UVIC MATHEMATICS COMPETITION 2013

- All the necessary work to justify an answer and all the necessary steps of a proof should be shown clearly to obtain full credit.
- Partial credit will be given only for substantial progress towards a solution.
- All **FOUR** questions are worth equal marks.

1) There are 8!=40320 different permutations of the numbers 1,2,3,4,5,6,7,8. For example 4,1,3,8,2,7,5,6 is one of them.

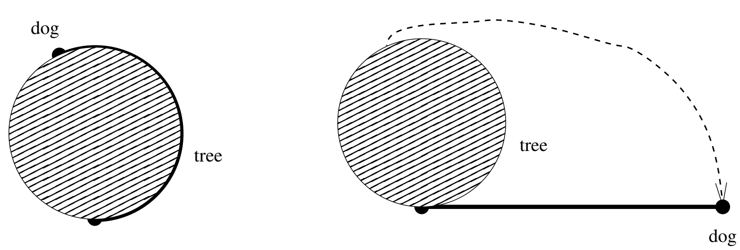
If these are put into dictionary ('lexicographic') order 1,2,3,4,5,6,7,8 is the first; 1,2,3,4,5,6,8,7 is the second and 8,7,6,5,4,3,2,1 is the 40320th.

Which is the 12000th permutation in the lexicographically-ordered list?

2) A dog is tied to a tree trunk of radius 1 by a rope of length 10 attached at a fixed point on the trunk of the tree. The rope is initially taut and fully wound around the trunk. The dog runs around the tree unwinding the rope and keeping the rope taut until the rope is tangential to the tree trunk.

This is illustrated (not to scale) in the figures below: the left panel shows the initial position and the right panel shows the final position, with the thick line representing the rope and the dashed line representing the path of the dog.

What is the total distance run by the dog? As indicated in the figure, you should assume that all motion takes place in the horizontal plane



3) A *cross* is the union of two perpendicular line segments of the same finite length with a common midpoint. The *size* of a cross is the length of the line segments making it up.

For instance, a cross of size $\sqrt{2}/5$ is formed by the line segments joining (2.3,3.1) to (2.5,2.9); and (2.5,3.1) to (2.3,2.9).

There are uncountably many distinct crosses. For instance, you can centre a cross of size 1 at any point in the plane.

Suppose that S is a set of crosses in the plane with the property that any two distinct crosses in S are *disjoint*. Prove that S is countable.

4) The Fibonacci numbers are given by $F_1 = 1$, $F_2 = 2$ and for $n > 2$, $F_n = F_{n-1} + F_{n-2}$.

Prove that every positive integer can be expressed in the form $F_{n_1} + F_{n_2} + \ldots + F_{n_k}$ for some $k \geq 1$, where $n_i \geq n_{i+1} + 2$ for each $i < k$.