On equivariant triangularization of matrix cocycles

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Outline

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Example: Coin flipping. Heads: A_H, Tails: A_T.
The string of flips $HHTHT$ gives the matrix $A_T A_H A_T A_H A_H$.
(Really just a two-sided $\frac{1}{2}$-$\frac{1}{2}$ Bernoulli shift.)

Skew Products

$(X, \mathcal{B}, \mu, T)$ as before. Let $(Y, \mathcal{A})$ be a measurable space, with invertible measurable maps indexed by X, S_x.

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Example: $Gr_1(\mathbb{F}^2)$ is the *Grassmannian* of dimension 1 subspaces of $\mathbb{F}^2$. If A is a 2-by-2 invertible matrix cocycle, then $P(x, V) = (T(x), A(1, x)V)$ is an invertible skew product.

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If each Y has a measure ν and S_x is measure-preserving, then P preserves the product measure $\mu \times \nu$.

Multiplicative Ergodic Theorem

Theorem (Multiplicative Ergodic Theorem, Invertible Case, Block Diagonalization Version)

$(X, \mathcal{B}, \mu, T)$ *ergodic, invertible base dynamics; A real invertible n-by-n matrix cocycle over* T*, with:*

$$\int_X \log^+ \|A(1,x)\| \ d\mu(x) < \infty, \ \int_X \log^+ \left\|A(1,x)^{-1}\right\| \ d\mu(x) < \infty.$$

Then there exist:

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Then there exist:

- $\lambda_1 > \lambda_2 > \cdots > \lambda_k > -\infty$;
- *positive integers* $m_1, m_2, \ldots, m_k$ *with* $m_1 + \cdots + m_k = n$;
- $C : X \to GL_n(\mathbb{R})$ *measurable such that for almost every* $x \in X$*:*

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- $\lambda_1 > \lambda_2 > \cdots > \lambda_k > -\infty$;
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- *$C : X \to GL_n(\mathbb{R})$ measurable such that for almost every $x \in X$:*
  1. **Equivalariance:** *$C(T(x))^{-1}A(x,1)C(x)$ is block diagonal, with the i^{th} block of size m_i;*
  2. **Growth:** *For $v \neq 0$ in the columnspace of the i^{th} block, $\lim_{n\to\infty} \frac{1}{n} \log \|A(n,x)v\| = \lambda_i$ and $\lim_{n\to\infty} \frac{1}{n} \log \|A(-n,x)v\| = -\lambda_i$.*

Motivation

Oseledets, 1968: proved the MET by extending the base space for the cocycle by $SO_n(\mathbb{R})$ and constructed a triangular cocycle for this extended space, in order to use nice properties of such a cocycle. Perhaps it is possible to triangularize the cocycle without extending the base?

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Single matrices: block triangularization would be better than block diagonalization with the same block sizes. We might have to pass to complex matrices instead of staying with real ones, but that's okay; we know how to get the complex one: Jordan form!

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Remark

These results are only about real-valued conjugation and normal forms.

A Better Question

Question

Can we always block upper-triangularize a matrix cocycle, over $\mathbb{C}$? That is, find $C : X \to GL_n(\mathbb{C})$ such that $C(T(x))^{-1}A(1,x)C(x)$ is block upper-triangular over $\mathbb{C}$?

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Remark

Using the MET, the problem reduces to triangularizing each block separately.

An Answer

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Answer

Not always!

The Main Theorem

$(X, \mathcal{B}, \mu, T)$ as before. Let $A(1, \cdot) : X \to O_2(\mathbb{R})$ be a cocycle of **orthogonal** matrices over T. Then:

$$A(1, x) = \begin{cases} rot(\alpha_x) & x \in X_r, \\ refl(\beta_x) & x \in X_f. \end{cases}$$

Denote $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, and equip it with Lebesgue measure as usual; equip $\mathbb{Z}_2$ with Haar measure (which is normalized counting measure).

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$$f_x(y) = \begin{cases} y + \dfrac{\alpha_x}{\pi} & x \in X_r, \\ \dfrac{2\beta_x}{\pi} - y & x \in X_f, \end{cases} \qquad g_x(a) = \begin{cases} a & x \in X_r, \\ a + 1 & x \in X_f. \end{cases}$$

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Define $S : X \times \mathbb{T} \to X \times \mathbb{T}$ and $R : X \times \mathbb{Z}_2 \to X \times \mathbb{Z}_2$ by:

$$S(x, y) = (T(x), f_x(y)), \qquad R(x, a) = (T(x), g_x(a)).$$

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Remark

S captures the action of A on the *angle*, and R captures the action of A on the *hemisphere*.

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- Get new objects that are invariant under S and R; use them to construct invariant sets of positive measure or non-constant invariant functions.
- Ergodicity of S and R is contradicted, therefore $V(x)$ cannot exist.

An Example: Irrational Rotation on Torus

$(X, \mathcal{B}, \mu, T) = (\mathbb{T}, \mathcal{B}, \lambda, T_\eta)$, where λ is Lebesgue and T_η is an irrational rotation: $T_\eta(x) = x + \eta$.

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$$A(1,x) = \begin{cases} \begin{bmatrix} \cos(\pi\alpha) & -\sin(\pi\alpha) \\ \sin(\pi\alpha) & \cos(\pi\alpha) \end{bmatrix} & x \in [0, 1-\eta), \\ \\ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} & x \in [1-\eta, 1). \end{cases}$$

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$$S(x,y) = \begin{cases} (x+\eta, y+\alpha) & x \in [0, 1-\eta), \\ (x+\eta, 1-y) & x \in [1-\eta, 1), \end{cases}$$

$$R(x,a) = \begin{cases} (x+\eta, a) & x \in [0, 1-\eta), \\ (x+\eta, a+1) & x \in [1-\eta, 1). \end{cases}$$

Another Example: Two-Symbol Bernoulli Shift

$(X, \mathcal{B}, \mu, T)$ is the two-sided left Bernoulli shift on symbols $\{0, 1\}$ with $p_0 = p_1 = \frac{1}{2}$ (coin flipping made rigorous!).

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$$S(x, y) = \begin{cases} (T(x), y + \alpha) & x \in C(x_0 = 0) \\ (T(x), 1 - y) & x \in C(x_0 = 1), \end{cases}$$

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Approach: Break the cocycle into its rotation part and its flipping part, and work on the factors.

The End

Thank you!

A Reasonable Topology

$\mathcal{C} = \{\tilde{S} : X \times \bar{\mathbb{C}} \to X \times \bar{\mathbb{C}} \ : \ \tilde{S}(x,z) = (T(x), S_x(z))\}.$
$\mathcal{R} = \{\tilde{S} : X \times \mathbb{T} \to X \times \mathbb{T} \ : \ \tilde{S}(x,y) = (T(x), S_x(y))\}.$
$\mathcal{F} = \{\tilde{S} : X \times \mathbb{Z}_2 \to X \times \mathbb{Z}_2 \ : \ \tilde{S}(x,a) = (T(x), S_x(a))\}.$

Topologies:
$\mathcal{T}_{\mathcal{C}}$: $N(\tilde{S}, \epsilon, \delta) = \{\tilde{P} \ : \ \mu\{x \ : \ \max_{z \in \bar{\mathbb{C}}}\{d(P_x(z), S_x(z))\} > \delta\} < \epsilon\}$
$\mathcal{T}_{\mathcal{R}}$: $N(\tilde{S}, \epsilon, \delta) = \{\tilde{P} \ : \ \mu\{x \ : \ \max_{y \in \mathbb{T}}\{d(P_x(y), S_x(y))\} > \delta\} < \epsilon\}$
$\mathcal{T}_{\mathcal{F}}$: $N(\tilde{S}, \epsilon, a) = \{\tilde{P} \ : \ \mu\{x \ : \ P_x(z) \neq S_x(z)\} < \epsilon\}$