later work and is calculated in Proposition 3.4. Some simple binomial identities prove that $[N_0^{-1}]_{ij} = (-1)^{i+j}\binom{j}{i}$. From this, computing $M_0 N_0^{-1}$ is an easy application of Proposition 1.6.

Lemma 2.9. *With the matrices defined as above,* $[M_0 N_0^{-1}]_{ij} = (-1)^{i+j}\binom{j}{i}\binom{k-j}{t-j}$.

The equivalence between the moment equations and the cone condition for W_Ω^* can now be established.

Theorem 2.10. $\mathbf{b}^* \in \mathtt{cone}(M)$ *if and only if* $DQ\tilde{\mathbf{b}} \in \mathtt{cone}(N)$.

Proof. It must be shown that the equations $M\mathbf{z} = \mathbf{b}^*$ and $N\mathbf{z} = DQ\tilde{\mathbf{b}}$ either both have or both do not have nonnegative solutions $\mathbf{z}$ for each choice of $\tilde{\mathbf{b}}$. By

Lemma 2.9 and equation (1.11), it follows that $M = M_0 N_0^{-1} N$. So, it suffices to prove $M_0 N_0^{-1} DQ = \mathrm{diag}(\binom{w}{i}\binom{v-w}{t-i})$. Using Lemma 2.9 again gives

$$M_0 N_0^{-1} D = \left[(-1)^{i+j} \binom{w}{j} \binom{j}{i} \right]_{\substack{i \in \{0,1,\ldots,t\} \\ j \in \{0,1,\ldots,t\}}}.$$

Now

$$\begin{aligned}
(M_0 N_0^{-1} DQ)_{ij} &= \sum_{\ell=i}^{j} (-1)^{i+\ell} \binom{w}{\ell} \binom{\ell}{i} \binom{w-\ell}{j-\ell} \binom{v-w}{t-j} \\
&= \binom{w}{i} \binom{v-w}{t-j} \sum_{\ell=i}^{j} (-1)^{i+\ell} \binom{w-i}{\ell-i} \binom{w-\ell}{j-\ell} \\
&= \binom{w}{i} \binom{v-w}{t-j} \binom{0}{j-i},
\end{aligned}$$

by equations (1.10) and (1.12). It is evident that the off-diagonal entries of $M_0 N_0^{-1} DQ$ vanish, and the proof is complete. $\square$

Chapter 3

Bipartitions

This chapter explores certain structures in t-designs which are most naturally or easily handled by considering a bipartition of the points. The cone condition is used along with the polynomial correspondence discussed in the previous chapter.

It should be noted that many inequalities presented here are already known for t-designs. Indeed, the main result of Section 2.4 is an equivalence between the cone condition for bipartitions and the well studied method of moments, with which any of the results here have either already been proved, or can be proved. Regardless, there are various reasons for considering the cone in this context. It is interesting, for example, to understand a description of the supporting vectors $\mathbf{y}_f$ and associated polynomials f which produce certain inequalities.

3.1 The Raghavarao-Wilson inequality

In [29], the moment equations (1.2) are used with the method of orthogonal projection to prove a variety of inequalities concerning block density in t-designs. One general result along these lines is an upper bound on the cardinality of the intersection of n blocks in a t-design. This is given in Theorem 3.1 below, which we have taken the liberty to name the 'Raghavarao-Wilson inequality'. Wilson first proved this as Corollary 1 in [29].

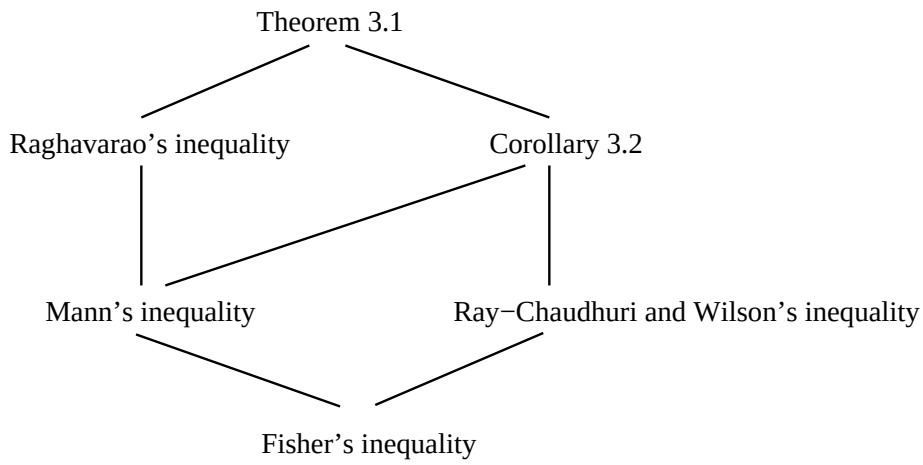

Figure 3.1: Hierarchy of some inequalities for block designs.

There are important and interesting special cases. For $t = 2$, Raghavarao's inequality [25] is recovered. This, in turn, implies Mann's inequality [20] on repeated blocks and Fisher's famous inequality $b \geq v$. For $t = 2s$, another special case is Ray-Chaudhuri and Wilson's extension [24] that, for $v \geq k + s$, one has $b \geq \binom{v}{s}$. Figure 3.1 illustrates these dependencies among block density results. Yet another intermediate result is discussed later in Corollary 3.2.

Here, a new proof of the Raghavarao-Wilson inequality for t-designs is given using bipartitions and Theorem 2.3. Like Wilson's original proof, we use several binomial identities and facts on polynomials $g^w_{s,k}(x)$. The interested reader will want to consult Section 1.3 and [29].

Theorem 3.1. ([29]) *Let $t \geq 2s$, and suppose $v \geq k + s$. In a t-(v, k, λ) design with a configuration of n blocks containing w points in their intersection, $(s \leq w \leq v - s)$,*

$$\frac{n}{\lambda} \leq \binom{v}{t}\binom{v}{s}^{-1}\binom{k}{t}^{-1}\binom{k}{s}\binom{v-w}{s}(g^w_{s,k}(w))^{-1}. \tag{3.1}$$

Proof. Consider the bipartition $\Omega = (U, V \setminus U)$ of the pointset V, where U is the intersection in question. Then $|\Omega| = (w, v - w)$, and $(x, k - x)$ will be used as the variables for intersection of a k-set with Ω.

Let $f(x) = (g^w_{s,k}(x))^2$, which is certainly nonnegative on $\{0, 1, \ldots, w\}$, and consider the supporting t-vector $\mathbf{y}_f$ for $\texttt{cone}(W_t)$. By the cone condition, we

have

$$\lambda\langle \mathbf{y}_f, \mathbf{1}\rangle \geq \sum_B \langle \mathbf{y}_f W_t, \mathbf{1}_B\rangle, \tag{3.2}$$

where the sum on the right is over all n blocks in the configuration. By Lemma 2.2,

$$\langle \mathbf{y}_f W_t, \mathbf{1}_B\rangle = f(|B\cap U|) = f(w) = (g^w_{s,k}(w))^2. \tag{3.3}$$

It remains to compute $\langle \mathbf{y}_f, \mathbf{1}\rangle$. Let $(a_0,\ldots,a_t)$ be such that

$$f(x) = \sum_{j=0}^{t} a_j \binom{x}{j}\binom{k-x}{t-j}$$

and let

$$F(x) = \sum_{j=0}^{t} a_j \binom{x}{j}\binom{v-x}{t-j}.$$

Then $\langle \mathbf{y}_f, \mathbf{1}\rangle = F(w)$ and in light of (3.2-3.3), we must prove

$$F(w) = g^w_{s,k}(w)\binom{v}{t}\binom{v}{s}^{-1}\binom{k}{t}^{-1}\binom{k}{s}\binom{v-w}{s}. \tag{3.4}$$

Now using equation (1.15),

$$\begin{aligned} f(x) &= (-1)^s g^w_{s,k}(x) g^{v-w+1}_{s,k}(k-x) \\ &= \sum_{r=0}^{t}(-1)^{s-r}\sum_i \frac{\binom{v-s}{i}\binom{v-s}{r-i}\binom{k-i}{s-i}\binom{k-r+i}{s-r+i}\binom{w-1-i}{s-i}\binom{v-w-r+i}{s-r+i}}{\binom{s}{i}\binom{s}{r-i}}\binom{x}{i}\binom{k-x}{r-i}, \end{aligned}$$

where the sum on i is from $\max\{0, r-s\}$ to $\min\{r,s\}$. It follows from equation (1.13) with $y = k-x$ that

$$a_j = \sum_{r=0}^{t}(-1)^{s-r}\sum_{i=\max\{0,r-s\}}^{\min\{r,s\}} \frac{\binom{v-s}{i}\binom{v-s}{r-i}\binom{k-i}{s-i}\binom{k-r+i}{s-r+i}\binom{w-1-i}{s-i}\binom{v-w-r+i}{s-r+i}\binom{j}{i}\binom{t-j}{r-i}}{\binom{s}{i}\binom{s}{r-i}\binom{k-r}{t-r}}. \tag{3.5}$$

One now has an expression for $F(x)$ in terms of these coefficients. Applying equation (1.10) and equation (1.13) with $y = v-x$ permits the simplification

$$F(w) = \binom{v-w}{s}\sum_{r=0}^{t}(-1)^{s-r}\frac{\binom{v-r}{t-r}}{\binom{k-r}{t-r}}\sum_{i=0}^{s}\frac{\binom{v-s}{i}\binom{v-s}{r-i}\binom{k-i}{s-i}\binom{k-r+i}{s-r+i}\binom{w-1-i}{s-i}\binom{w}{i}}{\binom{s}{i}}.$$

Changing back the indices of summation with $r = i + j$ and applying equation (1.10) again gives

$$F(w) = \frac{\binom{v-w}{s}}{\binom{v-t}{k-t}} \sum_{i,j=0}^{s} (-1)^{s-i-j} \frac{\binom{v-i-j}{k-i-j}\binom{v-s}{i}\binom{v-s}{j}\binom{k-i}{s-i}\binom{k-j}{s-j}\binom{w-1-i}{s-i}\binom{w}{i}}{\binom{s}{i}}.$$

The summation indexed by j is handled directly by Lemma 1.8. So one has

$$\begin{aligned} F(w) &= \frac{\binom{v-w}{s}\binom{v-s}{k-s}}{\binom{v-t}{k-t}} \sum_{i=0}^{s} (-1)^{s-i} \frac{\binom{v-s}{i}\binom{k-i}{s-i}\binom{w-1-i}{s-i}\binom{w}{i}}{\binom{s}{i}} \\ &= \binom{v}{t}\binom{k}{s}\binom{k}{t}^{-1}\binom{v}{s}^{-1}\binom{v-w}{s} g^{w}_{s,k}(w), \end{aligned}$$

as required, where two more applications of (1.10) have been used. □

It should be noted that Theorem 3.1 applied to the supplement of the given design produces a bound on the size of a union of n blocks, or the size of a set disjoint from each of n blocks (Corollaries 2 and 3 of [29].) When $w = k$ in the theorem, equation (1.16) recovers this generalization of Mann's inequality $b \geq nv$.

Corollary 3.2. ([29]) *Let $t \geq 2s$, and suppose $v \geq k + s$. In a t-(v, k, λ) design with an n-fold repeated block,*

$$\frac{n}{\lambda} \leq \binom{v}{t}\binom{k}{t}^{-1}\binom{v}{s}^{-1}. \tag{3.6}$$

For equality to hold in (3.1), the given supporting vector $\mathbf{y}_f$ must annihilate all characteristic vectors of blocks $\mathbf{1}_B$ of the design which are not among the n given blocks.

Corollary 3.3. *If equality holds in* (3.1)*, there are at most s possible intersection sizes for a pair of different blocks, and these are the roots of $g^{w}_{s,k}(x)$.*

Proof. See Theorem 1.5 and Lemma 2.2. □

It is shown in [24] that *at least* $s + 1$ intersection sizes occur in any $2s$-design. Thus, the roots of $g^{w}_{s,k}$ being integral and distinct forms a surprisingly

stringent necessary condition for the existence of designs meeting the bound with equality. For example, this observation has been used in [2] to disprove the existence of all but possibly finitely many *tight* t-designs for each $t \geq 12$. With further details, tight 6-designs were completely ruled out in [23]. Better understanding the distribution of roots of these polynomials would appear to be a crucial step toward more sophisticated inequalities and nonexistence results for designs.

3.2 Enclosings of designs

An *enclosing* of a t-(w, k, λ') design $(U, \mathcal{B}')$ is a t-(v, k, λ) design $(V, \mathcal{B})$ such that $U \subseteq V$ and $\mathcal{B}'$ is a subcollection of $\mathcal{B}$. When $w = k$ and $\lambda' = n$, this is equivalent to the existence of an n-fold block in a t-(v, k, λ) design. Since this case is of particular interest, it is considered in further detail here and in the next section. It should be noted that the inequalities in Theorem 3.1 and Corollary 3.2 apply to enclosings and n-fold blocks via the polynomials $g^k_{s,w}(x)$. The spirit of this section is that the facet-defining polynomials of Theorem 2.6 can be used to obtain sharper inequalities for enclosings. In fact, one obtains necessary and sufficient conditions for $\lambda\mathbf{1} - \lambda'\mathbf{1}_U$ to belong to $\texttt{cone}(W_t)$, since this vector is invariant under a bipartition. It is worth mentioning that improvements to Theorem 3.1, though sporadic and not in general optimal, can also be obtained in a similar manner.

In what follows, the cleaner case of t even will be assumed when necessary. Recall from Section 2.4 the $(t+1) \times (k+1)$ matrix

$$M = \left[\binom{j}{i} \binom{k-j}{t-i} \right]_{\substack{i \in \{0,1,\ldots,t\} \\ j \in \{0,1,\ldots,k\}}}.$$

By the discussion in Chapter 2, supporting vectors of $\texttt{cone}(W_{(U,V\setminus U)})$ are of the form

$$\mathbf{y} = \sum_{i=0}^{t} a_i \sum_{|T \cap U| = i} \mathbf{1}_T,$$

where $\mathbf{a} = (a_0, a_1, \ldots, a_t) \neq \mathbf{0}$ is such that $\mathbf{a}M \geq \mathbf{0}$. The corresponding polynomial is

$$f(x) = \sum_{i=0}^{t} a_i \binom{x}{i} \binom{k-x}{t-i},$$

which supports a facet by Theorem 2.6 if and only if f vanishes on an even set $Z \subset \{0, 1, \ldots, k\}$ of size t, and has $f(r) > 0$ for any $r \in \{0, 1, \ldots, k\} \setminus Z$. Let M_Z denote a square submatrix of M formed from the columns indexed by $Z^* = Z \cup \{r\}$, for some even Z and $r \notin Z$. (In many cases, $0 \notin Z$, and $r = 0$ is a nice choice for the computations which follow.) Any $t+1$ columns of M are linearly independent, so the facets are simply described (up to a positive multiple) by the vector $(M_Z^{-1})_r$, i.e., the row of M_Z^{-1} indexed by r.

For interest, the task of computing these facets explicitly is now briefly considered. Following the convention in Section 2.2, define $M_0 = M_{\{1,\ldots,t\}}$ (with 'positive coordinate' taken arbitrarily to be $r = 0$).

Proposition 3.4.

$$(M_0)^{-1} = \left[(-1)^{i+j} \frac{k-t}{k-t+j-i} \binom{j}{i} \binom{k-i}{t-j}^{-1} \right]_{i,j \in \{0,1,\ldots,t\}}.$$

Proof. Both M_0 and the given matrix are upper triangular, so it suffices to consider inner products of row i of M_0 with column j of the asserted inverse when $i \leq j$. For $i = j$, this is evidently equal to

$$\sum_{\ell=0}^{t} (-1)^{\ell+i} \binom{\ell}{i} \binom{i}{\ell} \binom{k-\ell}{t-i} \frac{k-t}{k-t-\ell+i} \binom{k-\ell}{t-j}^{-1} = (-1)^{2\ell} \frac{k-t}{k-t} = 1.$$

For $i < j$, the inner product is

$$\begin{aligned}
& (k-t) \binom{j}{i} \binom{t-i}{t-j}^{-1} \sum_{\ell=i}^{j} (-1)^{\ell+j} \frac{1}{k-\ell-t+j} \binom{j-i}{\ell-i} \binom{k-\ell-t+j}{j-i} \\
= \; & \frac{k-t}{j-i} \binom{j}{i} \binom{t-i}{t-j}^{-1} \sum_{\ell=i}^{j} (-1)^{\ell+j} \binom{j-i}{\ell-i} \binom{k-\ell-t+j-1}{k-\ell-t+i} \\
= \; & \frac{k-t}{j-i} \binom{j}{i} \binom{t-i}{t-j}^{-1} \binom{k-t-1}{k-t} = 0,
\end{aligned}$$

where equation (1.10) is used three times along with the summation identity (1.12). □

Now the matrix $(M_Z)^{-1}$ can, in principle, be computed for general Z by making use of the $(t+1) \times (t+1)$ *Vandermonde matrix* V_Z defined by $V_Z(i,j) =$

j^i, where $i \in \{0, 1, \ldots, t\}$ and $j \in Z^*$. Observe that $M_Z = EV_Z$, where E is defined by the polynomial equations

$$\binom{x}{i}\binom{k-x}{t-i} = \sum_{\ell=0}^{t} E_{i\ell} x^\ell.$$

Now

$$M_Z^{-1} = V_Z^{-1} E^{-1} = V_Z^{-1} V_{\{1,\ldots,t\}} M_0^{-1}.$$

By Proposition 3.4 and a known formula [18] for the inverse of a general Vandermonde matrix, the required row of M_Z^{-1} can be expressed concretely, if desired. As Section 2.2, all facets for a bipartition arise in this way.

Theorem 3.5. *Suppose $t \le |U| = w \le v$. Then $\lambda\mathbf{1} - \lambda'\mathbf{1}_U \in \mathtt{cone}(W_{tk}^v)$ if and only if, for all even t-sets $Z \subset \{0, 1, \ldots, k\}$ and some $r \notin Z$,*

$$\frac{\lambda'}{\lambda} \le \frac{\sum_{j=0}^{t} (M_Z^{-1})_{rj} \binom{w}{j}\binom{v-w}{t-j}}{(M_Z^{-1})_{rt}\binom{w}{t}}.$$

Proof. By the cone condition and Theorem 2.6, $\lambda\mathbf{1} - \lambda'\mathbf{1}_U \in \mathtt{cone}(W_t)$ if and only if

$$\lambda\langle \mathbf{y}_Z, \mathbf{1}\rangle \ge \lambda'\langle \mathbf{y}_Z, \mathbf{1}_U\rangle$$

for all even $Z \subset \{0, 1, \ldots, k\}$, where

$$\mathbf{y}_Z = \sum_{i=0}^{t} (M_Z^{-1})_{ri} \sum_{|T\cap U|=i} \mathbf{1}_T.$$

The result follows from

$$\begin{aligned} \langle \mathbf{y}_Z, \mathbf{1}_U\rangle &= (M_Z^{-1})_{rt}\binom{w}{t} \quad \text{and} \\ \langle \mathbf{y}_Z, \mathbf{1}\rangle &= \sum_j (M_Z^{-1})_{rj}\binom{w}{j}\binom{v-w}{t-j}. \end{aligned} \tag{3.7}$$

□

Observe that for some constant C – depending on the choice of $r \in \{0, 1, \ldots, t\} \setminus Z$, – we have

$$C\prod_{\zeta\in Z}(x-\zeta) = \sum_{j=0}^{t} (M_Z^{-1})_{rj}\binom{x}{j}\binom{k-x}{t-j}.$$

When some such r is understood, define

$$F_Z(x) = \sum_{j=0}^{t} (M_Z^{-1})_{rj} \binom{x}{j} \binom{v-x}{t-j}.$$

This notation is useful when referencing the right side of (3.7).

The following nonexistence result, first proved by Delsarte in [7], is a rather striking use of Theorem 3.5.

Example 3.1. There does not exist a 4-$(17, 8, 5)$ design. For these parameters, one has $f(x) = g_{2,k}^k(x)^2 \approx C(x-2.48)^2(x-4.52)^2$ for some constant C. Suppose there is an n-fold block B, and consider the test of $\lambda\mathbf{1} - n\mathbf{1}_B \in \mathcal{C}W_{4k}^v$. From the supporting vector $\mathbf{y}_f$, the Wilson-Mann bound of $n/\lambda \leq 1/4$ results, which permits $n = 1$. Instead, consider the polynomial $(x-2)(x-3)(x-4)(x-5)$. The bound from $F_{\{2,3,4,5\}}(k)$ in Theorem 3.5 is $n/\lambda < 4/25$. This rules out even $n = 1$. In other words, a design with these parameters cannot exist.

In general, the upper bound on λ'/λ from Theorem 3.5 is obtained by minimizing a certain quantity over even sets Z. We now present some preliminary steps toward understanding the optimum such Z.

Example 3.2. When $t = 2$, the only possible extremal polynomials (up to a positive multiple) are $x(k-x)$ or $(x-c)(x-c-1)$ for some $c = 0, \ldots, k-1$. The various cases for Z and corresponding facet weights a_0, a_1, a_2 are computed and presented in the table below.

Z	a_0	a_1	a_2
$\{k-1, k\}$	1	0	0
$\{0, k\}$	0	1	0
$\{0, 1\}$	0	0	1
$\{c, c+1\}$	$\frac{c+1}{k-c-1}$	-1	$\frac{k-c}{c}$

Note the last line in the table is for $1 \leq c \leq k-2$, and this case yields the only nontrivial family of facets for $\texttt{cone}(W_{2k}^v)$ that are invariant under a bipartition.

Corollary 3.6. *Suppose $t \leq |U| = w < v-1$. Then $\lambda\mathbf{1} - \lambda'\mathbf{1}_U \in \texttt{cone}(W_{2k}^v)$ if and only if*

$$\frac{\lambda'}{\lambda} \leq \frac{c(c+1)(v-w)(v-w-1)}{(k-c)(k-c-1)w(w-1)} - \frac{2c(v-w)}{(k-c)(w-1)} + 1,$$

where

$$c = \left\lfloor \frac{w(k-1)}{v-1} \right\rfloor.$$

Proof. Using the table above, a concrete restatement of Theorem 3.5 for $t = 2$ is

$$\frac{\lambda'}{\lambda} \leq \min_{c=1,\ldots,k-2} \frac{\frac{c+1}{k-c-1}\binom{v-w}{2} - w(v-w) + \frac{k-c}{c}\binom{w}{2}}{\frac{k-c}{c}\binom{w}{2}}. \tag{3.8}$$

(Note that the other three facets give no meaningful bound on λ'/λ.) Now, let $c \in (0, k-1)$ be a continuous parameter, and define $h(c)$ to be the rational function in c on the right side of (3.8). Using calculus and some factoring, the minimum of h on $(0, k-1)$ is seen to be achieved at

$$c_0 = \frac{(k-1)(v+w-1) - \sqrt{(k(v-w-1)+w)^2 - (v-1)^2}}{2(v-1)}.$$

The square root lies in the open interval with endpoints $k(v-w-1)+w \pm (v-1)$. After some simplification, it follows that $c_0 \in (\gamma - 1, \gamma)$, where $\gamma = \frac{w(k-1)}{v-1}$. Now the function h is strictly decreasing on $(0, c_0)$ and strictly increasing on $(c_0, k-1)$. Furthermore, a calculation shows $h(\gamma - 1) = h(\gamma)$. So, the (1 or 2) integers in $(0, k-1)$ which minimize h must belong to the interval $[\gamma - 1, \gamma]$. Thus over integers, $h(c)$ is minimized at $c = \lfloor \gamma \rfloor$. □

Remarks: For $w = v - 1$, the inequality in (3.8) reduces to

$$\frac{\lambda'}{\lambda} \leq \min_{c=1,\ldots,k-2} 1 - \frac{2c}{(k-c)(v-2)} = 1 - \frac{k-2}{v-2}. \tag{3.9}$$

Enclosings with $w = v - 1$ are said to be *minimal*, and the smallest gap between λ' and λ is of interest. Some nice constructions of such enclosings for $t = 2$ and $k = 3$ are found in [14], along with an alternate proof of the bound in (3.9). Similar inequalities concerning enclosings of *group divisible designs* are considered (along with several constructions) in work in progress by Hurd, Purewal, and Sarvate, and these bounds can also be proved with a modification of Corollary 3.6.

It is interesting to note that γ above is the root of $g^k_{1,w}(x) = (v-1)x - w(k-1)$. Roughly speaking, a sharper inequality results because, in the cone of quadratics nonnegative on $\{0, 1, \ldots, k\}$, the polynomial $(x - \gamma)^2$ is closest to the extremal ray generated by $(x - \lfloor \gamma \rfloor)(x - \lfloor \gamma \rfloor - 1)$. In fact, it can be shown (see the techniques in Section 6.2) that, among *square* polynomials, the optimal bounds for enclosings of $(2s)$-designs arise from supporting vectors $\mathbf{y}_f$ of $\texttt{cone}(W_{2s})$ corresponding to $f(x) = (g^k_{s,w}(x))^2$.

On the other hand, it is curious that Corollary 3.6 fails for $t \geq 4$. Specifically, it is not always the case that the minimizing even set Z of Theorem 3.5 is obtained from the floor and ceiling of the roots of $g^k_{s,w}$; however, such sets appear to be 'very close' to optimal. See Example 3.3 and and Figure 3.2 below.

Given this unexpected behaviour of optimal even sets Z, we devote the next section to a closer investigation for the case $w = k$ of n-fold blocks.

3.3 Repeated blocks in t-designs

We only consider here the existence of an n-fold block in a t-(v, k, λ) design, specializing again to the case of t even. For $t = 2$, Corollary 3.6 provides, in closed form, the most strict upper bound on n/λ possible from the cone condition. However, the situation is less clear for $t > 2$, as illustrated by the following example.

Example 3.3. Consider upper bounds on n/λ, where it is assumed that there exists an n-fold block in a 4-$(24, 12, \lambda)$ design. With $v = 24$,

$$g^{12}_{2,12}(x) = \frac{11}{2}(21x^2 - 241x + 660) \approx C(x - 4.51)(x - 6.96).$$

However, $F_{\{4,5,6,7\}}(w) \geq F_{\{4,5,7,8\}}(w)$.

The optimal even set $Z = \{4, 5, 7, 8\}$ of Example 3.3 is shown as one column at $v = 24$ in Figure 3.2.

In general, an even 4-subset of $\{0, \ldots, k\}$ has the form $Z = \{\alpha, \alpha+1, \beta, \beta+1\}$, where $0 < \alpha + 1 < \beta < k$. For the polynomial

$$f(x) = (x - \alpha)(x - \alpha - 1)(x - \beta)(x - \beta - 1),$$

one obtains after tedious calculations

$$\frac{n}{\lambda} \leq \frac{c_0 + c_1(\alpha + \beta) + c_{11}\alpha\beta + c_2(\alpha^2 + \beta^2) + c_{12}\alpha\beta(\alpha + \beta) + c_{22}\alpha^2\beta^2}{f(k)k(k-1)(k-2)(k-3)},$$

where

$$c_0 = k^2(k-1)^2(2v^2 + 4k^2v - 16kv + 6v + k^4 - 10k^3 + 25k^2 - 12k),$$

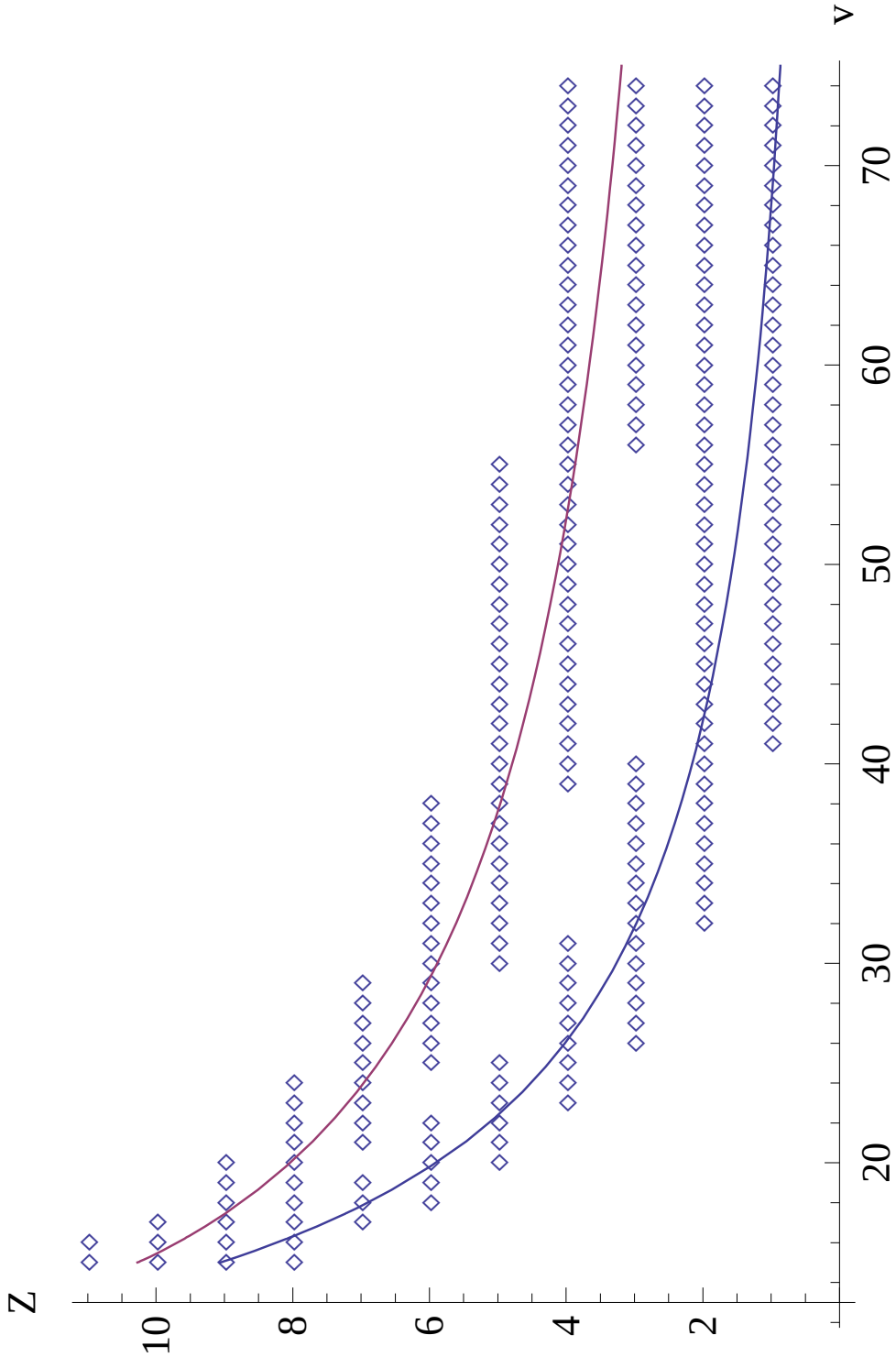

Figure 3.2: Optimal even sets (◇) and roots of $g^{12}_{2,12}$ (i.e. $t = 4$, $k = 12$) versus v.

$$\begin{aligned}
c_1 &= (v-3)(3v+2k^2-8k-2)k^2(k-1)^2, \\
c_{11} &= (v-2)(v-3)(v(v-1)+4k^2(k-1)^2), \\
c_2 &= (v-2)(v-3)k^2(k-1)^2, \\
c_{12} &= (v-1)(v-2)(v-3)(v-2k^2), \\
c_{22} &= v(v-1)(v-2)(v-3).
\end{aligned} \tag{3.10}$$

Unfortunately, we presently see no general pattern in (3.10) for the purpose of optimizing over α, β. However, it is easy to find optimum even sets for small parameters, such as those in Figure 3.2.

In fact, even $n = 1$ is sometimes ruled out, as was seen in Example 3.1. For each of the hypothetical 4-designs in Table 3.1, a lower bound on λ/n is given which exceeds the minimum possible admissible index $\lambda_{\min}$. (Any other such admissible λ must, of course, be a positive multiple of $\lambda_{\min}$.) As a consequence, the given designs do not exist for the first one or more positive admissible indices. Column 'Z' provides an optimal even set Z furnishing the inequality.

When $k - t$ is small, the choices for Z are limited. In such cases, it may be possible to obtain, in closed form, a reasonable bound from Theorem 3.5. One such example is given next.

Corollary 3.7. *Let $t = 2s$. In a t-$(2t+2, t+1, \lambda)$ design with an n-fold block,*

$$n(t+2) \leq 2\lambda.$$

Proof. This follows from Theorem 3.5 with $Z^* = \{0, 1, \ldots, t\}$, $w = k = t+1$ and $\lambda' = n$. By Proposition 3.4 and equation (1.10), it follows that

$$(M_Z^{-1})_{0j} = (-1)^j \frac{1}{j+1}\binom{t+1}{j+1}^{-1} = (-1)^j \frac{1}{t+1}\binom{t}{j}^{-1}.$$

So

$$\begin{aligned}
\frac{n}{\lambda} &\leq \frac{1}{\binom{t+1}{t}} \sum_{j=0}^{t} (-1)^j \binom{t}{j}^{-1} \binom{t+1}{j}\binom{t+1}{t-j} \\
&= \left(\frac{1}{t+1}\right) \frac{2(t+1)}{2t+2-t} = \frac{2}{t+2},
\end{aligned}$$

where equation (1.14) is invoked to simplify the sum. □

v	k	$\lambda_{\min}$	$\lambda/n \geq$	Z
12	6	2	3.33333	$\{1,2,3,4\}$
17	7	2	3.0	$\{1,2,3,4\}$
17	8	5	6.25	$\{2,3,4,5\}$
23	8	2	2.8	$\{1,2,3,4\}$
23	11	6	12.5373	$\{3,4,6,7\}$
24	12	15	15.6716	$\{4,5,7,8\}$
38	10	2	2.57143	$\{1,2,3,4\}$
42	16	14	17.2531	$\{4,5,7,8\}$
47	11	2	2.5	$\{1,2,3,4\}$
47	23	35	60.3844	$\{9,10,12,13\}$
57	12	1	2.44444	$\{1,2,3,4\}$
57	21	19	27.8667	$\{5,6,9,10\}$
58	22	19	30.4	$\{6,7,9,10\}$
66	30	87	91.8333	$\{11,12,15,16\}$
71	35	88	140.8	$\{14,15,19,20\}$
78	21	6	14.4655	$\{3,4,7,8\}$
79	22	5	16.4163	$\{4,5,7,8\}$
80	14	2	2.36364	$\{1,2,3,4\}$
87	17	4	4.48921	$\{1,2,4,5\}$
87	35	88	94.6156	$\{11,12,16,17\}$
87	36	33	103.636	$\{12,13,16,17\}$
93	15	2	2.33333	$\{1,2,3,4\}$
93	28	15	33.3487	$\{6,7,10,11\}$
95	47	165	272.4	$\{20,21,25,26\}$

Table 3.1: Nonexistence of some 4-designs via the cone condition.

Remarks: This inequality is actually strict, because the associated supporting vector fails to annihilate t-sets fully contained in the specified block or its complement. Note that Corollary 3.2 applied to a design with these parameters is the weaker statement

$$\frac{n}{\lambda} \leq \frac{\binom{2t+2}{t}}{(t+1)\binom{2t+2}{s}}.$$

On the other hand, Delsarte's inequalities [7] have been applied by Chan and Wilson [5] to small $k-t$ and $n=1$ to obtain bounds similar to that in Corollary 3.7.

Even when k is a bit larger than $t+1$, many results can be compiled with the help of the computer. By automating Theorem 3.5, the best upper bound (via the cone condition) on n/λ in t-designs was computed for $t = 4, 6$ and 8, $t < k \leq 12$, and $2k \leq v \leq 24$. In the tables which follow, the column labeled '$n \leq$' gives the sharpest bound from Theorem 3.5 with $\lambda = \lambda_{\min}$. The corresponding optimal even set of roots Z is included in the adjacent column. A missing parameter pair (v, k) within range indicates that the cone condition permits $n = \lambda$ in that case, thereby yielding no information. Note that for all t, k, this eventually occurs for sufficiently large v.

Example 3.4. Consider 4-$(23, 11, \lambda)$ designs. Since $0.4786 < 1/2$, we see from Table 3.2 that there does not exist such a design for $\lambda = 6$ or $\lambda = 12$. Further any such design with $\lambda = 18$ or 24 is required to be simple.

Example 3.5. Table 3.3 actually asserts the nonexistence of 6-$(19, 9, 2)$ and 6-$(20, 10, 7)$ designs.

We close by remarking that the smallest parameter pair $(v, k) = (2t+2, t+1)$ in each of the tables above corresponds to the bound in Corollary 3.7.

v	k	$\lambda_{\min}$	$n \leq$	Z	v	k	$\lambda_{\min}$	$n \leq$	Z
10	5	6	2	$1,2,3,4$	21	8	70	25.06	$1,2,3,4$
12	6	2	0.6	$1,2,3,4$	22	8	30	10.6	$1,2,3,4$
13	6	6	1.8	$1,2,3,4$	23	8	2	0.7143	$1,2,3,4$
14	6	15	5.4	$1,2,3,4$	24	8	5	1.857	$1,2,3,4$
15	6	5	2.6	$1,2,3,4$	18	9	14	1.6	$3,4,5,6$
16	6	6	5	$1,2,3,4$	19	9	21	3.143	$2,3,5,6$
14	7	20	3.6	$2,3,4,5$	20	9	168	29.71	$2,3,4,5$
15	7	5	1.8	$1,2,3,4$	21	9	14	2.429	$2,3,4,5$
16	7	20	6.8	$1,2,3,4$	22	9	252	46.23	$2,3,4,5$
17	7	2	0.6667	$1,2,3,4$	23	9	18	3.8	$2,3,4,5$
18	7	28	9.667	$1,2,3,4$	24	9	24	5.75	$1,2,4,5$
19	7	35	13.4	$1,2,3,4$	20	10	28	2.857	$3,4,5,6$
20	7	140	63.73	$1,2,3,4$	21	10	28	2.857	$3,4,5,6$
21	7	10	5.733	$1,2,3,4$	22	10	42	5	$3,4,5,6$
22	7	4	3	$1,2,3,4$	23	10	42	5.321	$2,3,5,6$
16	8	15	2.4	$2,3,4,5$	24	10	60	8.375	$2,3,5,6$
17	8	5	0.8	$2,3,4,5$	22	11	72	5.439	$3,4,6,7$
18	8	7	1.333	$2,3,4,5$	23	11	6	0.4786	$3,4,6,7$
19	8	105	28	$2,3,4,5$	24	11	120	11.17	$3,4,6,7$
20	8	70	22.05	$1,2,4,5$	24	12	15	0.9571	$4,5,7,8$

Table 3.2: Bounds on n-fold repeated blocks for $t = 4$.

v	k	$\lambda_{\min}$	$n \leq$	Z	v	k	$\lambda_{\min}$	$n \leq$	Z
14	7	4	1	$1,2,3,4,5,6$	23	9	20	6.5	$1,2,3,4,5,6$
16	8	15	3.429	$1,2,3,4,5,6$	24	9	24	10.25	$1,2,3,4,5,6$
17	8	5	1.143	$1,2,3,4,5,6$	20	10	7	0.7143	$2,3,4,5,6,7$
18	8	6	1.714	$1,2,3,4,5,6$	21	10	105	10.71	$2,3,4,5,6,7$
19	8	6	2.857	$1,2,3,4,5,6$	22	10	70	8.929	$2,3,4,5,6,7$
20	8	7	6.571	$1,2,3,4,5,6$	23	10	70	14.29	$2,3,4,5,6,7$
18	9	20	2.286	$2,3,4,5,6,7$	24	10	90	27.08	$1,2,3,4,5,6$
19	9	2	0.5714	$1,2,3,4,5,6$	22	11	168	10.71	$3,4,5,6,7,8$
20	9	28	7.571	$1,2,3,4,5,6$	23	11	14	1.611	$2,3,4,5,7,8$
21	9	35	9.286	$1,2,3,4,5,6$	24	11	252	29.76	$2,3,4,5,6,7$
22	9	280	78.04	$1,2,3,4,5,6$	24	12	42	2.381	$3,4,5,6,7,8$

Table 3.3: Bounds on n-fold repeated blocks for $t = 6$.

v	k	$\lambda_{\min}$	$n \leq$	Z
18	9	10	2	$1, 2, 3, 4, 5, 6, 7, 8$
20	10	6	1.111	$1, 2, 3, 4, 5, 6, 7, 8$
21	10	6	1.111	$1, 2, 3, 4, 5, 6, 7, 8$
22	10	7	1.667	$1, 2, 3, 4, 5, 6, 7, 8$
23	10	105	47.22	$1, 2, 3, 4, 5, 6, 7, 8$
22	11	28	2.222	$2, 3, 4, 5, 6, 7, 8, 9$
23	11	35	8.333	$1, 2, 3, 4, 5, 6, 7, 8$
24	11	280	63.33	$1, 2, 3, 4, 5, 6, 7, 8$
24	12	70	5	$2, 3, 4, 5, 6, 7, 8, 9$

Table 3.4: Bounds on n-fold repeated blocks for $t = 8$.

Chapter 4

Finer Partitions

The previous chapter offered a fairly thorough investigation of $\texttt{cone}(W)$, when simplified using an invariant bipartition of the points. From the viewpoint of design configurations, this is a rather limited approach. Here, finer invariant partitions are considered. The primary focus is again applications to t-designs. The first section shows how a classical result on the intersection size of two blocks (and its generalization due to Wilson) follows from partitions of size four. Improvements are possible using easy facets and linear programming techniques. Still finer partitions are then used to study uniform intersection of three blocks in a 2-design.

4.1 The Connor-Wilson inequalities

Suppose in a t-(v, k, λ) design that two blocks B_1 and B_2 intersect in μ points. In [30], Wilson establishes conditions on μ generalizing Connor's inequalities [6], which give upper and lower bounds on μ for $t = 2$. Here, Wilson's result is reproduced with the cone condition. The t-vector under consideration is $\mathbf{b} = \lambda\mathbf{1} - \mathbf{1}_{B_1} - \mathbf{1}_{B_2}$, in which every entry is either $\lambda, \lambda - 1$, or $\lambda - 2$. Such $\mathbf{b}$ are clearly invariant under the partition

$$\Omega = (B_1 \setminus B_2, B_2 \setminus B_1, B_1 \cap B_2, V \setminus (B_1 \cup B_2)) .$$

It is required to consider polynomials $f(x_1, x_2, y)$ in three variables which are nonnegative on $0 \leq x_1, x_2 \leq k - \mu$, $0 \leq y \leq \mu$, and $x_1 + x_2 + y \leq k$.

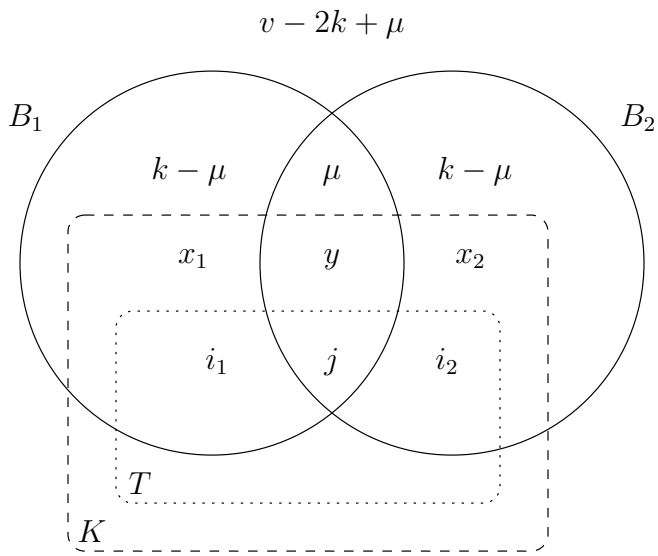

Figure 4.1: Cardinalities for a typical k-set K and t-set T meeting B_1, B_2.

The polynomials from Section 1.3 will play an important role once again. Recall that to simplify notation, we define $g_s(x) = g^k_{s,k}(x)$.

Theorem 4.1. ([30]) *Let $t \geq 2s$, and suppose $v \geq k+s$. In a t-(v,k,λ) design with two blocks intersecting in μ points,*

$$\binom{k}{s}\binom{v-k}{s} \pm g_s(\mu) \leq \lambda \binom{v}{t}\binom{v}{s}^{-1}\binom{k}{t}^{-1}\binom{k}{s}\binom{v-k}{s}. \tag{4.1}$$

Proof. Theorem 2.3 is used with the partition described above and the (non-negative) polynomials $f(x_1,x_2,y) = [g_s(x_1+y) \pm g_s(x_2+y)]^2$. For $\{p,q\} \subset \{1,2\}$, define the weights $a_{pq}(i_1,i_2,j)$ by

$$g_s(x_p+y)g_s(x_q+y) = \sum_{0\leq i_1+i_2+j\leq t} a_{pq}(i_1,i_2,j)\binom{x_1}{i_1}\binom{x_2}{i_2}\binom{y}{j}\binom{k-x_1-x_2-y}{t-i_1-i_2-j}.$$

It is immediate that $a_{12}(i_1,i_2,j) = a_{12}(i_2,i_1,j)$ and $a_{11}(i_1,i_2,j) = a_{11}(i_1,i_2',j)$ for any i_2, i_2', and similarly for a_{22} with i_1, i_1'. Define

$$a(i_1,i_2,j) = a_{11}(i_1,i_2,j) + a_{22}(i_1,i_2,j) \pm 2a_{12}(i_1,i_2,j).$$

The condition $\langle \mathbf{y}_f, \mathbf{b} \rangle \geq 0$ is seen to be equivalent to

$$\left[\sum_{i_1+j=t, i_2=0} + \sum_{i_2+j=t, i_1=0} \right] A(i_1, i_2, j) \leq \lambda \sum_{0 \leq i_1+i_2+j \leq t} A(i_1, i_2, j), \tag{4.2}$$

where

$$A(i_1, i_2, j) = a(i_1, i_2, j) \binom{k-\mu}{i_1} \binom{k-\mu}{i_2} \binom{\mu}{j} \binom{v-2k+\mu}{t-i_1-i_2-j}.$$

It remains to simplify the sums on the left and right sides of this inequality, which are denoted by Σ^L and Σ^R, respectively. Define Σ^L_{pq} and Σ^R_{pq} to be these sums with 'a_{pq}' taking the place of 'a', so that $\Sigma^L = \Sigma^L_{11} + \Sigma^L_{22} \pm 2\Sigma^L_{12}$, and similarly for Σ^R. It is a straightforward observation that

$$\Sigma^L_{11} = \Sigma^L_{22} = (g_s(k))^2 + (g_s(\mu))^2 \quad \text{and} \quad \Sigma^L_{12} = 2g_s(k)g_s(\mu).$$

And computing as in the proof of Theorem 3.1 yields

$$\Sigma^R_{11} = \Sigma^R_{22} = \binom{v}{t}\binom{v}{s}^{-1}\binom{k}{t}^{-1}(g_s(k))^2, \quad \Sigma^R_{12} = \binom{v}{t}\binom{v}{s}^{-1}\binom{k}{t}^{-1} g_s(k)g_s(\mu).$$

The inequality (4.2) can now be rewritten as

$$2(g_s(k) \pm g_s(\mu))^2 \leq \lambda \binom{v}{t}\binom{v}{s}^{-1}\binom{k}{t}^{-1} 2g_s(k)(g_s(k) \pm g_s(\mu)).$$

Canceling $2(g_s(k) \pm g_s(\mu))$, which is evidently positive if the inequality holds, and using the ${}_2F_1$ identity $g_s(k) = \binom{k}{s}\binom{v-k}{s}$ completes the proof. □

Remarks: The case $\mu = k$ of the above reduces to the case $n = 2$ of Corollary 3.2. In general, however, Theorem 4.1 is a more stringent condition on μ than Theorem 3.1 is on w for $n = 2$. When $t = 2$, Theorem 4.1 reduces (after some arithmetic) to Connor's inequalities for a pair of blocks:

$$k - \lambda\left(\frac{v-k}{k-1}\right) \leq \mu \leq \frac{2k(k-1)}{v-1} - k + \lambda\left(\frac{v-k}{k-1}\right). \tag{4.3}$$

The left inequality results from the polynomial with the '$-$' sign and the right inequality arises from the polynomial with the '$+$' sign.

It is of interest when equality occurs in the bounds of this section. By Lemma 2.2, this happens for bounds corresponding to f if and only if for every

block B distinct from B_1 and B_2 with $|B \cap \Omega| = (x_1, x_2, y, k - x_1 - x_2 - y)$, it is the case that $f(x_1, x_2, y) = 0$. So the lower bound of (4.3) is met with equality if and only if $g_1(x_1+y) - g_1(x_2+y) = 0$ for every block B meeting the partition as above. Since g_1 is linear, this is simply equivalent to $x_1 = x_2$. Therefore, equality results in the lower bound of Connor's inequalities for $\mu = |B_1 \cap B_2|$ if and only if every other block meets the given pair of blocks in the same number of points. In this case, there are only two possible intersection sizes in the 2-design for a disjoint pair of blocks. This observation was first made by Majindar [19]. Similarly, equality occurs in the upper bound of (4.3) if and only if $g_1(x_1+y) + g_1(x_2+y) = 0$, or $|B \cap B_1| + |B \cap B_2| = 2k(k-1)/(v-1)$ for all B distinct from B_1 and B_2. The following gives a flavor of the conditions for equality when $t > 2$.

Proposition 4.2. *When $t = 4$, equality occurs in the '$-$' bound of Theorem 4.1 if and only if, for every block B distinct from B_1 and B_2, either*

(*i*) $|B \cap B_1| = |B \cap B_2|$, *or*

(*ii*) $|B \cap B_1| + |B \cap B_2| = 1 + \frac{2(k-1)(k-2)}{v-3}$.

Proof. This follows from the remarks above with the factorization

$$g_2(x_1+y) - g_2(x_2+y) = \frac{1}{4}(v-2)(x_1-x_2)\left(x_1 + x_2 + 2y - 1 - \frac{2(k-1)(k-2)}{v-3}\right).$$

(By comparison, the polynomial $g_2(x_1 + y) + g_2(x_2 + y)$ does not split in $\mathbb{R}[x_1, x_2, y]$.) □

With only a minor modification to the proof of Theorem 4.1, a generalization to blocks with higher multiplicity follows.

Theorem 4.3. *Let $t \geq 2s$ and suppose $v \geq k + s$. Suppose in a t-(v, k, λ) design that an n-fold block B_1 meets the (different) block B_2 in μ points. Then*

$$n\binom{k}{s}\binom{v-k}{s} \pm g_s(\mu) \leq \lambda\binom{v}{t}\binom{v}{s}^{-1}\binom{k}{t}^{-1}\binom{k}{s}\binom{v-k}{s}. \tag{4.4}$$

Note that this result can be applied to the intersection of an n_1-fold block with a distinct n_2-fold block upon multiplication by n_2. Undoubtedly, inequalities concerning the still more general $\mathbf{b} = \lambda\mathbf{1} - n_1\mathbf{1}_{W_1} - n_2\mathbf{1}_{W_2}$ for $s \leq |W_1|, |W_2| \leq v - s$ can be established by merging the proofs of Theorems 3.1 and 4.1.

4.2 Improvements from linear programming

The polynomials used to establish Theorem 4.1 are squares; hence they do not define facets of $\texttt{cone}(W_t)$. As before, tighter inequalities arise from facets, but at the expense of losing a concise closed form. Rather than looking for the best extremal polynomials in several variables, the approach will be to find optimal facets of $\texttt{cone}(W_t)$ using the LP viewpoint outlined in Section 1.2. Computational restrictions force this discussion to the case $t = 2$.

First, consider bounds on the intersection μ of two blocks B_1, B_2 in a 2-(v, k, λ) design. The vector $\mathbf{b} = \lambda\mathbf{1} - \mathbf{1}_{B_1} - \mathbf{1}_{B_2}$ is invariant under the partition Ω of size four described in the last section. The further symmetry between $B_1 \setminus B_2$ and $B_2 \setminus B_1$ allows a reduction to the seven orbits of pairs, or edges, described below.

edge orbit for $\{x, y\}$	$x \in$	$y \in$	number of edges
E_1	$(B_1 \cup B_2)^c$	$(B_1 \cup B_2)^c$	$\binom{v-2k+\mu}{2}$
E_2	$B_1 \triangle B_2$	$(B_1 \cup B_2)^c$	$2(k-\mu)(v-2k+\mu)$
E_3	$B_1 \cap B_2$	$(B_1 \cup B_2)^c$	$\mu(v-2k+\mu)$
E_4	$B_1 \setminus B_2$	$B_2 \setminus B_1$	$(k-\mu)^2$
E_5	$B_i \setminus B_j$	$B_i \setminus B_j$	$2\binom{k-\mu}{2}$
E_6	$B_1 \cap B_2$	$B_1 \triangle B_2$	$2\mu(k-\mu)$
E_7	$B_1 \cap B_2$	$B_1 \cap B_2$	$\binom{\mu}{2}$

It is now an LP problem to determine edge weights $a_1, \ldots, a_7$ so that the 2-vector $\mathbf{y} = \sum_{i=1}^{7} a_i \sum_{T \in E_i} \mathbf{1}_T$ minimizes the quantity $\langle \mathbf{y}, \mathbf{b} \rangle$ subject to some normalization (say $a_1 = 1$) and the constraints $\langle \mathbf{y}, \mathbf{1}_K \rangle \geq 0$ for all k-sets K. If this minimum is negative for some v, k, λ, μ, it follows that a 2-(v, k, λ) design cannot have two blocks with intersection μ.

Table 4.1 summarizes the best results of implementing this on computer for small parameters. Specifically, we present all parameters from the BIBD table in [22] with 'replication number' $r = \lambda\frac{v-1}{k-1} \leq 25$, and for which the linear program above improves Connor's inequalities. By Theorem 4.1, the LP does at least as well as Connor's inequalities. The column labeled '#' gives the corresponding number of the parameter set in [22]. The column 'LP' gives intervals for allowable μ linear programming bound.

#	v	k	λ	LP	#	v	k	λ	LP
4	6	3	2	$\{1,2\}$	226	96	16	3	$[0,4]$
10	10	4	2	$\{1,2\}$	227	153	17	2	$\{1,2\}$
16	15	5	2	$\{1,2\}$	255	61	10	3	$[0,8]$
24	9	4	3	$[0,3]$	257	37	10	5	$[0,8]$
25	21	6	2	$\{1,2\}$	259	21	10	9	$[0,9]$
33	10	5	4	$[1,4]$	260	111	12	2	$[0,7]$
36	28	7	2	$\{1,2\}$	262	141	15	2	$[0,5]$
49	21	7	3	$[0,3]$	263	57	15	5	$[0,6]$
50	36	8	2	$\{1,2\}$	265	76	16	4	$[0,5]$
59	45	9	2	$\{1,2\}$	266	171	18	2	$\{1,2\}$
75	21	6	3	$[0,4]$	291	57	9	3	$[0,8]$
77	13	6	5	$[0,5]$	293	22	11	10	$[1,10]$
78	22	8	4	$[0,4]$	297	40	14	7	$[0,8]$
79	33	9	3	$[0,3]$	301	85	17	4	$[0,5]$
80	55	10	2	$\{1,2\}$	302	120	18	3	$[0,4]$
89	14	7	6	$[1,6]$	303	190	19	2	$\{1,2\}$
92	66	11	2	$\{1,2\}$	326	45	11	5	$[0,9]$
107	29	7	3	$[0,5]$	331	133	19	3	$[0,4]$
110	78	12	2	$\{1,2\}$	332	210	20	2	$\{1,2\}$
131	21	9	6	$[0,6]$	344	70	10	3	$[0,9]$
136	91	13	2	$\{1,2\}$	347	231	21	2	$\{1,2\}$
158	17	8	7	$[0,7]$	388	89	12	3	$[0,9]$
165	105	14	2	$\{1,2\}$	392	25	12	11	$[0,11]$
177	35	7	3	$[0,6]$	393	105	14	3	$[0,9]$
179	18	9	8	$[1,8]$	397	115	20	4	$[0,5]$
180	52	13	4	$[0,5]$	398	161	21	3	$[0,4]$
181	120	15	2	$\{1,2\}$	400	253	22	2	$\{1,2\}$
205	49	9	3	$[0,7]$	424	26	13	12	$[1,12]$
209	55	10	3	$[0,7]$	426	51	15	7	$[0,10]$
210	100	12	2	$[0,5]$	429	76	19	6	$[0,8]$
213	136	16	2	$\{1,2\}$	433	176	22	3	$[0,4]$
225	39	13	6	$[0,7]$	434	276	23	2	$\{1,2\}$

Table 4.1: LP improvements on Connor's inequalities.

Example 4.1. Consider a hypothetical 2-$(22, 8, 4)$ design, for which existence was only recently ruled out in [4]. Connor's inequalities state that two blocks can meet in 0 through 5 points. With $\mu = 5$, the minimum L.P. bound $\langle \mathbf{y}, \mathbf{b} \rangle = -\frac{28}{15}$ is achieved with

$$(a_1, \ldots, a_7) = \left(1, -\frac{1}{3}, -\frac{7}{5}, \frac{1}{5}, \frac{1}{5}, 1, \frac{11}{3}\right).$$

Thus two blocks of such a design cannot meet in more than 4 points. This is indicated in Table 4.1.

4.3 Intersection of several blocks

In principle, the same approach as in Section 4.1 can be applied to the pairwise intersection sizes among n blocks.

Theorem 4.4. ([30]) *Let $t \geq 2s$ and $v \geq k+s$. Suppose $B_1, \ldots, B_n$ are blocks in a t-(v, k, λ) design with $|B_i \cap B_j| = \mu_{ij}$ for all i, j. Define the $n \times n$ matrix $G = [g_s(\mu_{ij})]_{ij}$. Then*

$$\det(\lambda\gamma G - G^2) \geq 0,$$

where $\gamma = \binom{v}{t}\binom{v}{s}^{-1}\binom{k}{t}^{-1}\binom{k}{s}\binom{v-k}{s}$.

Proof outline. It is enough to show that $\lambda\gamma G - G^2$ is positive semidefinite. Consider the partition Ω of V into 2^n subsets defined by intersection with either B_i or B_i^c for all i, and with associated variables $\{x_S : S \in \Omega\}$. Let $X_i = \sum_{S \subset B_i} x_S$ and $\mathbf{X} = (X_1, \ldots, X_n)$. Define the vector of polynomials

$$\mathbf{g}(\mathbf{X}) = (g_s(X_1), \ldots, g_s(X_n)).$$

For $\mathbf{u} \in \mathbb{R}^n$, consider the nonnegative polynomial $f(\mathbf{X}) = \langle \mathbf{u}, \mathbf{g}(\mathbf{X}) \rangle^2$. With similar computations and notation as in Theorem 4.1, one has the cone condition

$$\lambda \langle \mathbf{y}_f, \mathbf{1} \rangle \geq \sum \langle \mathbf{y}_f, \mathbf{1}_{B_i} \rangle$$

equivalent to

$$\lambda \mathbf{u}^\top [\Sigma_{ij}^R] \mathbf{u} \geq \mathbf{u}^\top [\Sigma_{ij}^L] \mathbf{u},$$

where $\Sigma_{ij}^L = \sum_{m=1}^n g_s(\mu_{im}) g_s(\mu_{mj})$ and $\Sigma_{ij}^R = \gamma g_s(\mu_{ij})$. Since $\Sigma_{ij}^L = G_{ij}^2$, it follows that $\mathbf{u}^\top(\lambda\gamma G - G^2)\mathbf{u} \geq 0$. Since $\mathbf{u}$ was arbitrary, this shows that the given matrix is positive semidefinite. $\square$

Remark: The statement $\det(\lambda\gamma I - G) \geq 0$ is proved in [30] and follows from Theorem 4.4 if it is known that G has positive determinant.

Example 4.2. Consider 2-$(56, 12, 3)$ designs, for which existence is known, and suppose some three blocks meet pairwise in $\mu = 4$ points. The above determinant inequality for $n = 3$ fails, so this block intersection pattern is not allowed. Connor's inequalities (4.3) for two blocks permit $\mu = 4$, however.

It is unfortunate that the bound in Theorem 4.4 is independent of the threewise intersection numbers $\nu_{hij} = |B_h \cap B_i \cap B_j|$. If the variable z represents the threewise intersection, the simple polynomial $(g_s(z))^2$ yields an upper bound on ν_{hij}; however, this inequality is implied by the case $n = 3$ of Theorem 3.1.

Convinced of additional structure at the threewise level, we implemented an analogous LP as in Section 4.2. Here, attention is again restricted to 2-designs, and also to patterns of three blocks intersecting uniformly; that is, all three pair-intersections of blocks are some common value μ. Even after symmetry, there are 13 variables and many more constraints. Table 4.2 is a compilation of results obtained by computer. It has similar headings as in Table 4.1, with ν representing threewise intersection.

Example 4.3. In a 2-$(26, 6, 3)$ design, three blocks which meet pairwise in 4 points can only have threewise intersection $\nu = 3$ or 4. Note $\nu = 2$ is ruled out in Table 4.2.

An important possible continuation of this work is an exploration of supporting 2-vectors – and their polynomials – which produce meaningful statements on m-wise intersection patterns of n blocks, $2 \leq m \leq n$.

#	v	k	λ	μ	forbidden ν by LP
4	6	3	2	2	1
9	7	3	2	2	1
15	8	4	3	3	$2, 3$
23	13	4	2	2	0
24	9	4	3	3	$2, 3$
35	16	6	3	4	$2, 3, 4$
46	21	5	2	3	1
49	21	7	3	0	0
56	12	4	3	4	4
74	31	6	2	3	0
75	21	6	3	4	$2, 3, 4$
78	22	8	4	1	1
79	33	9	3	0	0
90	27	9	4	5	1
91	40	10	3	0	0
91	40	10	3	4	$0, \ldots, 4$
92	66	11	2	0	0
92	66	11	2	3	$0, 1$
106	43	7	2	4	1
107	29	7	3	4	1
107	29	7	3	5	$3, 4, 5$
107	29	7	3	6	$5, 6$
107	29	7	3	7	7
127	26	6	3	4	2
127	26	6	3	5	$4, 5$
127	26	6	3	6	6
133	46	10	3	5	$0, \ldots, 5$
133	46	10	3	6	$2, \ldots, 6$
135	56	12	3	0	0
135	56	12	3	4	$0, \ldots, 4$

Table 4.2: Some forbidden threewise intersections in 2-designs.

Chapter 5

Variations on Block Designs

It is clear by now that the set-up in chapter 1 can be modified to consider other combinatorial objects. Here, we consider a few modifications to the inclusion matrix equation (1.3). We have taken the approach of merely outlining some possible variations, each accompanied with perhaps one example or nontrivial statement. Although the included material is admittedly very cursory, it is quite likely that further usefulness of the cone condition is possible in these contexts.

5.1 Different block sizes

Consider a tBD with block sizes $k_1 < k_2 < \dots$. Our corresponding inclusion matrix is a row-concatenation

$$W_t = \left[W^v_{tk_1} | W^v_{tk_2} | \cdots\right]$$

of the matrices introduced in Section 1.1. As before, define $\mathbf{1}_K$ to be the 0-1 characteristic t-vector for incidence of t-subsets with the k-subset K. Note that for $t \leq l < k$,

$$\mathbf{1}_K = \frac{1}{\binom{k-t}{l-t}} \sum_{L \subset K, |L|=l} \mathbf{1}_L.$$

It follows that $\mathtt{cone}(W^v_{tk_1}) \subseteq \mathtt{cone}(W^v_{tk_2}) \subseteq \dots$ and so

$$\mathtt{cone}(W) = \bigcup_j \mathtt{cone}(W^v_{tk_j}) = \mathtt{cone}(W^v_{tk_1}).$$

A *generalized incomplete t-wise balanced design* (or GItBD) with index λ and *hole* H of *strength* m is a triple $(V, H, \mathcal{B})$ such that $H \subset V$ and $\mathcal{B}$ is a collection of blocks of V with the property that every t-subset T occurs in exactly λ blocks if $|T \cap H| < t - m$, and exactly 0 blocks otherwise. A GItBD is called *proper* if all block sizes are strictly between t and v. The case $m = 0$ is the well studied *incomplete* tBD (ItBD) with hole H.

Theorem 5.1. *Suppose $t - m$ is even. In a GItBD $(V, H, \mathcal{B})$ with $|V| = v$, $|H| = h \geq t$, and hole strength m, it is necessary that $v \geq 2h+m+1$. Equality holds if and only if every block $B \in \mathcal{B}$ satisfies $|B \setminus H| \leq t$.*

Proof. Define the t-vector $\mathbf{b}$ by

$$\mathbf{b}(T) = \begin{cases} 1 & \text{if } |T \cap H| < t - m, \\ 0 & \text{otherwise.} \end{cases}$$

By the remarks starting this section, it is required that $\mathbf{b} \in \texttt{cone}(W^v_{t,t+1})$. Clearly, $\mathbf{b}$ is invariant under the bipartition $(H, V \setminus H)$. Define the polynomial $f(x) = (-1)^t \binom{x-1}{t}$, so that $f(x) \geq 0$ for $0 \leq x < t$. The alternate expression

$$f(x) = (t+1) \sum_{j=0}^{t} (-1)^j \binom{t}{j}^{-1} \binom{x}{j} \binom{t+1-x}{t-j}$$

is implicit from the matrix M_0^{-1} of Proposition 3.4. By the cone condition and Theorem 2.3, $\langle \mathbf{y}_f, \mathbf{b} \rangle \geq 0$. Equivalently,

$$\begin{aligned} 0 &\leq \sum_{j=0}^{t-m-1} \binom{t}{j}^{-1} \binom{h}{j} \binom{v-h}{t-j} \qquad (5.1) \\ &= \frac{t+1}{v-t} \left[\binom{v-h}{t+1} - (-1)^{t-m} \binom{h}{t-m} \binom{v-h}{m+1} \binom{t+1}{m+1}^{-1} \right], \end{aligned}$$

where the closed form arises from identity (1.14). Using equation (1.10), this is equivalent to

$$\binom{v-h-m-1}{t-m} \geq \binom{h}{t-m},$$

or $v \geq 2h + m + 1$. For equality to occur in (5.1), no $B \in \mathcal{B}$ can contain a $(t+1)$-set X disjoint from H; for otherwise $\langle \mathbf{y}_f, \mathbf{b} \rangle \geq \langle \mathbf{y}_f, \mathbf{1}_X \rangle = f(0) > 0$. In other words, it must be that $m < |B \setminus H| \leq t$ for all blocks B. $\square$

Remarks: The case $m = 0$ was recently proved in [16], and an argument similar to the one given here is presented in [32]. Either proof can be modified for $m > 0$. Note that the condition on equality, also discussed in [32] for $m = 0$, implies that all block sizes are between $t + 1$ and $2t - m + 1$. It is curious that the inequality in Theorem 5.1 is the same constraint as on an ItBD with $v+m$ points and hole size $h+m$, yet there appears to be no easy combinatorial equivalence between these objects and GItBDs with v points, hole size h, and hole strength m.

5.2 Packings and coverings

A t-(v, k, λ) *packing* (*covering*) is a family of blocks on v points such that

- each block is a k-subset of the points, and
- each pair of distinct points is contained in **at most** (**at least**) λ blocks.

Packings and coverings are important in applications which would ideally use designs, but for which the design don't exist. On one hand, necessary conditions may be infeasible. For instance, a card tournament might desire that 12 participants play, each pair meeting at least 5 times, at tables of 4. A 2-$(12, 4, 5)$ design is desired, but the necessary conditions fail. A covering must be used. On the other hand, a design may fail to exist, even when the parameters are admissible. A wireless network of 57 nodes may wish to schedule time slots for broadcasts of 12 nodes at a time so that – perhaps to minimize interference – every 4-subset of nodes is broadcasting together at most once. A 4-$(57, 12, 1)$ design would be nice; but, as seen in Table t4-ruledout, it does not exist. Here, a packing must be used.

Instead of the equation $W_{tk}^v \mathbf{x} = \lambda \mathbf{1}$ being relevant, we have inequalities

$$\begin{aligned} W_t \mathbf{x} &\leq \lambda \mathbf{1} \qquad \text{for packings} \\ \text{and} \quad W_t \mathbf{x} &\geq \lambda \mathbf{1} \qquad \text{for coverings.} \end{aligned} \tag{5.2}$$

Of course, by elementary counting we know that a packing (covering) contains at most (at least) as many blocks

$$\lambda \binom{v}{t} \binom{k}{t}^{-1}.$$

as a design would with those parameters. Thus, coverings in particular inherit the standard lower bounds on b such as Corollary 3.2.

Conversely, maximizing the number of blocks in a t-(v, k, λ) packing can be approached using (5.2) and the LP formulation in Section 1.2. Suppose B is a block in the packing. Then 'subtracting' B, we obtain the primal problem

$$\begin{array}{rl} \text{maximize} & \langle \mathbf{1}, \mathbf{x} \rangle \\ \text{subject to} & W_t \mathbf{x} \leq \lambda \mathbf{1} - \mathbf{1}_B, \quad \mathbf{x} \geq \mathbf{0}. \end{array}$$

Its dual is

$$\begin{array}{rl} \text{minimize} & \lambda \langle \mathbf{y}, \mathbf{1} \rangle - \langle \mathbf{y}, \mathbf{1}_B \rangle \\ \text{subject to} & \mathbf{y} W_t \geq \mathbf{1}, \quad \mathbf{y} \geq \mathbf{0}. \end{array}$$

Therefore, this minimum serves as an upper bound on the number of blocks in the packing.

Packings in a different association scheme (see Chapter 6) translate into *codes*. The matrix W_t needs to change, but in principle this 'dualized' cone condition works in the same way. Various coding-theoretic bounds can be reproduced, just as we have done for design-theoretic bounds. We leave details of this to future work.

5.3 Resolvability

A set of v/k blocks which partitions the points of a 2-(v, k, λ) design is called a *parallel class*. Note that a parallel class is also a 1-$(v, k, 1)$ subdesign. An α-parallel class is a 1-(v, k, α) subdesign, although we do not pursue this here.

A 2-design is *resolvable* if the block family admits a partition into r parallel classes. Resolvability is sometimes extended to t-designs by insisting that the blocks partition into s-designs for $s < t$. Nevertheless, both existence and structural results for resolvable designs are somewhat harder.

For instance, the well known modification

$$b \geq v + r - 1$$

of Fisher's inequality on the number of blocks in a resolvable 2-design can actually be proved using only the fact that the design has a *single* parallel

class. Whether the blocks of a design partition into parallel classes is not easily captured by the inclusion matrix equation $W_t\mathbf{x} = \lambda\mathbf{1}$. However, we are able to discuss an easy application of the cone condition under an assumption of one parallel class.

A block B' is *transverse* to a parallel class $\{B_1, \ldots, B_{v/k}\}$ if $|B' \cap B_i| \leq 1$ for all i.

Proposition 5.2. *Suppose a 2-(v, k, λ) design with $v > (k+1)\binom{k-1}{2} + k$ contains a parallel class P. Then some block is transverse to P.*

Proof. Suppose no block is transverse to P. The vector $\mathbf{b} = \lambda\mathbf{1} - \sum_{B \in P} \mathbf{1}_B$ is of course invariant under the partition defined by P. It is necessary that $\mathbf{b} \in \mathtt{cone}(W_{tk}^v|\mathcal{K})$, where we restrict the columns to the family $\mathcal{K}$ of k-subsets not transverse to P. Define the 2-vector $\mathbf{y}$ by

$$\mathbf{y}(T) = \begin{cases} \binom{k}{2} - 1 & \text{if } T \text{ is a pair within some } B \in P, \\ -1 & \text{otherwise.} \end{cases}$$

By assumption, at least one pair of points in every $K \in \mathcal{K}$ is contained in some $B \in P$, so $\mathbf{y}$ supports the cone. Let $p = |P| = v/k$. Then $\langle \mathbf{y}, \mathbf{b} \rangle \geq 0$ implies

$$1 > 1 - \frac{1}{\lambda} \geq \frac{k^2\binom{p}{2}}{\left(\binom{k}{2} - 1\right)\binom{k}{2}p} = \frac{k(p-1)}{(k+1)\binom{k-1}{2}},$$

from which the result follows. □

It should be remarked that the proof above essentially just relies on counting pairs within and across blocks of P. Nonetheless, this is yet another structure which can be formulated in terms of the cone condition. Further study of the structure of parallel classes is left to the ambitious reader.

5.4 Automorphisms

So far, our treatment of the inclusion matrices W_{tk}^v has focused on nonexistence results for designs. This is historically misleading. These inclusion matrices were perhaps first used for the purpose of constructing designs. In order to reduce the dimensions, group actions can be imposed.

The *automorphism group* of a t-(v, k, λ) design on points V and block family $\mathcal{B}$ is the maximal subgroup G of $\mathcal{S}_V$ such that $\mathcal{B}^\sigma = \mathcal{B}$ for every $\sigma \in G$. Let the induced G-orbits of $\binom{V}{t}$ and $\binom{V}{k}$ index rows and columns of a matrix $W^v_{tk}(G)$, where the (i, j)-entry is the number of k-subsets in the jth orbit of $\binom{V}{k}$ which contain a t-subset in the ith orbit of $\binom{V}{t}$. With this set-up, a design $\boldsymbol{\phi}$ having G as a subgroup of its automorphism group exists is equivalent to a nonnegative integral solution of

$$W^v_{tk}(G)\boldsymbol{\phi} = \lambda\mathbf{1}. \tag{5.3}$$

In Section 2.1, we essentially considered these matrices for $G = \mathcal{S}_{U_1} \times \cdots \times \mathcal{S}_{U_d}$, leaving an invariant partition $(U_1, \ldots, U_d)$.

A 'basis reduction algorithm' was used by Kreher and Radziszowski [15] to find a few previously unknown t-designs. In this manuscript, we are interested in neither existence results nor lattice (integrality) techniques; however, the success in solving (5.3) is certainly important to mention.

Important further work on the cone condition should include an investigation of supporting vectors for $W^v_{tk}(G)$. Put another way, if $\mathbf{y}$ is a t-vector supporting W_t, and $\mathbf{y}$ is known to be G-invariant, then a constraint arises on t-designs with automorphisms in G. Existence of such $\mathbf{y}$ for cyclic groups $G = \mathbb{Z}/(v)$ is probably an important point of departure from our work.

5.5 Graph decompositions

A *t-uniform hypergraph* is a pair (V, E), where V is a set of points (here, a finite set) and E is a set of t-subsets of V called *edges*. Suppose G is a t-uniform hypergraph and $\mathcal{H}$ is a finite set of t-uniform hypergraphs on the vertices V. The incidence matrix $W = W^G_{\mathcal{H}}$ has rows indexed by all edges T of G and columns indexed by all $H \in \mathcal{H}$. The definition follows that in Section 1.1. We have

$$W(T, H) = \begin{cases} 1 & \text{if } T \text{ is an edge of } H, \\ 0 & \text{otherwise.} \end{cases}$$

As usual, let $|V| = v$. When G is the complete t-uniform hypergraph on V and $\mathcal{H}$ is the set of all $\binom{v}{k}$ complete t-uniform hypergraphs on some k points of V, the matrix $W^V_{\mathcal{H}}$ coincides with W^v_{tk} introduced earlier. It is natural to say that nonnegative integral solutions $\boldsymbol{\phi}$ of $W^G_{\mathcal{H}}\boldsymbol{\phi} = \lambda\mathbf{1}$ correspond to *hypergraph decompositions*.

Usually, $\mathcal{H}$ is defined as the set of all possible embeddings of a collection of isomorphism types H_1, $H_2, \ldots$. In fact, most literature so far concerns graph decompositions in the classical sense ($t = 2$) and with a fixed isomorphism type H for blocks.

Example 5.1. A k-cycle system of index λ on a v-set V of points is a collection $\mathcal{B}$ of cycles of length k in V, such that every pair of points is an edge of exactly λ members of $\mathcal{B}$. Let W' be the incidence matrix of 2-subsets of V with k-cycles in V. Then W' is a $\binom{v}{2} \times \binom{v}{k}(k-1)!$ matrix. By 'averaging' k-cycles over a k-subset of V, it follows that $\texttt{cone}(W^v_{2k}) \subset \texttt{cone}(W')$. This containment of cones is proper, since there certainly exist supporting vectors of $\texttt{cone}(W^v_{2k})$ which have negative inherited weight on some k-cycle. Suppose $\mathbf{1}_C$ is a characteristic 2-vector for some generic k-cycle C on the ponts of V. There are k ones and $\binom{v}{2} - k$ zeros in $\mathbf{1}_C$. The condition $\lambda\mathbf{1} - n\mathbf{1}_C \in \texttt{cone}(W')$ generates a family of inequalities on the existence of an n-fold cycle in a cycle system with the given parameters. An automorphism group under which this vector is invariant is isomorphic to $\mathbb{Z}_k \times \mathcal{S}_{v-k}$.

By contrast, there is a seldom-studied alternative viewpoint. Suppose the graph G we wish to decompose is not complete, but that the blocks are. This scenario actually arises in important applications.

For example, in statistical sampling applications, one may wish to avoid simultaneously sampling adjacent population units. Combinatorially, we require a nonnegative rational decomposition of a *circulant graph* of degree $v-2$, missing the distances ± 1. Existence of these sampling plans was considered in [10] with a modification of the inclusion matrix. One restricts to $W^v_{tk}|\mathcal{K}$, where $\mathcal{K}$ consists of all k-subsets not containing cyclically adjacent units. The $\mathbb{Z}/(v)$ group action can also be incorporated as in Section 5.4.

We have not pursued any nonexistence results of these objects via the cone condition; however, this is a potentially fruitful approach.

Chapter 6

Association Schemes

As the reader may now suspect, the cone condition for t-designs can be abstracted still further. Here, we consider a more algebraic viewpoint. This investigation must begin with a summary of the theory of association schemes. Chapter 30 of [26] is a nice introductory starting point. Chris Godsil's notes [13] are excellent as a deeper resource for the interested reader.

6.1 Definitions, identites and the Johnson scheme

A *k-class association scheme* on a set X consists of $k+1$ nonempty symmetric binary relations $R_0, \ldots, R_k$ which partition $X \times X$, such that

- R_0 is the identity relation, and
- for any $x, y \in X$ with $(x, y) \in R_h$, the number of $z \in X$ such that $(x, z) \in R_i$ and $(z, y) \in R_j$ is the *structure constant* p_{ij}^h depending only on h, i, j.

Let $|X| = n$. For $h = 0, \ldots, k$, define the $n \times n$ *adjacency matrix* A_h, indexed by entries of X, to have (x, y)-entry 1 if $(x, y) \in R_h$, and 0 otherwise. Each A_h can be viewed as a regular graph of degree n_h, where $n_0 = 1$ for the graph A_0 with n disjoint loops.

By definition of the structure constants, $A_iA_j = \sum_h p_{ij}^h A_h$. Therefore, the adjacency matrices form a basis for a matrix algebra, called the *Bose-Mesner algebra*.

Interestingly, the adjacency matrices are orthogonal idempotents with respect to *entrywise* multiplication, and their sum is J, the all ones matrix. The Bose-Mesner algebra also has a basis of orthogonal idempotents $E_0, \ldots, E_k$ with respect to *ordinary* matrix multiplication, and such that $E_0 + \cdots + E_k = I$. A convention is adopted so that $E_0 = n^{-1}J$, which is necessarily one of these idempotents. The rank of E_h is denoted m_h, so that in particular $m_0 = 1$. From an orthogonal decomposition, we have $\sum_h m_h = n$. The m_h, $h = 0, \ldots, k$, are known as the *multiplicities* of the scheme.

The *first* (respectively, *second*) *eigenmatrix* P (respectively, Q) has rows and columns indexed by $\{0, 1, \ldots, k\}$, with (i, j)-entry written $P_j(i)$ (respectively $Q_j(i)$). They provide basis-change coefficients as follows.

$$\begin{aligned} A_j &= \sum_{i=0}^{k} P_j(i) E_i, \quad \text{and} \\ nE_j &= \sum_{i=0}^{k} Q_j(i) A_i. \end{aligned}$$

Column j of P consists of the eigenvalues of A_j.

For later use, we summarize some important facts concerning the eigenmatrices, degrees and multiplicities.

$$\begin{aligned} P_h(0) &= n_h & \qquad Q_h(0) &= m_h \\ P_0(x) &= 1 & \qquad Q_0(x) &= 1 \\ P &= nQ^{-1} & \qquad Q &= nP^{-1} \end{aligned} \tag{6.1}$$

Particularly useful is the relation

$$Q^\top \begin{bmatrix} n_0 & & \\ & \ddots & \\ & & n_k \end{bmatrix} Q = \begin{bmatrix} m_0 & & \\ & \ddots & \\ & & m_k \end{bmatrix}. \tag{6.2}$$

Consider now $X = \binom{V}{k}$, with $A, B \in X$ declared in R_h if and only if $|A \cap B| = k - h$. By symmetry, these relations form an association scheme on

X, called the *Johnson scheme*, or $J(k,v)$. For now it is assumed that $J(k,v)$ is our underlying scheme. It has degrees $n_h = \binom{k}{h}\binom{v-k}{h}$, multiplicities $m_h = \binom{v}{h} - \binom{v}{h-1}$, and (under appropriate ordering of the idempotents) columns of Q given by

$$Q_j(x) = m_j \sum_{h=0}^{j} (-1)^h n_h^{-1} \binom{j}{h} \binom{v+1-j}{h} \binom{x}{h}. \tag{6.3}$$

This is a polynomial of degree j in x, also known as the jth *Hahn polynomial*.

6.2 The cone condition and Delsarte's inequalities

Let us now return to the cone condition and supporting t-vectors $\mathbf{y}$ for $\texttt{cone}(W_t)$. Such vectors $\mathbf{y}$ afford an inequality

$$\lambda\langle \mathbf{y}, \mathbf{1}\rangle \geq \langle \mathbf{y}, \mathbf{1}_B\rangle \tag{6.4}$$

necessary for the existence of a t-(v,k,λ) design. Here, we are using Theorem 1.5, where the configuration in question is $\boldsymbol{\psi} = (1,0,\ldots,0)$ consisting of a single block B.

We now reproduce some of the material in Chapter 2, this time for an 'algebraic' look at the cone condition.

Put $\mathbf{y}(T) = y_j$, where $|T \cap B| = t - j$. We regard this as a weighting on $\binom{V}{t}$ which is invariant under $\mathcal{S}_B \times \mathcal{S}_{V\setminus B}$. Since $\mathbf{1}_B$ is invariant under this same permutation group, we can do no better in (6.4) by choosing a finer partition for $\mathbf{y}$.

Now define

$$f(x) = \sum_{j=0}^{t} y_j \binom{x}{j}\binom{k-x}{t-j}, \tag{6.5}$$

a polynomial of degree at most t in x. The reader is cautioned that this f has a somewhat different definition than in Chapter 3, since y_j is taking the place of a_{t-j}.

In any case, the inherited weight on a k-subset $K \subset V$ is calculated by

$$\mathbf{y}W_t(K) = \sum_{j=0}^{t} y_j \binom{i}{j}\binom{k-i}{t-j} = f(i), \tag{6.6}$$

where $|K \cap B| = k - i$. Conveniently, $\mathbf{y}$ supports $\texttt{cone}(W_t)$ if and only if $f(i) \geq 0$ for all $i \in \{0, 1, \ldots, k\}$. Since $\mathbf{1}_B$ is actually a column of W_t, a special case of this calculation allows us to compute the right side of (6.4).

Lemma 6.1. *With* $\mathbf{y}$ *as in* (6.5), $\langle \mathbf{y}, \mathbf{1}_B \rangle = f(0)$.

The left side is somewhat more delicate. In what follows, let w_t denote the rowsum of W_t.

Lemma 6.2. *With* $\mathbf{y}$ *as in* (6.5),

$$\langle \mathbf{y}, \mathbf{1} \rangle = \frac{1}{w_t} \sum_{i=0}^{k} n_i f(i).$$

Proof. Since $W_t\mathbf{1} = w_t\mathbf{1}$, it follows that $\langle \mathbf{y}, \mathbf{1} \rangle = w_t^{-1}\langle \mathbf{y}, W_t\mathbf{1} \rangle = w_t^{-1}\langle \mathbf{y}W_t, \mathbf{1} \rangle$, the latter inner product being for k-vectors. In (6.6), there are n_i k-subsets receiving weight $f(i)$. So $\langle \mathbf{y}, \mathbf{1} \rangle = w_t^{-1} \sum_{i=0}^{k} n_i f(i)$, as required. □

This, along with Lemma 6.1 reduces (6.4) to

$$\lambda w_t^{-1} \sum_{i=0}^{k} n_i f(i) \geq f(0), \tag{6.7}$$

for any polynomial f, $\deg(f) \leq t$, where $f \geq 0$ on $\{0, 1, \ldots, k\}$.

Any function $f : \{0, 1, \ldots, k\} \to \mathbb{R}$ is a polynomial of degree at most k. From (6.3), f can be expressed in the basis consisting of the columns of Q. That is,

$$f(x) = \sum_{i=0}^{k} f_i^Q Q_i(x), \tag{6.8}$$

or more concisely $\mathbf{f} = Q(\mathbf{f}^Q)$, where $\mathbf{f} = [f(x)]_{x=0,\ldots,k}$ and $\mathbf{f}^Q = [f_i^Q]_{i=0,\ldots,k}$. Also, from the expression in (6.3), $\deg(f) \leq t$ is equivalent to $f_i^Q = 0$ for $i \notin \{0, 1, \ldots, t\}$.

Lemma 6.3. *With f as in* (6.8),

$$\sum_{i=0}^{k} n_i f(i) = |X| f_0^Q.$$

Proof. Using (6.1) and (6.2),

$$\begin{aligned}
\langle (n_0, \ldots, n_k), \mathbf{f} \rangle &= [n_0 \ \ldots \ n_k] Q \mathbf{f}^Q \\
&= [1 \ 0 \ \ldots \ 0] Q^\top \begin{bmatrix} n_0 & & \\ & \ddots & \\ & & n_k \end{bmatrix} Q \mathbf{f}^Q \\
&= |X| [1 \ 0 \ \ldots \ 0] \begin{bmatrix} m_0 & & \\ & \ddots & \\ & & m_k \end{bmatrix} \mathbf{f}^Q \\
&= |X| f_0^Q.
\end{aligned}$$

□

By linearity in (6.8), we may normalize f so that $f_0^Q = 1$. Then (6.4) becomes $\lambda |X| w_t^{-1} \geq f(0)$. It is straigtforward counting that for any t-(v, k, λ) design $\boldsymbol{\phi}$, one has

$$|\boldsymbol{\phi}| = \lambda \frac{\binom{v}{t}}{\binom{k}{t}} = \lambda \frac{|X|}{w_t}.$$

We can now summarize the 'cone condition' for t-designs.

Theorem 6.4. *Let $\boldsymbol{\phi}$ be a t-design. Then*

$$|\boldsymbol{\phi}| \geq f(0),$$

where $f : \{0, 1, \ldots, k\} \to \mathbb{R}$ satisfies

- $f_0^Q = 1$,
- $f_i^Q = 0$ *for* $i \notin \{0, 1, \ldots, t\}$,
- $f(i) \geq 0$ *for* $i \in \{0, 1, \ldots, k\}$.

It is curious that this is *almost* as strong a bound as *Delsarte's linear programming bound* for designs in association schemes. One form of these famous inequalities appears in [17] and is restated below.

Theorem 6.5 (Delsarte's Inequalities)**.** *Let $\boldsymbol{\phi}$ be a simple t-design. Then*

$$|\boldsymbol{\phi}| \geq f(0),$$

where $f : \{0, 1, \ldots, k\} \to \mathbb{R}$ satisfies

- $f_0^Q = 1$,
- $f_i^Q \leq 0$ *for* $i \notin \{0, 1, \ldots, t\}$,
- $f(i) \geq 0$ *for* $i \in \{0, 1, \ldots, k\}$.

The only differences from Theorem 6.4 are that (1) $\boldsymbol{\phi}$ be simple – a detail that can be weakly addressed in Delsarte's inequalities – and that (2) the Q-coefficients f_i^Q are permitted to be negative for $i > t$.

Example 6.1. Consider the possible existence of a 4-$(17, 8, 5)$ design. It is easy to see that the necessary conditions (1.1) hold for these parameters. Both Theorems 6.4 and 6.5 rule out the design identically, with the same optimum function $\widehat{f}$. After some tedious calculations (or a simple exercise on computer software),

$$5\binom{17}{4}\binom{8}{4}^{-1} = 170 = |\boldsymbol{\phi}| < \widehat{f}(0) = 212.5.$$

Even though the goal is to maximize $f(0)$, allowing in Delsarte's inequalities $f_i^Q < 0$ for $i > t$ can be substantial.

Example 6.2. Consider 6-$(19, 9, \lambda)$ designs. The necessary conditions (1.1) permit any nonnegative even λ. Theorem 6.4 offers a bound $\lambda \geq 3.5$, with optimum

$$\mathbf{f}^Q = \left(1, 0, \tfrac{459}{988}, -\tfrac{54}{2717}, \tfrac{945}{5434}, -\tfrac{28}{2717}, \tfrac{35}{836}, 0, 0, 0\right).$$

On the other hand, Delsarte's Inequalities give $\lambda \geq 11$ with Q-coefficients

$$(1, \tfrac{33}{52}, \tfrac{33}{52}, \tfrac{9}{26}, \tfrac{9}{26}, \tfrac{7}{52}, \tfrac{7}{52}, 0, 0, -\tfrac{3}{52}).$$

Amazingly, it is this last coefficient which accounts for the improvement! The resulting $f : \{0, 1, \ldots, k\} \to \mathbb{R}$ does not correspond to a supporting t-vector for $\texttt{cone}(W_t)$.

6.3 Revisiting Ray-Chaudhuri and Wilson's inequality

Recall the Hahn polynomials g_s defined in Section 1.3. In terms of (6.3),

$$g_s(x) \equiv \sum_{i=0}^{s} (-1)^{s-i} \frac{\binom{v-s}{i}\binom{k-i}{s-i}\binom{k-1-i}{s-i}}{\binom{s}{i}} \binom{x}{i} = \sum_{j=0}^{s} Q_j(x). \tag{6.9}$$

Here, we take a second look at the inequality of Ray-Chaudhuri and Wilson, which was proved with tedious calculations in Chapter 3.

Theorem 6.6 (Ray Chaudhuri and Wilson's bound)**.** *Let $t \geq 2s$ and suppose $\boldsymbol{\phi}$ is a t-design. Then*

$$|\boldsymbol{\phi}| \geq m_0 + \cdots + m_s. \tag{6.10}$$

Equality occurs in (6.10) if and only if $\boldsymbol{\phi}$ has exactly s distinct intersection numbers, on which the polynomial g_s vanishes.

Proof. Take $f(x) = (g_s(x))^2$ in (6.7). Recomputing the sum on the left side using (6.2) and (6.9), we obtain

$$\begin{aligned}
\langle (n_0, \ldots, n_k), \mathbf{f} \rangle &= [\overbrace{1 \ \ldots \ 1}^{s} \ 0 \ \ldots \ 0] Q^\top \operatorname{diag}(n_0, \ldots, n_k) \, Q [\overbrace{1 \ \ldots \ 1}^{s} \ 0 \ \ldots \ 0]^\top \\
&= |X| \cdot [\overbrace{1 \ \ldots \ 1}^{s} \ 0 \ \ldots \ 0] \operatorname{diag}(m_0, \ldots, m_k) \, [\overbrace{1 \ \ldots \ 1}^{s} \ 0 \ \ldots \ 0]^\top \\
&= |X|(m_0 + \cdots + m_s)
\end{aligned}$$

Therefore, the cone condition becomes

$$\frac{\lambda |X|}{w_t}(m_0 + \cdots + m_s) \geq f(0) = (m_0 + \cdots + m_s)^2,$$

from which the bound easily follows. $\square$

Remarks: For the Johnson scheme, multiplicities are $m_i = \binom{v}{i} - \binom{v}{i-1}$. Therefore, the bound in (6.10) simply becomes

$$|\boldsymbol{\phi}| \geq \binom{v}{s}.$$

If a t-design $\boldsymbol{\phi}$ contains repeated blocks, then the same method can reprove Corollary 3.2 as well.

6.4 The general setting

Consider now an arbitrary association scheme on X. There is an important definition of 'design' in this general context. We say $\boldsymbol{\phi} \in \{0,1\}^X$ is a *Delsarte t-design* if $E_i\boldsymbol{\phi} = 0$ for $i = 1, \ldots, t$. As one would hope, the definition declares as in (1.1) that a t-design is also an i-design for $i < t$. Unfortunately – except in various specific schemes – Delsarte t-designs are usually difficult to interpret. With a sufficiently 'combinatorial' scheme, an alternative formulation is possible.

The notation and terminology below essentially follows that in [8]. Let $(\mathcal{P}, \preceq)$ be a semilattice with rank function $\rho : \mathcal{P} \to \{0, 1, \ldots, k\}$. Define $\mathcal{P}^i$ to be the ith *fiber* of $\mathcal{P}$, namely the set

$$\mathcal{P}^i = \{x \in \mathcal{P} : \rho(x) = i\}$$

of elements of rank i. Suppose $z \in \mathcal{P}^j$ and $x \in X$. If the quantities

$$\alpha_{ij} = |\{y \in \mathcal{P}^i : z \preceq y \preceq x\} \quad \text{and} \quad \beta_{ij} = |\{y \in \mathcal{P}^i : z \preceq y\}|$$

are constants independent of x and z, it is said that $(\mathcal{P}, \preceq)$ is a *regular* semilattice.

Define the inclusion matrix W_t, whose rows and columns are indexed by $\mathcal{P}^t$ and $\mathcal{P}^k$, respectively, by

$$W(x, y) = \begin{cases} 1 & \text{if } x \preceq y, \\ 0 & \text{otherwise.} \end{cases}$$

Now suppose the top fiber $X = \mathcal{P}^k$ carries an association scheme. The vector $\boldsymbol{\phi} \in \mathbb{N}_0^X$ is a *combinatorial t-design* in $(\mathcal{P}, \preceq)$ if there exist $\lambda_0, \ldots, \lambda_t \in \mathbb{N}_0$ such that

$$W_t\boldsymbol{\phi} = \lambda_j\mathbf{1}.$$

As usual, $\boldsymbol{\phi}$ is *simple* if $\boldsymbol{\phi} \in \{0,1\}^X$. This definition is extended beyond regular semilattices to more general 'Q-posets' for association schemes in [21].

It is not hard to see that (simple) combinatorial t-designs are also Delsarte t-designs in X. For let $\mathbf{e}$ be a row of E_j, for some $j = 1, \ldots, t$. The rowspace of W_t admits an orthogonal decomposition $V_0 \oplus V_1 \oplus \cdots \oplus V_k$, where E_j is the matrix of projection onto each V_j. See [13] for details. So for some vector $\mathbf{z}$,

$$\langle \mathbf{e}, \boldsymbol{\phi} \rangle = \mathbf{z} W_j \boldsymbol{\phi} = \lambda_j \langle \mathbf{z}, \mathbf{1} \rangle = 0.$$

The orthogonality of $\mathbf{z}$ and $\mathbf{1}$ follows from $E_j E_0 = E_j J = 0$.

Example 6.3. The *Hamming lattice* $(\mathcal{P}, \preceq)$ on a vertex set U (with $|U| = n$) has $\mathcal{P}$ given by the words of length k over the alphabet $U \cup \{*\}$. For $x, y \in \mathcal{P}$, define $x \preceq y$ if and only if $y_i = *$ implies $x_i = *$ and $x_i \neq *$ implies $x_i = y_i$, for each i. The rank function for this poset is $\rho(x) = |\{i : x_i \neq *\}|$. The top fiber $X = \mathcal{P}^k$ consists of words with no occurrence of $*$. The incidence matrix W_t for this poset has dimensions $\binom{k}{t} n^t \times n^k$. A t-design in this lattice is known as an *orthogonal array* of *strength* t and *index* λ, which is denoted here by $\mathrm{OA}_\lambda(t, k, n)$. Concretely, an $\mathrm{OA}_\lambda(t, k, n)$ is a $\lambda n^t \times k$ array (say A) with entries from U, such that in any selection of t columns, each of the n^t ordered t-tuples of vertices occurs in exactly λ rows. In [9], the cone condition for OAs is used to obtain inequalities useful in pseudo-random number generation.

Open problems

It seems appropriate to close with a short list of the most important open problems arising from this work.

1. Understand all facets of $\texttt{cone}(W_{2k}^v)$.
2. Determine the optimal even sets Z for all $t = 2s$.
3. Obtain a closed form general result on threewise intersection of blocks.
4. Pursue the cone condition with variations on W_t, and interpret results in terms of other combinatorial objects.
5. Solidify the connection between the cone condition, Delsarte theory, and Wilson's method of orthogonal projection in [29].

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