On equivariant triangularization of matrix cocycles

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Outline

1. Background

2. Equivariant Triangularization

3. The Main Result
(\(X, \mathcal{B}, \mu, T\)): invertible, measure-preserving system - “base dynamics”
Invertible Matrix Cocycles

\((X, \mathcal{B}, \mu, T)\): invertible, measure-preserving system - “base dynamics”

\(A(1, \cdot) : X \to GL_n(\mathbb{F})\) - measurable, each \(A(1, x)\) invertible
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Define \(A : \mathbb{Z} \times X \to GL_n(\mathbb{F})\) by:

\[
A(n, x) = A(1, T^{n-1}(x)) \ldots A(1, T(x))A(1, x)
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\(A\) is building a product of matrices while moving along the orbit of \(x\) under \(T\), by multiplying on the left by a matrix corresponding to \(T^i(x)\) (hence ‘cocycle over \(T\)’).
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Example: Coin flipping. Heads: \(A_H\), Tails: \(A_T\).
The string of flips \(HHTHT\) gives the matrix \(A_T A_H A_T A_H A_H\).
(Really just a two-sided \(\frac{1}{2}-\frac{1}{2}\) Bernoulli shift.)
Skew Products

\((X, \mathcal{B}, \mu, T)\) as before. Let \((Y, \mathcal{A})\) be a measurable space, with invertible measurable maps indexed by \(X, S_x\).

\[ P: X \times Y \rightarrow X \times Y, \quad P(x, y) = (T(x), S_x(y)) \]

Example: \(\text{Gr}_{1}(F_2)\) is the Grassmannian of dimension 1 subspaces of \(F_2\). If \(A\) is a 2-by-2 invertible matrix cocycle, then \(P(x, V) = (T(x), A(1, x)V)\) is an invertible skew product.

If each \(Y\) has a measure \(\nu\) and \(S_x\) is measure-preserving, then \(P\) preserves the product measure \(\mu \times \nu\).
(X, B, μ, T) as before. Let (Y, A) be a measurable space, with invertible measurable maps indexed by X, S_x.

*Invertible Skew Product:*

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Multiplicative Ergodic Theorem

Theorem (Multiplicative Ergodic Theorem, Invertible Case, Block Diagonalization Version)

\((X, \mathcal{B}, \mu, T)\) ergodic, invertible base dynamics; A real invertible \(n\)-by-
\(n\) matrix cocycle over \(T\), with:

\[
\int_X \log^+ \|A(1, x)\| \, d\mu(x) < \infty, \quad \int_X \log^+ \|A(1, x)^{-1}\| \, d\mu(x) < \infty.
\]

Then there exist:

\[\lambda_1 > \lambda_2 > \cdots > \lambda_k > -\infty;\]

positive integers \(m_1, m_2, \ldots, m_k\) with \(m_1 + \cdots + m_k = n\);

\(C: X \to \text{GL}_n(\mathbb{R})\) measurable such that for almost every \(x \in X\):

1. Equivariance: \(C(T(x))^{-1} A(x, 1) C(x)\) is block diagonal, with the
   \(i\)th block of size \(m_i\);

2. Growth: For \(v \neq 0\) in the columnspace of the \(i\)th block,
   \(\lim_{n \to \infty} \frac{1}{n} \log \|A(n, x)v\| = \lambda_i\) and
   \(\lim_{n \to \infty} \frac{1}{n} \log \|A(-n, x)v\| = -\lambda_i\).
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\]
Oseledets, 1968: proved the MET by extending the base space for the cocycle by $SO_n(\mathbb{R})$ and constructed a triangular cocycle for this extended space, in order to use nice properties of such a cocycle. Perhaps it is possible to triangularize the cocycle without extending the base?
Oseledets, 1968: proved the MET by extending the base space for the cocycle by $SO_n(\mathbb{R})$ and constructed a triangular cocycle for this extended space, in order to use nice properties of such a cocycle. Perhaps it is possible to triangularize the cocycle without extending the base?

Single matrices: block triangularization would be better than block diagonalization with the same block sizes. We might have to pass to complex matrices instead of staying with real ones, but that’s okay; we know how to get the complex one: Jordan form!
If we can block diagonalize in this equivariant manner (MET), can we do better? Say, upper-triangularization?
A Vague Question

Question
If we can block diagonalize in this equivariant manner (MET), can we do better? Say, upper-triangularization?

Answer

Remark
These results are only about real-valued conjugation and normal forms.
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These results are only about real-valued conjugation and normal forms.
Can we always block upper-triangularize a matrix cocycle, over $\mathbb{C}$? That is, find $C : X \to GL_n(\mathbb{C})$ such that $C(T(x))^{-1}A(1,x)C(x)$ is block upper-triangular over $\mathbb{C}$?
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Remark
Using the MET, the problem reduces to triangularizing each block separately.
An Answer

Answer

Not always!

Joseph Horan (UVic)  Equivariant triangularization  Apr. 9, 2015
An Answer

Answer
Not always!
The Main Theorem

\((X, \mathcal{B}, \mu, T)\) as before. Let \(A(1, \cdot) : X \to O_2(\mathbb{R})\) be a cocycle of \textbf{orthogonal} matrices over \(T\). Then:

\[
A(1, x) = \begin{cases} 
\text{rot}(\alpha_x) & x \in X_r, \\
\text{refl}(\beta_x) & x \in X_f.
\end{cases}
\]

Denote \(\mathbb{T} = \mathbb{R}/\mathbb{Z}\), and equip it with Lebesgue measure as usual; equip \(\mathbb{Z}_2\) with Haar measure (which is normalized counting measure).
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\[
f_x(y) = \begin{cases} 
y + \frac{\alpha_x}{\pi} & x \in X_r, \\
\frac{2\beta_x}{\pi} - y & x \in X_f,
\end{cases}
\]

\[
g_x(a) = \begin{cases} 
a & x \in X_r, \\
a + 1 & x \in X_f.
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a & x \in X_r, \\
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Define \(S : X \times T \to X \times T\) and \(R : X \times \mathbb{Z}_2 \to X \times \mathbb{Z}_2\) by:

\[
S(x, y) = (T(x), f_x(y)), \quad R(x, a) = (T(x), g_x(a)).
\]
The Main Theorem

**Theorem**

If $S$ is ergodic, then $A$ is not equivariantly triangularizable over $\mathbb{R}$. If both $S$ and $R$ are ergodic, then $A$ is not equivariantly triangularizable over $\mathbb{C}$.
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**Theorem**

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**Remark**

$S$ captures the action of $A$ on the *angle*, and $R$ captures the action of $A$ on the *hemisphere*.
By contradiction. Assume that $A$ may be equivariantly triangularized.
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- Obtain a family of equivariant family of dimension 1 subspaces $V(x)$. 

Project $V(x)$ forward to $T$ and $Z^2$ by noting that $Gr_1(C^2)$ is the Riemann sphere. Get new objects that are invariant under $S$ and $R$; use them to construct invariant sets of positive measure or non-constant invariant functions.

Ergodicity of $S$ and $R$ is contradicted, therefore $V(x)$ cannot exist.
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An Example: Irrational Rotation on Torus

\((X, \mathcal{B}, \mu, T) = (\mathbb{T}, \mathcal{B}, \lambda, T_\eta)\), where \(\lambda\) is Lebesgue and \(T_\eta\) is an irrational rotation: \(T_\eta(x) = x + \eta\).
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\[
A(1, x) = \begin{cases} 
\begin{bmatrix} 
\cos(\pi \alpha) & -\sin(\pi \alpha) \\
\sin(\pi \alpha) & \cos(\pi \alpha) 
\end{bmatrix} & x \in [0, 1 - \eta), \\
\begin{bmatrix} 
1 & 0 \\
0 & -1 
\end{bmatrix} & x \in [1 - \eta, 1).
\end{cases}
\]
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\(S(x, y) = \begin{cases}
(x + \eta, y + \alpha) & x \in [0, 1 - \eta),

(x + \eta, 1 - y) & x \in [1 - \eta, 1),
\end{cases}\)

\(R(x, a) = \begin{cases}
(x + \eta, a) & x \in [0, 1 - \eta),

(x + \eta, a + 1) & x \in [1 - \eta, 1).\end{cases}\)
Another Example: Two-Symbol Bernoulli Shift

\((X, \mathcal{B}, \mu, T)\) is the two-sided left Bernoulli shift on symbols \(\{0, 1\}\) with 
\(p_0 = p_1 = \frac{1}{2}\) (coin flipping made rigorous!).
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$$S(x, y) = \begin{cases} 
(T(x), y + \alpha) & x \in C(x_0 = 0) \\
(T(x), 1 - y) & x \in C(x_0 = 1),
\end{cases}$$

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A Conjecture

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The set of matrix cocycles into $O_2(\mathbb{R})$ which cannot be triangularized over $\mathbb{C}$ is topologically generic (i.e. contains a dense $G_\delta$ set), with respect to a reasonable topology.
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The set of matrix cocycles into $O_2(\mathbb{R})$ which cannot be triangularized over $\mathbb{C}$ is topologically generic (ie. contains a dense $G_\delta$ set), with respect to a reasonable topology.

Approach: Break the cocycle into its rotation part and its flipping part, and work on the factors.
Thank you!
\[ \mathcal{C} = \{ \tilde{S} : X \times \bar{C} \to X \times \bar{C} : \tilde{S}(x, z) = (T(x), S_x(z)) \}. \]
\[ \mathcal{R} = \{ \tilde{S} : X \times T \to X \times T : \tilde{S}(x, y) = (T(x), S_x(y)) \}. \]
\[ \mathcal{F} = \{ \tilde{S} : X \times \mathbb{Z}_2 \to X \times \mathbb{Z}_2 : \tilde{S}(x, a) = (T(x), S_x(a)) \}. \]

Topologies:
\[ \mathcal{T}_\mathcal{C} : N(\tilde{S}, \epsilon, \delta) = \{ \tilde{P} : \mu\{ x : \max_{z \in \bar{C}} \{ d(P_x(z), S_x(z)) \} > \delta \} < \epsilon \} \]
\[ \mathcal{T}_\mathcal{R} : N(\tilde{S}, \epsilon, \delta) = \{ \tilde{P} : \mu\{ x : \max_{y \in T} \{ d(P_x(y), S_x(y)) \} > \delta \} < \epsilon \} \]
\[ \mathcal{T}_\mathcal{F} : N(\tilde{S}, \epsilon, a) = \{ \tilde{P} : \mu\{ x : P_x(z) \neq S_x(z) \} < \epsilon \} \]