Ergodicity of a Furstenberg-Type Map

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Consider the space $X = [0,1) \times [0,1)$, with the usual product Borel σ-algebra $\mathcal{B}$ and product Lebesgue measure, which we shall denote by λ. Let $\eta \in [0,1) \setminus \mathbb{Q}$. Define the map

$$T : X \to X,\ T(x,y) = (x+\eta, y+x),$$

where the operations are assumed to be modulo 1. This map arises as a particular skew product of $[0,1)$ with itself.

Theorem 0.1. *T is an ergodic map, with respect to the product Lebesgue measure.*

This has been proved by many, first by Furstenburg [1]. Here, we shall give a geometric proof, using a particular characterization of ergodicity.

Proof. Recall that T is ergodic if and only if for any A, B in a generating semi-algebra for $\mathcal{B}$, we have

$$\frac{1}{n}\sum_{k=0}^{n-1} \lambda(T^{-k}(A) \cap B) \underset{n\to\infty}{\to} \lambda(A)\lambda(B).$$

Note that T is invertible; its inverse is given by $T^{-1}(x,y) = (x-\eta, y-x+\eta)$. For proof, behold!

$$T(T^{-1}(x,y)) = T(x-\eta, y-x+\eta) = (x-\eta+\eta, y-x+\eta+x-\eta) = (x,y)$$
$$T^{-1}(T(x,y)) = T^{-1}(x+\eta, y+x) = (x+\eta-\eta, y+x-(x+\eta)+\eta) = (x,y).$$

Both T and T^{-1} are measure-preserving. Then, recall that T is ergodic if and only if T^{-1} is ergodic; to see this, note that

$$(T^{-1})^{-1}(A) = T(A) = A \iff T^{-1}(T(A)) = A = T^{-1}(A),$$

and so our claim is easily verified. Thus the characterization of ergodicity of T is equivalent to saying that for all A, B in a generating semi-algebra for $\mathcal{B}$:

$$\frac{1}{n}\sum_{k=0}^{n-1} \lambda(T^{k}(A) \cap B) \underset{n\to\infty}{\to} \lambda(A)\lambda(B).$$

We shall show that this holds for any open rectangles

$$A = (a,b) \times (c,d),\ B = (e,f) \times (g,h),$$

since we know that the open rectangles generate the Borel σ-algebra on X. Assume that

$$0 < a < b < 1,\ 0 < c < d < 1,\ 0 < e < f < 1,\ 0 < g < h < 1,$$

since any other rectangle can be formed as a disjoint union of these rectangles. Our job, then, is to estimate what $\lambda(T^k(A) \cap B)$ is.

First, we determine what $T^k(A)$ looks like. We compute T^k essentially by inspection, though we could do an induction argument:

$$T^k(x,y) = (x + k\eta, y + kx + \frac{(k-1)k}{2}\eta).$$

This seems mildly daunting, but it's less so if we consider it first as a map on $\mathbb{R}^2$. We'll look at $T(A)$ to start. A is the rectangle bounded by the vertical lines segments

$$L_1 = \{(a, y) \ : \ y \in (c, d)\}, \ L_2 = \{(b, y) \ : \ y \in (c, d)\},$$

and the horizontal line segments

$$L_3 = \{(x, c) \ : \ x \in (a, b)\}, \ L_4 = \{(x, d) \ : \ x \in (a, b)\}.$$

What does T do to these lines? We see that:

$$\begin{aligned} T(a, y) &= (a + \eta, y + a), \ T(b, y) = (b + \eta, y + b) \\ T(x, c) &= (x + \eta, c + x), \ T(x, d) = (x + \eta, d + x). \end{aligned}$$

In particular, T shifts the rectangle to the right by η, and then *shears* the rectangle into a parallelogram, by raising the right edge of the rectangle $b - a$ more than the left edge. As the second line of computation shows, the horizontal line segments are mapped to line segments of slope 1, with appropriate endpoints. Similarly, we obtain that:

$$\begin{aligned} T^k(a, y) &= ((a + k\eta, y + ka + \frac{(k-1)k}{2}\eta)), \ T^k(b, y) = ((b + k\eta, y + kb + \frac{(k-1)k}{2}\eta)) \\ T^k(x, c) &= ((x + k\eta, c + kx + \frac{(k-1)k}{2}\eta)), \ T^k(x, d) = ((x + k\eta, d + kx + \frac{(k-1)k}{2}\eta)). \end{aligned}$$

So after k iterations, the rectangle is mapped to a very sheared parallelogram, with sides of slope k. This means that while it takes up the same width in the x-axis, it stretches over more of the y-axis. The exact value of that stretch is the height of the vertical sides plus the height attributable to the shear:

$$H_k = (d - c) + y + kb + \frac{(k-1)k}{2}\eta - y + ka + \frac{(k-1)k}{2}\eta = (d - c) + k(b - a).$$

We might also wish to know how wide the parallelogram is, parallel to the x-axis, while it is sloping upwards. We can figure this out by some trigonometry; observe that the width W_k lies on the top of right-angled triangle opposite the angle θ, where the hypotenuse has slope k and the other side is of length $d - c$. That yields

$$\frac{W_k}{d - c} = \tan(\theta) = \tan(\arctan(\frac{1}{k})) = \frac{1}{k},$$

which tells us that $W_k = \frac{d-c}{k}$.

This describes the parallelogram, ignoring the wrap-around that actually occurs while on the torus. Taking this into account, we see that the (not-necessarily-integer-valued) number of times $T^k(A)$ wraps completely around the vertical direction of the torus is given by H_k. Note that we ignore the vertical shifting, because it's meaningless considering the extreme wraparound; it doesn't exactly matter immensely how high up the parallelogram starts.

We'd like to estimate the intersection of $T^k(A)$ and B. The former is a collection of stripes, of width W_k. Any full intersection of a stripe with B is of height $h - g$. We estimate the number of stripes by considering the intersection of $\sigma^k(a, b)$ and (e, f), where $\sigma(x) = x + \eta$. The distance between left end-points of stripes is exactly $\frac{1}{k}$, because the parallelograms have slope k, and the torus has vertical height 1. Then the number of stripes in the intersection is approximated by:

$$\frac{\lambda(\sigma^k(a, b) \cap (e, f))}{\frac{1}{k}},$$

with error $\mathcal{O}(1)$, because we may be missing part of a stripe depending on vertical location of $T^k(a, b)$. Then we may approximate the area of intersection of $T^k(A)$ and B by:

$$(h - g)\frac{(d - c)}{k}\frac{\lambda(\sigma^k(a, b) \cap (e, f))}{\frac{1}{k}} = (h - g)(d - c)\lambda(\sigma^k(a, b) \cap (e, f)),$$

with error $\mathcal{O}(\frac{1}{k})$, because the leftover area from the extra partial stripe would be at most $(h-g)\frac{(d-c)}{k}$. The quantity $\lambda(\sigma^k(a,b)\cap(e,f))$ converges to $(b-a)(f-e)$, by Birkhoff's Theorem; this allows to conclude:

$$\begin{aligned}
\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\lambda(T^k(A)\cap B) &= \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\Big((h-g)(d-c)\lambda(\sigma^k(a,b)\cap(e,f))+\mathcal{O}(\frac{1}{k})\Big)\\
&= (h-g)(d-c)\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\lambda(\sigma^k(a,b)\cap(e,f))+\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\mathcal{O}(\frac{1}{k})\\
&= (h-g)(d-c)(b-a)(f-e)+\lim_{n\to\infty}\frac{1}{n}\mathcal{O}(\log(n))\\
&= (b-a)(d-c)(f-e)(h-g).
\end{aligned}$$

Therefore T is ergodic. □

References

[1] H. Furstenberg. Strict ergodicity and transformation of the torus. *Amer. J. Math.*, 83:573–601, 1961.