THE UNIVERSITY OF VICTORIA MATHEMATICAL COMPETITION
September 21, 2010

- No calculators, books or notes are allowed.
- Write solutions in the booklets provided. Clearly separate rough workings from solutions.
- All the necessary work to justify an answer and all the necessary steps of a proof must be shown clearly to obtain full credit.
- Partial credit will be given only for substantial progress toward a solution.
- Questions are of equal value.

Duration: 2 hours

Question 1. Let $x_0 = 1/2$ and set $x_{n+1} = x_n - x_n^2$. Prove that $x_{2010} < 1/2012$.

Question 2. If seven points can be placed arbitrarily in the plane, what is the maximum possible number of lines that each intersect at least three of the points?

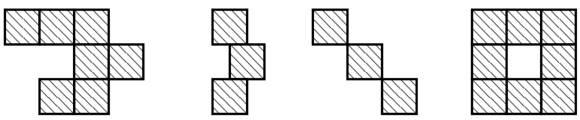
Question 3. Let α and β be real numbers satisfying

$$\begin{aligned} \alpha^3 - 3\alpha^2 + 5\alpha - 17 &= 0, \\ \beta^3 - 3\beta^2 + 5\beta + 11 &= 0. \end{aligned}$$

Find $\alpha + \beta$.

Question 4. A polyomino is a shape that can be obtained by gluing together 1×1 squares full-edge-to-full-edge in such a way that the resulting shape is connected (i.e. you can move from any square to any other through edges that are connected) and simply-connected (i.e. the shape has no holes).

For example in the following figure the first is a polyomino; the second is not (the edges do not fully line up); the third is not (same reason) and the fourth is not (it fails to be simply connected).



Prove that if $P(n)$ denotes the number of polyominoes that can be built from n unit squares (identifying two polyominoes if one can be transformed into another by a rigid motion), then $P(n)$ satisfies $A^n < P(n) < B^n$ for all $n \geq 3$ for suitable constants $1 < A < B$.