Math 101 - Calculus II
Euler's relation and complex numbers

Complex numbers are numbers that are constructed to solve equations where square roots of negative numbers occur. These numbers look like

$$1+i, \quad 2i, \quad 1-i$$

They are added, subtracted, multiplied and divided with the normal rules of algebra with the additional condition that $i^2=-1$. The symbol i is treated just like any other algebraic variable. So,

$$(2+i)^2=(2+i)(2+i)=2\cdot 2+2\cdot i+2\cdot i+i^2=4+4i-1=3+4i$$

and

$$(x-i)(x+i)=x^2+x\cdot i-x\cdot i-i^2=x^2+1.$$

This last example shows that when complex numbers are used, we are able to factor more expressions than we can with real numbers. These numbers are also useful with the quadratic formula

$$x^2+x+1=0 \quad \Rightarrow \quad x=\frac{-1\pm\sqrt{1-4}}{2}=\frac{-1\pm i\sqrt{3}}{2}. \tag{1}$$

The real and imaginary parts of a complex number are given by

$$\text{Re}(3-4i)=3 \quad \text{and} \quad \text{Im}(3-4i)=-4.$$

This means that if two complex numbers are equal, their real and imaginary parts must be equal.

Next we investigate the values of the exponential function with complex arguments. This will leaf to the well-known Euler formula for complex numbers. We start with the power series for e^x and substitute $x=i\theta$. Therefore

$$\begin{aligned} e^x &= \sum_{n=0}^{\infty}\frac{x^n}{n!}=\sum_{n=0}^{\infty}\frac{(i\theta)^n}{n!} \\ &= e^{i\theta}=\sum_{n=0}^{\infty}\frac{i^n\theta^n}{n!}=\underbrace{\left(1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\frac{\theta^6}{6!}+\cdots\right)}_{\text{for } n \text{ even}}+\underbrace{\left(\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\frac{\theta^7}{7!}+\cdots\right)}_{\text{for } n \text{ odd}}i \end{aligned}$$

where we used $i^2=-1$ and arranged the terms into real and imaginary parts. Now if we recall the power series for sine and cosine,

$$\begin{aligned} \sin\theta &= \sum_{n=0}^{\infty}\frac{(-1)^n\theta^{2n+1}}{(2n+1)!}=\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\frac{\theta^7}{7!}+\cdots \\ \cos\theta &= \sum_{n=0}^{\infty}\frac{(-1)^n\theta^{2n}}{(2n)!}=1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\frac{\theta^6}{6!}+\cdots \end{aligned}$$

we see that

$$e^{i\theta}=\cos\theta+i\sin\theta$$

which is Euler's formula and has extensive uses in physics, electronics, electrical engineering and mathematics. For example, if we substitute $\theta=\pi$, we obtain

$$e^{i\pi}+1=0$$

which combines five of the most important mathematical constants: 0, 1, i, π and e.

Trigonometric identities can easily be proved using Euler's formula. Here is a demonstration for the formulas $\cos(\theta_1+\theta_2)$ and $\sin(\theta_1+\theta_2)$:

$$\begin{aligned}
e^{i(\theta_1+\theta_2)} &= e^{i\theta_1+i\theta_2} = e^{i\theta_1}\cdot e^{i\theta_2} \\
\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2) &= (\cos\theta_1+i\sin\theta_1)(\cos\theta_2+i\sin\theta_2) \\
&= \cos\theta_1\cos\theta_2-\sin\theta_1\sin\theta_2+(\sin\theta_1\cos\theta_2+\cos\theta_1\sin\theta_2)i
\end{aligned}$$

If we now equate the real and imaginary parts, we have

$$\begin{aligned}
\cos(\theta_1+\theta_2) &= \cos\theta_1\cos\theta_2-\sin\theta_1\sin\theta_2, \\
\sin(\theta_1+\theta_2) &= \sin\theta_1\cos\theta_2+\cos\theta_1\sin\theta_2.
\end{aligned}$$

Next we give on application of Euler's formula in finding roots of numbers. As an example, we will find the three cube roots of one. Euler's formula yields

$$e^{i2\pi n} = \cos 2\pi n + i\sin 2\pi n = 1, \quad \text{for} \quad n=0,1,2,3,\ldots$$

since sine and cosine both are 2π-periodic. Now this means by the property of exponentiation that

$$(1)^{\frac{1}{3}} = \left(e^{i2\pi n}\right)^{\frac{1}{3}} = e^{\frac{2n\pi}{3}i} = \cos\frac{2n\pi}{3}+i\sin\frac{2n\pi}{3}, \quad \text{for} \quad n=0,1,2.$$

The right hand side of this expression gives the three different values for $n=0,1,2$, namely

$$\begin{aligned}
(1)^{\frac{1}{3}} &= \cos 0 + i\sin 0 = 1 && (n=0) \\
&= \cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3} = -\frac{1}{2}+\frac{\sqrt{3}}{2}i && (n=1) \\
&= \cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3} = -\frac{1}{2}-\frac{\sqrt{3}}{2}i && (n=2)
\end{aligned}$$

The other values for different n just cycle through these same three values. You may check that the cube of any one of these values simplifies to one of the three above. One may also obtain exactly the same roots by factoring the polynomial $x^3-1=0$, and using the quadratic formula in

$$x^3-1=(x-1)(x^2+x+1)=0,$$

where the roots of x^2+x+1 was obtained in (1).

One should not use the expression $\sqrt{-1}$ in place of the symbol i. The expression has many pitfalls, for example

$$\begin{aligned}
\sqrt{-1} &= \sqrt{-1} \\
\sqrt{\frac{-1}{1}} &= \sqrt{\frac{1}{-1}} \\
\frac{\sqrt{-1}}{\sqrt{1}} &= \frac{\sqrt{1}}{\sqrt{-1}} \\
\sqrt{-1}\sqrt{-1} &= \sqrt{1}\sqrt{1} \quad \Rightarrow \quad -1=1.
\end{aligned}$$

Which step is wrong in the above argument?

Sample problems on complex numbers

1. Calculate each of the following and express the result in the form $a + bi$:

   (a) $(1 + 2i) - (1 + 3i)$

   (b) $(1 + 2i)(1 + 3i)$

   (c) $(1 + 2i)^2$

   (d) $e^{\frac{\pi}{3}i}$

   (e) $e^{\frac{\pi}{6}i}$

   (f) $ie^{\frac{\pi}{6}i}$

   (g) $\dfrac{1+2i}{1+3i}$ (HINT: Let $\dfrac{1+2i}{1+3i} = a + bi$ and multiply both sides by $1 + 3i$.)

2. Find the three cube roots of -8 by solving the equation $x^3 + 8 = 0$. One of the solutions is real and the other two are complex. HINT: $x^3 + 8 = (x + 2)(x^2 - 2x + 4)$.

3. Find the 6 sixth roots of 1 by examining the values of $1^{\frac{1}{6}} = \left(e^{i2n\pi}\right)^{\frac{1}{6}}$ for various values of the integer n.

Solutions

1. (a) $(1 + 2i) - (1 + 3i) = (1 - 1) + (2 - 3)i = -i$

   (b) $(1 + 2i)(1 + 3i) = 1(1 + 3i) + 2i(1 + 3i) = 1 + 3i + 2i + 6i^2 = 1 + 5i - 6 = -5 + 5i$

   (c) $(1 + 2i)^2 = (1 + 2i)(1 + 2i) = 1(1 + 2i) + 2i(1 + 2i) = 1 + 2i + 2i + 4i^2 = 1 + 4i - 4 = -3 + 4i$

   (d) $e^{\frac{\pi}{3}i} = \cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3} = \dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i$

   (e) $e^{\frac{\pi}{6}i} = \cos\dfrac{\pi}{6} + i\sin\dfrac{\pi}{6} = \dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i$

   (f) $ie^{\frac{\pi}{6}i} = i\left(\dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i\right) = \dfrac{\sqrt{3}}{2}i + \dfrac{1}{2}i^2 = -\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i$

   (g) Let $\dfrac{1+2i}{1+3i} = a + bi$. Then we have that

   $$1 + 2i = (1 + 3i)(a + bi) = 1(a + bi) + 3i(a + bi) = (a - 3b) + (3a + b)i,$$

   and consequently require that $a - 3b = 1$ and $3a + b = 2$ (on equating the real and imaginary components in the above expression). After solving for $a = \frac{7}{10}$ and $b = -\frac{1}{10}$, we have conclude that

   $$\frac{1+2i}{1+3i} = \frac{7}{10} - \frac{1}{10}i.$$

2. Because $x^3 + 8 = (x + 2)(x^2 - 2x + 4) = 0$ we have that either

   $$x + 2 = 0 \quad \text{or} \quad x^2 - 2x + 4 = 0.$$

   Therefore

   $$x = -2, \quad \text{or} \quad x = \frac{2 \pm \sqrt{4 - 16}}{2} = \frac{2 \pm \sqrt{-12}}{2} = 1 \pm \sqrt{3}i$$

   when utilizing the quadratic formula for finding roots of $x^2 - 2x + 4 = 0$. Therefore, the three cube roots of -8 are

   $$-2, \quad 1 + \sqrt{3}i, \quad \text{and} \quad 1 - \sqrt{3}i.$$

3. It follows from Euler's formula that $\left(e^{i2n\pi}\right)^{\frac{1}{6}} = e^{\frac{n\pi}{3}i} = \cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3}$. Due to the periodicity of the sine and cosine functions, the 6 sixth roots are given by

n	$\cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3}$
0	1
1	$\frac{1}{2} + \frac{\sqrt{3}}{2}i$
2	$-\frac{1}{2} + \frac{\sqrt{3}}{2}i$
3	-1
4	$-\frac{1}{2} - \frac{\sqrt{3}}{2}i$
5	$\frac{1}{2} - \frac{\sqrt{3}}{2}i$