An Introduction to Ergodic Theory

Normal Numbers: We Can't See Them, But They're Everywhere!

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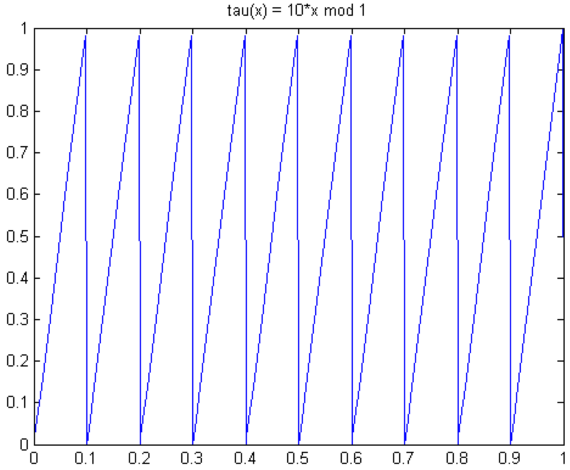
Together, this is called a *dynamical system*: $(X, \mathfrak{B}, \lambda, \tau)$. One can think of it like a state space, which evolves over time by way of iterating τ.


tau(x) = 10*x mod 1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

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If we don't have an ergodic pair (λ, τ), then if $\tau^{-1}(A) = A$ with non-zero measure, we could study τ just on A instead, and so decompose our space. Here, our (λ, τ) are ergodic.

A Specific Case of Birkhoff's Ergodic Theorem

Theorem (Birkhoff)

Let everything be as above, and $f \in L^1([0,1])$. Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(\tau^i(x)) = \int_{[0,1]} f \; d\lambda,$$

where the left-hand side converges almost everywhere with respect to λ.

Briefly, *time average* = *space average*.

The set of $x \in [0,1]$ for which this is true depends on τ and on f.

f is an observable on the state space, so it samples points.

Normal Numbers

What do we mean by "normal"? Essentially, a real number x is normal if for any base b, the frequency of finite strings of a fixed length in the base b representation of x is uniform, ie. whenever a word w has length n, the frequency with which it shows up is $\frac{1}{b^n}$, independent of which word it is.

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Conjecture

There exists at least one normal number.

Borel Normal Number Theorem

Theorem (Borel, 1909)

Almost every real number, with respect to the Lebesgue measure, is normal.

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- $\tau(0.x_1x_2\ldots) = x_1.x_2x_3\ldots \mod 1 = 0.x_2x_3\ldots$
- $\chi_{I_k}(x)$ checks if the first digit of x is k. That is:

$$\chi_{I_k}(x) = \begin{cases} 1, & x = 0.kx_2x_3\ldots \\ 0, & \text{otherwise} \end{cases}$$

Proof continued

$\chi_{I_k} \in L^1([0,1])$, so we can apply the Birkhoff Ergodic Theorem:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_{I_k}(\tau^i(x)) = \int_{[0,1]} \chi_{I_k} \, d\lambda$$
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In general, we do this for a more general word instead of a single digit, and then we use countable sub-additivity of the measure to conclude the proof. □

Conclusion

We introduced ergodic theory, and applied it to a neat problem that seemed far removed from the abstract theory. Turns out that ergodic theory has other such surprising applications!

Ask or see the extended abstract for references.