

THE UNIVERSITY OF VICTORIA MATHEMATICAL COMPETITION  
SEPTEMBER 2009

- All the necessary work to justify an answer and all the necessary steps of a proof must be shown clearly to obtain full credit.
- Partial credit will be given only for substantial progress toward a solution.
- All questions are worth equal marks.

NO CALCULATORS, NOTES OR BOOKS ARE ALLOWED

**Question 1:** If  $x$  and  $y$  are real numbers such that  $y \geq 0$  and  $y(y+1) \leq (x+1)^2$ , prove that  $y(y-1) \leq x^2$ .

**Question 2:** A sequence  $\{a_n\}$  of real numbers is defined by

$$a_0 = 1; \quad a_{n+1} = \frac{a_n}{1 + na_n}, \text{ for } n = 0, 1, 2, \dots$$

Find, with proof, an explicit formula for  $a_n$ .

**Question 3:** Consider a circle with centre at the origin and arc  $R$  of the circle lying entirely in the first quadrant. Let  $A$  be the area of the region lying below  $R$  and above the  $x$ -axis and let  $B$  be the area of the region lying to the right of the  $y$ -axis and to the left of  $R$ . Prove that  $A + B$  depends only on the arc length, and not on the position, of  $R$ .

**Question 4:** Can there be quadratic polynomials  $f, g, h$  such that the roots of the equation  $f(g(h(x))) = 0$  are  $1, 2, 3, 4, 5, 6, 7, 8$ ?